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ON THE KOSZUL ALGEBRA OF A LOCAL RING

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Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and $\mathbf{x}$ a minimal system of generators of $\mathfrak{m}$. The Koszul complex $K_{\cdot}(\mathbf{x})$ is essentially independent of the choice of $\mathbf{x}$, and thus an invariant of R (as an alternating algebra equipped with an anti-derivation of degree -1). Therefore one may write $H_{\cdot}(R)$ for its homology; it carries the structure of an alternating k-algebra and is called the *Koszul algebra* of R. By the universal property of the exterior algebra $\bigwedge H_1(R)$, there is always a natural map $\lambda_{\cdot}\colon \bigwedge H_1(R) \to H_{\cdot}(R)$ which extends the identity on $H_1(R)$. (We refer to Bourbaki [2], Ch. X for notation and results related to the Koszul complex, to [2], Ch. III for exterior algebra, and to Matsumura [5] for commutative algebra.)

Using the methods of Tate [8], Assmus [1] gave the following beautiful characterization of complete intersections.

THEOREM 1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then the following are equivalent*:

(a) *R is a complete intersection*;

(b) *$H_{\cdot}(R)$ is (isomorphic with) the exterior algebra of $H_1(R)$*;

(c) *$H_{\cdot}(R)$ is generated by $H_1(R)$*;

(d) $H_2(R) = H_1(R)^2$.

In particular, R is a complete intersection if (and only if) $\lambda_{\cdot}$ is surjective. In this note we want to describe complete intersections by the injectivity of $\lambda_{\cdot}$. More precisely, we shall prove the following theorem:

THEOREM 2. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring containing a field. Then*:

(a) $H_1(R)^i = 0$ *for* $i > \operatorname{emb}\dim R - \dim R$;

(b) *in particular, R is a complete intersection if (and only if) the natural map*

$$\lambda_{\cdot}\colon \bigwedge H_1(R) \to H_{\cdot}(R)$$

is injective.

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It is easy to see that part (a) of Theorem 2 implies part (b). In fact, if $\lambda.$ is injective, then (a) yields $\dim_k H_1(R) \leq \operatorname{emb}\dim R - \dim R$, and this holds if and only if R is a complete intersection (and $\dim_k H_1(R) = \operatorname{emb}\dim R - \dim R$); see [5], §21.

The crucial argument in proving part (a) of Theorem 2 will be the theorem of Evans-Griffith [3] on order ideals of minimal generators of syzygies. This explains the restriction to rings containing a field: the theorem of Evans-Griffith has not yet been proved in general. (Even if it should fail, Theorem 2 holds 'almost' for arbitrary local rings; cf. Remarks, (a).)

Since the Koszul algebra, the property of being a complete intersection, and the numerical invariants in Theorem 2 are stable under completion, we may assume that R is complete. Then R has a presentation $R = S/I$ in which $(S, \mathfrak{n}, k)$ is a regular local ring, and $I \subset \mathfrak{n}^2$ is an ideal of S. We choose a regular system of parameters $\mathbf{y}$ in S.

For the moment, let us consider more generally a (Noetherian) ring S, and ideals $I \subset \mathfrak{n}$ of S. Let $\mathbf{y} = y_1, \ldots, y_n$ generate $\mathfrak{n}$, and $\mathbf{a} = a_1, \ldots, a_m$ generate I. We write $a_i = \Sigma a_{ji} y_j$ with $a_{ji} \in S$.

Denote the canonical bases of S^n and S^m by $f_1, \ldots, f_n$ and $e_1, \ldots, e_m$ resp., and let $\varphi : S^m \to S^n$ be the map given by the matrix (a_{ji}). Setting $u_i = \varphi(e_i) \in S^n$ we have $d_{\mathbf{a}}(e_i) = a_i = d_{\mathbf{y}}(u_i)$. Here $d_{\mathbf{a}}$ and $d_{\mathbf{y}}$ are the differentials in the Koszul complexes $K.(\mathbf{a})$ and $K.(\mathbf{y})$. Furthermore,

$$\wedge \varphi : K.(\mathbf{a}) \to K.(\mathbf{y}).$$

is a chain map. The induced map $H.(\mathbf{a}, S/I) \to H.(\mathbf{y}, S/I)$ actually yields a homomorphism

$$\Lambda.: \wedge (S/\mathfrak{n})^m \cong H.(\mathbf{a}, S/\mathfrak{n}) \to H.(\mathbf{y}, S/I)$$

of $S/\mathfrak{n}$-algebras: note that $H.(\mathbf{a}, S/I) \cong K.(\mathbf{a}) \otimes S/I \cong \wedge(S/I)^m$ and that $H.(\mathbf{y}, M)$ is annihilated by $\mathfrak{n}$ for an arbitrary S-module M.

