# CONVERSION FROM NONSTANDARD MATRIX ALGEBRAS TO STANDARD FACTORS OF TYPE II₁

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## 1. Introduction

In the recent applications of nonstandard analysis the following method of research has been acknowledged to be useful: To find a construction of a standard object by taking a standard part of a nonstandard object which is a nonstandard extension of a standard object or a well-defined internal object. By this method we can construct a complicated mathematical structure from a much simpler structure in the nonstandard universe. Loeb's construction [7] of measure spaces from simpler internal measure spaces has been known as one of the most successful results along with this line. For Banach space theory Henson and Moore [5] have found a construction of Banach spaces, called nonstandard hulls, from internal Banach spaces, and succeeded in characterizing several deep properties of Banach spaces by simpler properties of the nonstandard hulls obtained from their nonstandard extensions. In this paper we will apply this method to the theory of operator algebras. We will give a construction of a factor of type $II_1$ from a much simpler internal matrix algebra, and investigate some properties of this factor by the methods of infinitesimal analysis and hyperfinite combinatrics.

To summarize our construction in advance, let $\nu$ be a nonstandard natural number in an $\aleph_1$-saturated nonstandard universe. Consider the internal algebra of $\nu \times \nu$ matrices over the internal complex numbers, and pay attention to two norms on this algebra. One is the operator norm and the other is the normalized Hilbert-Schmidt norm. Collect all matrices with finite operator norm, and identify two such matrices if the normalized Hilbert-Schmidt norm of their difference is infinitesimal. The resulting algebra, equipped with the quotient norm that comes from the operator norm, is our factor of type $II_1$. Section 2 covers the fact that this is really a von Neumann algebra and a factor of type $II_1$. Section 3 proves the nonseparability of its representations, and Section 4 proves that it is not approximately finite. The proofs of these two results simplifies considerably the corresponding proofs

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for the ultraproducts given by Feldman [4] and Widom [11]. In Section 5, the relation of our construction to the ultraproduct construction of factors of type $II_1$ due to Wright [12] and Feldman [3] will be discussed. An interesting open problem will be posed there: To what extent does the algebra constructed in our approach depend on the nonstandard integer $\nu$?

For the basic framework of nonstandard analysis we will refer to Stroyan-Luxemburg [8], Hurd-Loeb [6] and Chang-Keisler [1] as standard textbooks. Throughout this paper, we shall denote the nonstandard extension of $A$ by $^\star A$ instead of $^*A$ in order to reserve the symbol $*$ for denoting the involution of operator algebras. For the basic theorems and terminology of operator algebras, we will refer the reader to Dixmier [2] and Takesaki [9]. Most proofs will be written in more detail than required of specialists on operator algebras in order to appeal to nonspecialists interested in nonstandard analysis.

## 2. Construction of a factor of type $II_1$

A nonstandard universe is called $\kappa$-*saturated* if any family $\langle S_i | i \in I \rangle$ of internal sets with $\mathrm{card}(I) < \kappa$ satisfying the finite intersection property always has a nonempty intersection. Throughout this paper, we will work with a fixed $\aleph_1$-saturated nonstandard universe. Such a nonstandard universe is obtained, for instance, by the usual bounded ultrapower construction [1, page 284].

Let $\nu$ be a nonstandard natural number and $^\star C$ the internal complex number field. Let $^\star C^\nu$ be the $\nu$-dimensional internal unitary space with the natural internal inner product and the internal norm $\| \cdot \|$ derived by the inner product. Let $M = {}^\star M(\nu)$ be the internal algebra of $\nu \times \nu$ matrices over $^\star C$. Naturally, $M$ acts on $^\star C^\nu$ as the internal linear operators, and let $p_\infty$ be the operator norm on $M$, i.e., $p_\infty(x) = \sup\{\|x\xi\| \mid \|\xi\| \le 1, \xi \in {}^\star C^\nu\}$. Denote by $x^*$ the adjoint of $x \in M$. Let $\tau$ be the internal normalized trace on $M$, i.e.,

$$\tau(x) = (1/\nu) \sum_{i=1}^{\nu} x_{ii} \quad \text{for} \quad x = (x_{ij}) \in M.$$

Then $\tau$ defines an internal inner product $(\cdot \mid \cdot)$ on $M$ by $(x|y) = \tau(y^*x)$ for $x, y \in M$. Its derived norm called the normalized Hilbert-Schmidt norm is denoted by $p_2$, i.e., $p_2(x) = \tau(x^*x)^{1/2}$ for $x \in M$. Denote by $(M, p_\infty)$ and $(M, p_2)$ the normed linear space structures on $M$ equipped with these respective norms. The *principal galaxies* $\mathrm{fin}_\infty(M)$ of $(M, p_\infty)$, and $\mathrm{fin}_2(M)$ of $(M, p_2)$, are defined as follows.

$$\mathrm{fin}_\infty(M) = \{x \in M | p_\infty(x) \text{ is finite}\},$$
$$\mathrm{fin}_2(M) = \{x \in M | p_2(x) \text{ is finite}\}.$$

The *principal monads* $\mu_\infty(0)$ of $(M, p_\infty)$, and $\mu_2(0)$ of $(M, p_2)$, are defined as follows.

$$\mu_\infty(0) = \{x \in M | p_\infty(x) \text{ is infinitesimal}\},$$
$$\mu_2(0) = \{x \in M | p_2(x) \text{ is infinitesimal}\}.$$

By properties of norms these sets are linear spaces over $\mathbf{C}$. By the hull completeness theorem [6, page 155], the quotient space $\hat{M}_2 = \text{fin}_2(M)/\mu_2(0)$ turns out to be a Hilbert space, called the *nonstandard hull* of $(M, p_2)$, with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|_2$ defined by

$$\langle x + \mu_2(0) | y + \mu_2(0) \rangle = {}^\circ(x|y)$$

and

$$\| x + \mu_2(0) \|_2 = {}^\circ p_2(x)$$

for $x, y \in \text{fin}_2(M)$. In the rest of this paper, we will write $\hat{x} = x + \mu_2(0)$ for all $x \in \text{fin}_2(M)$. Similarly, the nonstandard hull $\hat{M}_\infty = \text{fin}_\infty(M)/\mu_\infty(0)$ of $(M, p_\infty)$ is a Banach *-algebra equipped with the operations inherited from the matricial operations and norm $\hat{p}_\infty$ defined by $\hat{p}_\infty(x + \mu_\infty(0)) = {}^\circ p_\infty(x)$ for $x \in \text{fin}_\infty(M)$. In what follows, we will write $\tilde{x} = x + \mu_\infty(0)$. Now, it is easy to see that the norm $\hat{p}_\infty$ satisfies the $C^*$-condition, i.e., $\hat{p}_\infty(\tilde{x}^* \tilde{x}) = \hat{p}_\infty(\tilde{x})^2$ for all $\tilde{x} \in \hat{M}_\infty$, and hence $(\hat{M}_\infty, \hat{p}_\infty)$ is a $C^*$-algebra.

By transfer principle, we have $p_2(x) \leq p_\infty(x)$ for all $x \in M$, and hence we have the following chain of linear subspaces,

$$\mu_\infty(0) \subseteq \mu_2(0) \cap \text{fin}_\infty(M) \subseteq \text{fin}_\infty(M) \subseteq \text{fin}_2(M) \subseteq M.$$

In the rest of this section, we will examine the structure of another quotient space $\hat{M}$ defined by

$$\hat{M} = \text{fin}_\infty(M)/(\mu_2(0) \cap \text{fin}_\infty(M)),$$

and prove that it is a factor of type $\text{II}_1$. Let $J_2$ be a linear subspace of $\hat{M}_\infty$ defined by

$$J_2 = (\mu_2(0) \cap \text{fin}_\infty(M))/\mu_\infty(0).$$

Then clearly the relation $\hat{M} = \hat{M}_\infty/J_2$ holds. Thus we can define the quotient norm $\| \cdot \|_\infty$ on $\hat{M}$ inherited from $\hat{M}_\infty$, i.e.,

$$\| \tilde{x} + J_2 \|_\infty = \inf_{\tilde{y} \in J_2} \hat{p}_\infty(\tilde{x} + \tilde{y}) = \inf_{y \in \mu_2(0) \cap \text{fin}_\infty(M)} {}^\circ p(x + y),$$

for $x \in \text{fin}_\infty(M)$. Since there is an obvious one-to-one correspondence between $\tilde{x} + J_2 \in \hat{M}$ and $\hat{x} \in \hat{M}_2$ for $x \in \text{fin}_\infty(M)$, we identify them so that we write $\hat{x} = \tilde{x} + J_2$ for $x \in \text{fin}_\infty(M)$ and that $\hat{M} \subseteq \hat{M}_2$.

LEMMA 2.1. *The space $J_2$ is a closed two-sided ideal of the $C^*$-algebra $\hat{M}_\infty$ and hence $(\hat{M}, \| \cdot \|_\infty)$ is a $C^*$-algebra.*

*Proof.* By transfer principle, we have $p_2(a^*) = p_2(a)$ and $p_2(xa) \leq p_\infty(x)p_2(a)$ for all $a, x \in M$, and hence if $\tilde{a} \in J_2$ and $\tilde{x} \in \hat{M}_\infty$ then $\tilde{a}^*, \tilde{x}\tilde{a} \in J_2$ so that $\tilde{a}\tilde{x} \in J_2$ by taking the adjoint. Thus $J_2$ is a two-sided ideal of $\hat{M}_\infty$. To show the closedness of $J_2$, suppose that $\hat{p}_\infty(\tilde{x}_n - \tilde{x}) \to 0$ for a sequence $\tilde{x}_n \in J_2$ and $\tilde{x} \in \hat{M}_\infty$. Then we have

$$°p_2(x) = °p_2(x - x_n) \leq °p_\infty(x - x_n) = \hat{p}_\infty(\tilde{x} - \tilde{x}_n),$$

for all $n$, so that by the assumption we have $°p_2(x) = 0$, i.e., $\tilde{x} \in J_2$. Thus $J_2$ is a closed two-sided ideal of $\hat{M}_\infty$ and hence $\hat{M} = \hat{M}_\infty/J_2$ is a $C^*$-algebra [9, page 31]. ☐