One has natural homomorphisms

$$\rho.: H.(\mathbf{a}, S/\mathfrak{n}) \to \operatorname{Tor}^S_.(S/I, S/\mathfrak{n}),$$
$$\sigma.: H.(\mathbf{y}, S/I) \to \operatorname{Tor}^S_.(S/\mathfrak{n}, S/I).$$

By a standard argument of homological algebra, $\operatorname{Tor}^S_.(S/I, S/\mathfrak{n}) = \operatorname{Tor}^S_.(S/\mathfrak{n}, S/I)$. So we have two maps from $H.(\mathbf{a}, S/\mathfrak{n})$ to $\operatorname{Tor}^S_.(S/I, S/\mathfrak{n})$, namely $\rho.$ and $\sigma. \circ \Lambda.$. The proof of Theorem 2 hinges on the fact that these maps are essentially equal—under the proper identification of $\operatorname{Tor}^S_.(S/I, S/\mathfrak{n})$ and $\operatorname{Tor}^S_.(S/\mathfrak{n}, S/I)$. This may be a well-known fact, but we do not have a reference, and the argument is short.

We choose free resolutions $F.$ and $G.$ of S/I and $S/\mathfrak{n}$ resp. Then there are chain maps $K.(\mathbf{a}) \to F. \to S/I$ and $K.(\mathbf{y}) \to G. \to S/\mathfrak{n}$. Taking tensor products yields a commutative diagram

$$\begin{array}{ccccc} K.(\mathbf{a}) \otimes S/\mathfrak{n} & \xleftarrow{\alpha} & K.(\mathbf{a}) \otimes K.(\mathbf{y}) & \xrightarrow{\beta} & S/I \otimes K.(\mathbf{y}) \\ \downarrow & & \downarrow & & \downarrow \\ F.\otimes S/\mathfrak{n} & \longleftarrow & F.\otimes G. & \longrightarrow & S/I \otimes G.. \end{array}$$

The standard argument referred to above is that the bottom row induces an isomorphism

$$H.(F.\otimes S/\mathfrak{n}) \overset{\cong}{\leftarrow} H.(F.\otimes G.) \overset{\cong}{\rightarrow} H.(S/I \otimes G.).$$

This is the identification of

$$\operatorname{Tor}^S_.(S/I, S/\mathfrak{n}) \cong H.(F.\otimes S/\mathfrak{n}) \quad \text{and} \quad \operatorname{Tor}^S_.(S/\mathfrak{n}, S/I) \cong H.(S/I \otimes G.)$$

which we will use in the following.

Lemma 1. *One has* $\rho_s = (-1)^s \sigma_s \circ \Lambda_s$.

Proof. Let $e_1, \dots, e_m$ and $f_1, \dots, f_n$ be bases of S^m and S^n and choose elements $u_i \in S^n$ with $d_{\mathbf{y}}(u_i) = d_{\mathbf{a}}(e_i)$. It is enough to show that

$$\rho_s(\bar{e}_{i_1} \wedge \dots \wedge \bar{e}_{i_s}) = (-1)^s \sigma_s(\bar{u}_{i_1} \wedge \dots \wedge \bar{u}_{i_s}),$$

and in view of the diagram above it suffices to exhibit a cycle $z \in K.(\mathbf{a}) \otimes K.(\mathbf{y})$ such that $\alpha(z) = \bar{e}_{i_1} \wedge \dots \wedge \bar{e}_{i_s}$ and $\beta(z) = (-1)^s(\bar{u}_{i_1} \wedge \dots \wedge \bar{u}_{i_s})$. We choose

$$z = (e_{i_1} \otimes 1 - 1 \otimes u_{i_1}) \cdots (e_{i_s} \otimes 1 - 1 \otimes u_{i_s}).$$

In order to see that z is a cycle one uses that the product of cycles in $K.(\mathbf{a}) \otimes K.(\mathbf{y})$ is again a cycle. Thus it is enough to show that $e_i \otimes 1 - 1 \otimes u_i$ is a cycle, and this is immediate if one uses the definition of the differentiation on a tensor product of complexes. That $\alpha(z) = \bar{e}_{i_1} \wedge \dots \wedge \bar{e}_{i_s}$ and $\beta(z) = (-1)^s(\bar{u}_{i_1} \wedge \dots \wedge \bar{u}_{i_s})$ follows from the fact that α and β are algebra homomorphisms. □