To show that $\hat{M}$ is a $W^*$-algebra, i.e., a $C^*$-algebra which has a faithful *-representation as a von Neumann algebra on a Hilbert space, we consider the following representation. By transfer principle, we have $p_2(ax) \leq p_\infty(a)p_2(x)$ for any $a \in \text{fin}_\infty(M)$ and $x \in \text{fin}_2(M)$. Thus, given $a \in \text{fin}_\infty(M)$, the relation $\pi_\infty(\tilde{a})\hat{x} = \widehat{ax}$ for all $x \in \text{fin}_2(M)$ defines a bounded operator $\pi_\infty(\tilde{a})$ on $\hat{M}_2$ such that $\|\pi(\tilde{a})\| \leq \hat{p}_\infty(\tilde{a})$. By transfer principle, $\pi_\infty$ is an isometric and hence faithful representation of $\hat{M}_\infty$ on $\hat{M}_2$. Now, let $H$ be the $\pi_\infty(\hat{M}_\infty)$-invariant subspace generated by $\hat{1}$, where $1 \in M$ is the unit matrix. Obviously, $\hat{M}$ is dense in $H$. Since $\hat{1}$ is a cyclic vector for $H$, the subrepresentation of $\pi_\infty$ restricted to $H$ is unitarily equivalent to the GNS-representation of $\hat{M}_\infty$ induced by the positive linear functional $\tilde{\tau}$ such that $\tilde{\tau}(\tilde{x}) = \langle \hat{x}|\hat{1}\rangle = °\tau(x)$ [9, page 39, Theorem 9.14]. Since $\tilde{\tau}$ is a tracial state, i.e., $\tilde{\tau}(\tilde{x}\tilde{y}) = \tilde{\tau}(\tilde{y}\tilde{x})$ for all $x, y \in \text{fin}_\infty(M)$, the kernel of this representation is the set of all $\tilde{x}$ such that $\tilde{\tau}(\tilde{x}^*\tilde{x}) = 0$, and this is just $J_2$. Thus this subrepresentation induces the faithful representation $\pi$ of $\hat{M}$ on $H$ such that $\pi(\hat{a})\hat{x} = \widehat{ax}$ for all $\hat{a} \in \hat{M}$ and $\hat{x} \in H$. Denote by $\hat{\tau}$ the tracial state on $\hat{M}$ defined by $\hat{\tau}(\hat{x}) = °\tau(x)$ for $x \in \text{fin}_\infty(M)$. In what follows, we will draw no distinction between the $C^*$-algebra $\hat{M}$ and the operator algebra $\pi(\hat{M})$ acting on the Hilbert space $H$. We will refer to the representation $\{\pi, H\}$ as the canonical representation of $\hat{M}$.

THEOREM 2.2. *The representation $\{\pi, H\}$ is weakly closed and hence $\hat{M}$ is a von Neumann algebra.*

*Proof.* From [2, page 45, Theorem 2] and [2, page 307, Lemma 1], it suffices to show that the unit ball, denoted by $B$ below, of $(\hat{M}, \| \cdot \|_\infty)$ is complete with respect to the Hilbert space norm $\| \cdot \|_2$. For any standard positive integer $n$, consider the internal metric space $(A_n, d_n)$ defined by

$$A_n = \left\{ x \in M | p_\infty(x) \le 1 + \frac{1}{n} \right\} \quad \text{and} \quad d_n(x, y) = p_2(x - y).$$

Then by the hull completeness theorem [6, p. 155] its nonstandard hull is complete, and hence

$$B_n = \left\{ \hat{x} \in \hat{M} | p_\infty(x) \le 1 + \frac{1}{n} \right\}$$

is complete with respect to the Hilbert space norm. Therefore, so is $B = \bigcap_{n=1}^\infty B_n$. $\square$

Let $P$ be the internal lattice of all projections in $M$, and $L$ the lattice of all projections in $\hat{M}$. It is easy to see that the canonical map $x \mapsto \hat{x}$ maps $P$ into $L$. The following lifting theorem for projections will be useful in later discussions.

LEMMA 2.3. *The canonical map $x \in P \mapsto \hat{x} \in L$ is surjective; i.e., for any projection $\hat{x} \in \hat{M}$ there is a projection $e \in M$ such that $\hat{e} = \hat{x}$.*

*Proof.* Let $\hat{x}$ be a projection in $\hat{M}$ and let $f = x^*x \in M$. Then $f$ is an internal positive definite matrix and hence has the spectral form as follows:

$$f = \sum_{i=1}^\nu \lambda_i \xi_i \otimes \bar{\xi}_i \qquad \lambda_1 \ge \cdots \ge \lambda_\nu \ge 0,$$

where $\xi_i \otimes \bar{\xi}_i$ denotes the projection whose range is the subspace spanned by the proper vector $\xi_i \in {}^*\mathbf{C}^\nu$. On the other hand, since $\hat{x}$ is a projection in $\hat{M}$, both $x^* - x$ and $x^2 - x$ are in $\mu_2(0)$. It follows that $f - x$ is also in $\mu_2(0)$ from the relations,

$$f - x = x^*x - x = (x^* - x)x + x^2 - x.$$

Thus $\hat{f} = \hat{x}$. Let

$$\eta = \max\left\{ i | \lambda_i \ge \frac{1}{2} \right\} \quad \text{and} \quad e = \sum_{i=1}^\eta \xi_i \otimes \bar{\xi}_i.$$