Let us return to the special situation above in which S is a regular local ring, and $\mathbf{y}$ a regular system of parameters. Let $\mathbf{x}$ denote the sequence of residue classes of $\mathbf{y} = y_1, \dots, y_n$ in $R = S/I$. One has $H.(R) \cong H.(\mathbf{y}, R)$, and it is well known that the residue classes of the cycles u_i introduced above are a k-basis of $H_1(R)$, provided $\mathbf{a}$ is a minimal system of generators of I (cf. for example Scheja [6]). Therefore the maps $\lambda.$ and $\Lambda.$ differ only by an automorphism of $\bigwedge k^m$: both λ_1 and Λ_1 are isomorphisms $k^m \to H_1(R)$. Theorem 2 claims that $\lambda_i = 0$ for $i > \operatorname{emb\,dim} R - \dim R$. Since $\mathbf{y}$ is a regular sequence, $K.(\mathbf{y})$ is a free resolution of $k \cong S/\mathfrak{n}$, and so $\sigma.$ is an isomorphism. Summarizing our arguments, we have reduced the theorem to the fact that $\rho_i = 0$ for $i > \operatorname{emb\,dim} R - \dim R$. This follows from the next lemma since S/I has finite projective dimension over S. Moreover, one has

$$\operatorname{emb\,dim} R - \dim R = \dim S - \dim R = \operatorname{height} I.$$

LEMMA 2. *Let $(S, \mathfrak{n}, k)$ be a Noetherian local ring containing a field, and $I \subset \mathfrak{n}$ an ideal generated by a sequence* $\mathbf{a}$. *If* $\operatorname{proj\,dim} S/I < \infty$, *then the natural homomorphism*

$$H_i(\mathbf{a}, k) = K.(\mathbf{a}) \otimes k \to \operatorname{Tor}_i^S(S/I, k)$$

is zero for $i >$ height I.

Proof. The natural homomorphism $H_i(\mathbf{a}, k) \to \operatorname{Tor}_i^S(S/I, k)$ is induced by a chain map $\gamma.$ from $K.(\mathbf{a})$ to a free resolution $F.$ of S/I. It only depends on I and $\mathbf{a}$, so that we may assume that

$$F.: 0 \to F_s \xrightarrow{\varphi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

is a minimal free resolution. That $H_i(\mathbf{a}, k) = K.(\mathbf{a}) \otimes k$ and $\operatorname{Tor}_i^S(S/I, k) \cong F. \otimes k$, follows from the minimality of the complexes $K.(\mathbf{a})$ and $F.$. Thus the map

$$H.(\mathbf{a}, k) \to \operatorname{Tor}_.^S(S/I, k)$$

is just $\gamma. \otimes k$.

For an S-module M and $x \in M$ let $\mathcal{O}_M(x) = \{f(x)\colon f \in \operatorname{Hom}_S(M, S)\}$ denote its *order ideal*. We choose $M = \operatorname{Im} \varphi_i$. The theorem of Evans-Griffith says that

$$\operatorname{height} \mathcal{O}_M(\varphi_i(e)) \geq i \text{ for every element } e \in F_i,\ e \notin \mathfrak{n}F_i;$$

cf. [3], Proposition 1.6. We need the stronger assertion that height $\mathcal{O}_F(\varphi_i(e)) \geq i$ where $F = F_{i-1}$. (Of course, if $g_1, \ldots, g_w$ is a basis of F and $\varphi_i(e) = s_1 g_1 + \cdots + s_w g_w$ with $s_i \in S$, then $\mathcal{O}_F(\varphi_i(e))$ is the ideal generated by $s_1, \ldots, s_w$.)

In order to prove height $\mathcal{O}_F(\varphi_i(e)) \geq i$, we show that $\mathcal{O}_F(\varphi_i(e))_{\mathfrak{p}} = S_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ with height $\mathfrak{p} \leq i - 1$. Since $\operatorname{proj\,dim}(S/I)_{\mathfrak{p}} \leq i - 1$, the embedding $M_{\mathfrak{p}} \to F_{\mathfrak{p}}$ splits for such a prime ideal; furthermore the formation of order ideals commutes with localization. Therefore one has $\mathscr{O}_F(\varphi_i(e))_\mathfrak{p} = \mathscr{O}_M(\varphi_i(e))_\mathfrak{p}$, and that $\mathscr{O}_M(\varphi_i(e))_\mathfrak{p} = S_\mathfrak{p}$ is the result of Evans-Griffith.

The assertion of the lemma amounts to $\gamma_i(K_i(\mathbf{a})) \subset \mathfrak{n}F_i$ for $i >$ height I. Let $z \in K_i(\mathbf{a})$. If $\gamma_i(z) \notin \mathfrak{n}F_i$, then height $\mathscr{O}_F(\gamma_{i-1}(d_\mathbf{a}(z))) =$ height $\mathscr{O}_F(\varphi_i(\gamma_i(z))) \geq i$ as just explained. On the other hand, $\mathscr{O}_F(\gamma_{i-1}(d_\mathbf{a}(z))) \subset I$ since $\operatorname{Im} d_\mathbf{a} \subset IK_.(\mathbf{a})$. □

Remarks. (a) Suppose that $(S, \mathfrak{n}, k)$ is a regular local ring not containing a field. Let $p = \operatorname{char} k$, and $\bar{S} = S/(p)$. Then S is a Cohen-Macaulay local ring containing a field. Let I be an ideal of S, and $F_.$ a minimal free resolution of S/I. As in the proof of Lemma 2 we have a comparison map $K_.(\mathbf{a}) \to F_.$. Let $F'_.$ be the truncation

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to 0$$

of $F_.$. Then $F'_. \otimes \bar{S}$ is a minimal free resolution of $I \otimes \bar{S}$ over $\bar{S}$, and we can apply the theorem of Evans-Griffith to $F'_. \otimes \bar{S}$ over $\bar{S}$. With the notation of the proof of Lemma 2 it yields height $(\mathscr{O}_F(\varphi_i(e)) + (p))/(p) \geq i - 1$, and it follows easily that

$$\operatorname{height} \mathscr{O}_F(\varphi_i(e)) \geq i - 1.$$

This argument shows that Lemma 2 holds for regular rings not containing a field if we replace height I by height $I - 1$. Thus Theorem 2, (a) is valid without the hypothesis that R contains a field if $\operatorname{emb\,dim} R - \dim R$ is replaced by $\operatorname{emb\,dim} R - \dim R + 1$.

(b) The method we used to prove Theorem 2 also yields a quick proof of Theorem 1. Again one may assume that R is complete. If I is generated by an S-sequence $\mathbf{a}$, then $K_.(\mathbf{a})$ resolves R, and therefore $\rho_.$ is an isomorphism; it follows that $\lambda_.$ is an isomorphism, proving (a) ⇒ (b). While (b) ⇒ (c) ⇒ (d) is trivial, the implication (d) ⇒ (a) results from the fact that ρ_2 must be surjective if λ_2 is surjective. In order to conclude that (d) ⇒ (a) choose $F_.$ as a minimal free resolution of S/I. Then we have a commutative diagram

$$\begin{array}{ccccccc} \wedge^2 S^m & \longrightarrow & S^m & \longrightarrow & S & \longrightarrow & 0 \\ \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \cong} & & \| & & \\ F_2 & \longrightarrow & F_1 & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

The map ρ_2 is just $\gamma_2 \otimes k$, and $\gamma_2 \otimes k$ being surjective, γ is surjective itself. It follows immediately that $H_1(K_.(\mathbf{a})) = 0$, and this implies that $\mathbf{a}$ is an S-sequence ([5], Theorem 16.5).

(c) Lemma 2 is false without the hypothesis that proj dim $S/I < \infty$. In fact, Serre [7] showed that the map $H_i(\mathbf{a}, k) \to \operatorname{Tor}_i(S/I, k)$ is injective if $\mathbf{a}$ generates $I = \mathfrak{n}$. If S is not regular, this yields a counterexample.

(d) The reader may have noticed that Theorem 2 is trivial if R is a Cohen-Macaulay ring. Then dim R = depth R, and one always has $H_i(R) = 0$ for $i >$ emb dim R − depth R by the grade-sensitivity of the Koszul complex ([5], Theorem 16.8). On the other hand, if $H_1(R)^p \neq 0$ for p = emb dim R-depth R, then it follows easily from a theorem of Wiebe [9] that R is a complete intersection. Cf. Gulliksen-Levin [4], 3.5.3. (There the number n must be replaced by emb dim R − depth R; one first reduces to the case depth $R = 0$, and then applies Wiebe's theorem.)

(e) It is easy to find rings R which are not complete intersections, but for which λ_1: $H_1(R)^p \to H_p(R)$ is injective for p = emb dim R − dim R. This shows that Theorem 2 is optimal. □

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