Then $e$ is a projection in $M$ and we have $\hat{e} = \hat{f}$ by the following calculations:

$$p_2(e - f)^2 = \frac{1}{\nu} \sum_{i=1}^{\eta} |1 - \lambda_i|^2 + \frac{1}{\nu} \sum_{i=\eta+1}^{\nu} |\lambda_i|^2$$

$$= \frac{4}{\nu} \sum_{i=1}^{\eta} \left( \frac{1}{2} |1 - \lambda_i| \right)^2 + \frac{4}{\nu} \sum_{i=\eta+1}^{\nu} \left( \frac{1}{2} |\lambda_i| \right)^2$$

$$\leq \frac{4}{\nu} \sum_{i=1}^{\eta} |\lambda_i|^2 |1 - \lambda_i|^2 + \frac{4}{\nu} \sum_{i=\eta+1}^{\nu} |1 - \lambda_i|^2 |\lambda_i|^2$$

$$= \frac{4}{\nu} \sum_{i=1}^{\nu} |\lambda_i^2 - \lambda_i|^2$$

$$= 4p_2(f^2 - f)^2 \simeq 0.$$

It follows that $\hat{e} = \hat{x}$ and this completes the proof. $\square$

Now the following theorem concludes our construction of a *standard* factor $\hat{M}$ of type $II_1$ from a *nonstandard* matrix algebra $M$.

THEOREM 2.4.    *The von Neumann algebra $\hat{M}$ is a factor of type $II_1$.*

*Proof.* Let $\hat{e}$ be a projection in the center of $\hat{M}$. We will prove that $\hat{e} = 0$ or $\hat{e} = 1$. By Lemma 2.3, we may assume that $e$ is a projection in $M$ so that $e$ is of the form:

$$e = \sum_{i=1}^{\nu_0} \xi_i \otimes \bar{\xi}_i,$$

where $\langle \xi_i | 1 \leq i \leq \nu_0 \rangle$ is an orthonormal basis of the range of $e$. Let $\langle \xi_i | 1 \leq i \leq \nu \rangle$ be an orthonormal basis of $^*\mathbf{C}^\nu$ extending $\langle \xi_i | 1 \leq i \leq \nu_0 \rangle$. We may assume $\nu_0 \leq \nu/2$ without any loss of generality; otherwise, consider $1 - \hat{e}$. Let $w$ be the partial isometry such that

$$w\xi_i = \begin{cases} \xi_{\nu_0+i} & \text{if } 1 \leq i \leq \nu_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that $ew = 0$, $we = w$ and that $w^*w = e$, and hence $p_2(ew - we) = p_2(w)$. Since $\hat{e}$ is in the center of $\hat{M}$, we have $\|\hat{e}\hat{w} - \hat{w}\hat{e}\|_2 = 0$.

Thus we obtain

$$\|\hat{e}\|_2 = \sqrt[\nu]{\tau(e^*e)} = \sqrt[\nu]{\tau(e)} = \sqrt[\nu]{\tau(w^*w)}$$

$$= {}^\circ p_2(w) = {}^\circ p_2(ew - we) = \|\hat{e}\hat{w} - \hat{w}\hat{e}\|_2$$

$$= 0.$$

It follows that $\hat{e} = 0$ and therefore $\hat{M}$ is a factor. The restriction of the trace $\hat{\tau}$ to the projection lattice $L$ is a dimension function of $L$. Since the range of $\tau$ restricted to $P$ is the internal set $\{0, 1/\nu, \ldots, (\nu - 1)/\nu, 1\}$, the range of $\hat{\tau}$ restricted to $L$ is obtained by taking the standard parts to be $[0, 1]$. Therefore $\hat{M}$ is of type $\text{II}_1$.   $\square$

### 3. Nonseparability of representations of $\hat{M}$

In this section, we show that no non-trivial representations of $\hat{M}$ are on a separable Hilbert space.

LEMMA 3.1.  *The Hilbert space $H$ of the canonical representation $\{\pi, H\}$ of $\hat{M}$ is not separable.*

*Proof.*  Let $\langle e_k | 0 \le k \le \nu - 1 \rangle$ be an internal sequence of pairwise orthogonal minimal projections in $M = {}^*M(\nu)$. For any internal natural number $l$ with $0 \le l \le \nu - 1$, define $u(l) \in M$ by the following internal relation

$$u(l) = \sum_{k=0}^{\nu-1} \exp \frac{2kl\pi i}{\nu} e_k.$$

Then by transfer principle we have $p_\infty(u(l)) = 1$ and $\tau(u(l')^*u(l)) = \delta_{l,l'}$ for all $l, l'$ with $0 \le l, l' \le \nu - 1$. Thus $\widehat{u(l)} \in \hat{M} \subseteq H$ and $\|\widehat{u(l)} - \widehat{u(l')}\|_2 = \sqrt{2}$ if $l \ne l'$. Since the cardinality of the internal set $\{0, 1, \ldots, \nu - 1\}$ is at least $2^{\aleph_0}$, $H$ is not separable.   $\square$

THEOREM 3.2.  *Every non-trivial representation of $\hat{M}$ must be on a non-separable Hilbert space.*

*Proof.*  From [9, page 352, Theorem 5.1], every representation of a factor of type $\text{II}_1$ on a separable Hilbert space is normal. Since every non-trivial normal representation of a factor is faithful [2, page 46, Corollary 3], we can restrict our attention to faithful normal representations. Let $\mathcal{M}$ be a von Neumann algebra, on a Hilbert space $\mathcal{H}$, *-isomorphic to $\hat{M}$. It suffices to prove that $\mathcal{H}$ is not separable. The commutant $\mathcal{M}'$ is a factor either of type $\text{II}_1$ or of type $\text{II}_\infty$. If it is of type $\text{II}_\infty$, by [9, page 305, Proposition 1.40], $\mathcal{M}'$

can be spacially decomposed as

$$\mathcal{M}' \cong \mathcal{N} \hat{\otimes} \mathcal{L}(\mathcal{K}),$$

where $\dim(\mathcal{K}) = \infty$ and $\mathcal{N}$ is a factor of type $II_1$ on a Hilbert space $\mathcal{W}$. In this case, $\mathcal{N}'$ on $\mathcal{W}$ has a type $II_1$ commutant and is *-isomorphic to $\hat{M}$ by the commutation theorem for tensor products [9, page 226, Theorem 5.9]. Hence, the assertion for the type $II_1$ case will conclude that the Hilbert space $\mathcal{W}$ is not separable so that $\mathcal{H}$ is not separable. Thus we can assume that $\mathcal{M}'$ is of type $II_1$. It is well known that this case is completely classified up to unitary equivalence by the coupling constant $c$ of $\mathcal{M}$ [9, page 340, Theorem 3.11]. In the case where $c = 1$, $\{\mathcal{M}, \mathcal{H}\}$ is unitarily equivalent to the canonical representation $\{\pi(\hat{M}), H\}$, and hence, by Lemma 3.1, $\mathcal{H}$ is not separable. Consider the case where $c > 1$. Let $e'$ be a projection in $\mathcal{M}'$ with trace $1/c$. Then the reduced von Neumann algebra $\mathcal{M}_{e'}$ on $e'\mathcal{H}$ has the coupling constant 1 from [9, page 340, Proposition 3.10(ii)], and hence the assertion for $c = 1$ concludes that $e'\mathcal{H}$, and a fortiori $\mathcal{H}$, is not separable. Consider the case where $c < 1$. Let $e'$ be a projection in $\pi(\hat{M})'$ with trace $c$. Then the reduced von Neumann algebra $\pi(\hat{M})_{e'}$ on $e'H$ has the coupling constant $c$ and hence unitarily equivalent to $\mathcal{M}$ on $\mathcal{H}$. Since there is a subprojection $f'$ of $e'$ with trace $1/n$ for some positive integer $n$, the Hilbert space $H$ is covered by the ranges of $n$ pairwise orthogonal projections equivalent to $f'$. Thus obviously

$$\aleph_0 < \dim(H) \le n \times \dim(\mathcal{H}),$$

and hence $\mathcal{H}$ is not separable.

## 4. Non-approximate-finiteness of $\hat{M}$

In order to indicate the dependence of the nonstandard natural number $\nu$, we will denote by $\hat{M}(\nu)$ in this section the factor $\hat{M}$ of type $II_1$ constructed from the internal matrix algebra $*M(\nu)$. The purpose of this section is to prove that $\hat{M}(\nu)$ is not approximately finite. The approximate finiteness is usually defined for factors on separable Hilbert spaces as follows. A factor $\mathcal{M}$ is called approximately finite if there exists an increasing sequence $\mathcal{M}_n$ of finite type I subfactors of $\mathcal{M}$ which generate $\mathcal{M}$. A natural generalization of this definition to factors of type $II_1$ on arbitrary Hilbert spaces was proposed by Widom [11] as follows. A factor $\mathcal{M}$ of type $II_1$ is called *approximately finite* if given $a_i \in \mathcal{M}$ ($i = 1, \ldots, n$) and $\varepsilon > 0$ we can find a type I subfactor $\mathcal{N}$ of $\mathcal{M}$ containing elements $b_i$ ($i = 1, \ldots, n$) such that $\|a_i - b_i\|_2 < \varepsilon$, where for any $x \in \mathcal{M}$, $\|x\|_2 = \tau(x^*x)^{1/2}$ and $\tau$ stands for the normalized trace on $\mathcal{M}$.

Since $\hat{M}$ is of type $\mathrm{II}_1$ but not representable on a separable Hilbert space, as shown in the preceding sections, we adopt this definition.

For $n \in \mathbf{N}$, the algebra of $n \times n$ matrices over the complex number field $\mathbf{C}$ is denoted by $M(n)$; for this algebra, $\tau$ stands for the normalized trace, $p_\infty$ the operator norm, and $p_2$ the normalized Hilbert-Schmidt norm, just as for the nonstandard algebra ${}^\star M(\nu)$. An *embedding* of $M(m)$ into $M(n)$ is a *-isomorphism from $M(m)$ into $M(n)$ which maps the identity $1_m$ of $M(m)$ to the identity $1_n$ of $M(n)$. A system of $k$-th order *matrix units* in a von Neumann algebra $\mathscr{M}$ is a family of $k^2$ elements $c^{\alpha\beta}$ of $\mathscr{M}$ ($\alpha, \beta = 1, \ldots, k$) satisfying the following properties:

$$c^{\alpha\beta} c^{\gamma\delta} = \begin{cases} c^{\alpha\delta} & \text{if } \beta = \gamma, \\ 0 & \text{if } \beta \neq \gamma, \end{cases}$$

$$\left(c^{\alpha\beta}\right)^* = c^{\beta\alpha} \quad \text{and} \quad \sum_{\alpha=1}^{k} c^{\alpha\alpha} = 1.$$

Many well-known results on matrix algebras will play important roles in applications of nonstandard analysis to the theory of operator algebras. Among them we will refer to the following two lemmas [10].

LEMMA 4.1.   *Given $\delta > 0$ we can choose an $\varepsilon = \varepsilon(\delta) > 0$ so that for any $n \in \mathbf{N}$ there exists an $a = a(n, \delta) \in M(n)$ with $p_\infty(a) \leq 1$ such that, when $b \in M(n)$ commutes with a projection $e \in M(n)$ with $\delta < \tau(e) < 1 - \delta$, then $p_2(a - b) > \varepsilon$.*

LEMMA 4.2.   (1) *There exists an embedding of $M(m)$ into $M(n)$ if and only if $n$ is divisible by $m$.*

(2) *Let $n$ be divisible by $m$, say $n = mk$. Then $a \in M(n)$ belongs to the range of some embedding of $M(m)$ into $M(n)$ if and only if $a$ commutes with all elements of at least one system of $k$-th order matrix units in $M(n)$.*

The following lifting lemma will be useful for our purpose.

LEMMA 4.3.   (1) *Let $\langle \hat{e}_i | i = 1, \ldots, n \rangle$ be a resolution of the identity of $\hat{M}(\nu)$ with $\hat{\tau}(\hat{e}_i) = 1/n$ for all $i = 1, \ldots, n$ and suppose that $\nu$ is divisible by $n$. Then there exists an internal resolution of the identity $\langle f_i | i = 1, \ldots, n \rangle$ of ${}^\star M(\nu)$ such that $\tau(f_i) = 1/n$ and $\hat{f}_i = \hat{e}_i$ ($i = 1, \ldots, n$).*

(2) *Let $\langle \hat{c}^{\alpha\beta} \rangle$ be an $n$-th order matrix units of $\hat{M}(\nu)$ and suppose that $\nu$ is divisible by $n$. Then there exists an internal $n$-th order matrix units $\langle d^{\alpha\beta} \rangle$ of ${}^\star M(\nu)$ such that $\hat{d}^{\alpha\beta} = \hat{c}^{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, n$).*

*Proof.* (1). By Lemma 2.3, we may assume that $e_i$ $(i = 1, \ldots, n)$ are internal projections. Let $f_1$ be a projection such that $\tau(f_1) = 1/n$ and that $f_1 \geq e_1$ or $f_1 \leq e_1$. From $\hat{\tau}(\hat{e}_1) = 1/n$ we have $\hat{f}_1 = \hat{e}_1$. Let

$$e_2' = (1 - f_1)e_2(1 - f_1).$$

Then $0 \leq e_2' \leq 1 - f_1$. Since $\hat{f}_1 \perp \hat{e}_2$, we have $\hat{e}_2' = \hat{e}_2$. Moreover, let $e_2''$ be the projection obtained from the positive matrix $e_2'$ as in the proof of Lemma 2.3. Then $e_2'' \leq 1 - f_1$ and $\hat{e}_2'' = \hat{e}_2' = \hat{e}_2$. Since $\tau(e_2'') \leq \tau(1 - f_1)$ and $1/n \leq \tau(1 - f_1)$, we can choose a projection $f_2$ satisfying the following conditions: (1) $f_2 \perp f_1$, (2) $\tau(f_2) = 1/n$, and (3) $f_2 \geq e_2''$ or $f_2 \leq e_2''$. It is easily seen that $\hat{f}_2 = \hat{e}_2$. Similarly, we can inductively choose a projection $f_i$ satisfying (1) $f_i \perp f_1 + \cdots + f_{i-1}$, (2) $\tau(f_i) = 1/n$, and (3) $\hat{f}_i = \hat{e}_i$. Then $\langle f_i | i = 1, \ldots, n \rangle$ is the desired sequence.

(2) From the above discussion, we may assume that $\langle c^{\alpha\alpha} | \alpha = 1, \ldots, n \rangle$ is an internal resolution of the identity of ${}^*M(\nu)$ such that $\tau(c^{\alpha\alpha}) = 1/n$ $(\alpha = 1, \ldots, n)$. Thus it suffices to show that if $\hat{u}^*\hat{u} = \hat{e}_1$, $\hat{u}\hat{u}^* = \hat{e}_2$ and $e_1$ and $e_2$ are internal projections with $\tau(e_1) = \tau(e_2)$, then there exists an internal matrix $v$ such that $v^*v = e_1$, $vv^* = e_2$ and $\hat{u} = \hat{v}$. Let $y = e_2 u e_1$ and $h = y^*y$. Since $\hat{u}$ is the partial isometry with the initial projection $\hat{e}_1$ and the final projection $\hat{e}_2$, we have $\hat{y} = \hat{u}$ and $\hat{h} = \hat{e}_1$. Let $f$ be the projection obtained from the positive matrix $h$ as in the proof of Lemma 2.3. It follows from the construction of $f$ that $f \leq e_1$ and $\hat{f} = \hat{h} = \hat{e}_1$. By polar decomposition, we have $y = z\sqrt{h}$ for some partial isometry $z$. Let $w = zf$. Then, we obtain

$$\hat{w} = \hat{z}\hat{f} = \hat{z}\hat{h} = \hat{z}(\sqrt{h})^\wedge = \hat{y} = \hat{u}.$$

Moreover,

$$w^*w \leq f \leq rp(h) \leq e_1 \quad \text{and} \quad ww^* \leq rp(y) \leq e_2,$$

where $rp(x)$ stands for the range projection of $x$. Since $\tau(e_1) = \tau(e_2)$, we have

$$\tau(e_1 - w^*w) = \tau(e_2 - ww^*),$$

and hence there exists a matrix $s$ such that $s^*s = e_1 - w^*w$ and $ss^* = e_2 - ww^*$. Let $v = w + s$. Then $v^*v = e_1$, $vv^* = e_2$ and $\hat{v} = \hat{u}$. $\square$

The structure of $\hat{M}(\nu)$ is stable under finite perturbations of $\nu$ as follows.

LEMMA 4.4.   *If $\nu$ is nonstandard and $n$ is standard then $\hat{M}(\nu)$ and $\hat{M}(\nu + n)$ are *-isomorphic.*

*Proof.* It suffices to show that $\hat{M}(\nu)$ is *-isomorphic to $\hat{M}(\nu + 1)$. For $a = (a_{ij}) \in {}^{*}M(\nu)$, we define $\pi(a) = (\pi(a)_{ij}) \in M(\nu + 1)$ as follows:

$$
\pi(a)_{ij} = \begin{cases} a_{ij} & \text{if } i \neq \nu + 1 \text{ and } j \neq \nu + 1, \\ 0 & \text{otherwise.} \end{cases}
$$

Then the mapping $\pi$ is an internal injective *-homomorphism from $M(\nu)$ to $M(\nu + 1)$. If $p_\infty(a)$ is finite then $p_\infty(\pi(a))$ is also finite, and moreover

$$
\left\| \widehat{\pi(a)} \right\|_2^2 = {}^{\circ}\left( \frac{1}{\nu + 1} \sum_{i,j=1}^{\nu+1} |\pi(a)_{ij}|^2 \right) = {}^{\circ}\left( \frac{1}{\nu + 1} \sum_{i,j=1}^{\nu} |a_{ij}|^2 \right)
$$

$$
= {}^{\circ}\left( \frac{\nu}{\nu + 1} \cdot \frac{1}{\nu} \sum_{i,j=1}^{\nu} |a_{ij}|^2 \right) = \|\hat{a}\|_2^2.
$$

Therefore we can define a *-isomorphism $\hat{\pi}$ from $\hat{M}(\nu)$ into $\hat{M}(\nu + 1)$ by the relation $\hat{\pi}(\hat{a}) = \widehat{\pi(a)}$. It is easy to see that $\hat{\pi}(\hat{1}_\nu) = \hat{1}_{\nu+1}$. For any $\hat{c} \in \hat{M}(\nu + 1)$,

$$
\hat{c} = \hat{\pi}(\hat{1}_\nu)\hat{c}\hat{\pi}(\hat{1}_\nu) = \{\pi(1_\nu)c\pi(1_\nu)\}^{\char`\^}
$$

and $\pi(1_\nu)c\pi(1_\nu)$ is clearly in the range of $\pi$. It follows that $\hat{\pi}$ is onto. Therefore, $\hat{M}(\nu)$ is *-isomorphic to $\hat{M}(\nu + 1)$.  □

Now we prove the main theorem.

THEOREM 4.5.   $\hat{M}(\nu)$ *is not approximately finite.*

*Proof.* Let $\delta$ in Lemma 4.1 be $1/4$ and choose a standard $\varepsilon > 0$ satisfying the condition of that lemma. Then, by transfer principle, there exists an $a = {}^{*}a(\nu, 1/4) \in {}^{*}M(\nu)$ with $p_\infty(a) \leq 1$, such that when $b \in {}^{*}M(\nu)$ commutes with a projection $f \in {}^{*}M(\nu)$ with $1/4 < \tau(f) < 3/4$, then $p_2(a - b) > \varepsilon$, and hence $\|\hat{a} - \hat{b}\|_2 \geq \varepsilon$. Therefore it suffices to show that, for any $\hat{b} \in \hat{M}(\nu)$ which belongs to a finite type I subfactor, we can choose $b$ which commutes with some projection $f \in M$ with $1/4 < \tau(f) < 3/4$. Let $\mathcal{N}$ be an $I_n$-subfactor of $\hat{M}(\nu)$ and $\langle \hat{d}^{\alpha\beta}|\alpha, \beta = 1, \ldots, n \rangle$ an $n$-th order matrix units of $\mathcal{N}$. By Lemma 4.4, we may assume that $\nu$ is divisible by $n$, and hence by Lemma 4.3 we may assume that $\langle d^{\alpha\beta} \rangle$ is an internal $n$-th order matrix units. Let

$$
\mathcal{M}_n = \left\{ \sum_{\alpha,\beta=1}^{n} \lambda_{\alpha\beta} d^{\alpha\beta} | \lambda_{\alpha\beta} \in {}^{*}\mathbb{C}, \alpha, \beta = 1, \ldots, n \right\}.
$$

Suppose $\hat{b} \in \mathcal{N}$. We may assume that $b \in \mathcal{M}_n$. Since $\mathcal{M}_n$ determines an internal embedding of $^\star M(n)$ into $^\star M(\nu)$, $b$ commutes with all elements of at least one system of $\kappa(= \nu/n)$-th order matrix units $\langle c^{\alpha\beta} | \alpha, \beta = 1, \ldots, \kappa \rangle$ by Lemma 4.2 and transfer principle. Hence $b$ also commutes with $f = \sum_{\alpha=1}^{[\kappa/2]} c^{\alpha\alpha}$. By the definition of $\kappa$-th order matrix units, it is easily seen that $f$ is a projection and $\tau(c^{\alpha\alpha}) = \tau(c^{\beta\beta})$. Therefore,

$$°(\tau(f)) = °\left( \sum_{\alpha=1}^{[\kappa/2]} \tau(c^{\alpha\alpha}) \right) = °\left( \left[ \frac{\kappa}{2} \right] \frac{1}{\kappa} \right) = \frac{1}{2}.$$

It follows that $1/4 < \tau(f) < 3/4$. Thus $\hat{M}(\nu)$ is not approximately finite.
□

## 5. Remarks and problems

Although we have confined our attention to a nonstandard natural number $\nu$, if our construction is applied to a standard natural number $n$ instead, we have obviously the trivial relation $\hat{M}(n) \cong M(n)$. Thus, their structures are completely classified by the invariant $n$.

The structure of the factor $\hat{M}(\nu)$ depends in principle both on the underlying nonstandard universe and on the nonstandard natural number $\nu$. In Lemma 4.4, we have shown that $\hat{M}(\nu)$ and $\hat{M}(\mu)$ are *-isomorphic when $\nu - \mu$ is finite. However, a general problem as to when $\hat{M}(\nu)$ and $\hat{M}(\mu)$ are *-isomorphic has not been solved.

Suppose that the nonstandard universe is constructed as a bounded ultra-power of a given superstructure with respect to an index set $I$ and a countably incomplete ultrafilter $\mathcal{U}$ over $I$. Then the nonstandard natural number $\nu$ is represented by a family $\langle n_i | i \in I \rangle$ of natural numbers. In this case, the factor $\hat{M}(\nu)$ can be constructed also by the following ultraproduct construction due to Wright [12]. For each $i \in I$, let $\mathcal{M}_i = M(n_i)$ be the algebra of $n_i \times n_i$ matrices over $\mathbf{C}$ acting on the $n_i$ dimensional unitary space $\mathbf{C}^{n_i}$. Let $\mathcal{H} = \sum_{i \in I}^{\oplus} \mathbf{C}^{n_i}$ and let $\mathcal{M}$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ consisting of all operators which are of the form $x = \sum_{i \in I}^{\oplus} x_i$, where $x_i \in \mathcal{M}_i$, and $\|x\| = \sup_{i \in I} p_\infty(x_i) < \infty$. Let $\tau_i$ be the canonical trace on $\mathcal{M}_i$ normalized so that $\tau_i(1_i) = 1$, where $1_i$ is the identity in $\mathcal{M}_i$. Define a state $f$ on $\mathcal{M}$ by

$$f(x) = \lim_{\mathcal{U}} \tau_i(x_i),$$

for each $x = \sum_{i \in I}^{\oplus} x_i$ in $\mathcal{M}$. Then $f$ is a tracial state of $\mathcal{M}$ and

$$\mathcal{L}_f = \{ x \in \mathcal{M} | f(x^*x) = 0 \}$$

is a two-sided ideal. Wright [12] and Feldman [3] showed that $\mathscr{M}/\mathscr{L}_f$ is a factor of type $\text{II}_1$. In this case, we can observe that $\mathscr{M}/\mathscr{L}_f$ is *-isomorphic to our factor $\hat{M}(\nu)$. The detail goes beyond the scope of the present paper and it will be published elsewhere.

Turning to the isomorphism problem, even in the case of ultraproducts we have no clue as to whether all of them are isomorphic. Since our approach provides us with a route for discussing certain number theoretical characters of $\nu$, it is interesting whether there is an internal or external property of $\nu$ which distinguishes the structures of $\hat{M}(\nu)$.

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