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TANGENTIAL LIMITS AND EXCEPTIONAL SETS FOR HOLOMORPHIC BESOV FUNCTIONS IN THE UNIT BALL OF $\mathbf{C}^n$

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1. Introduction

Let B^n denote the unit ball in $\mathbf{C}^n$ with boundary S, the unit sphere. If f is holomorphic in B^n with homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

define a radial fractional derivative of order $\beta > 0$ by

$$R^{\beta} f(z) = \sum_{k=0}^{\infty} (k+1)^{\beta} f_k(z).$$

Note that for $\beta = 1$, $R^1 f = \mathscr{R} f + f$ where $\mathscr{R}$ is the usual radial derivative as defined in [R]. Define the Besov space $B^p_\beta(B^n)$, $p > 1$, $\beta > 0$, as

$$B^p_\beta(B^n) = \left\{ f \in H(B^n) : \int_{B^n} |R^{1+\beta} f(z)|^p (1 - |z|)^{p-1}\, dV(z) < \infty \right\},$$

so that

$$\|f\|^p_{p,\beta} = \int_{B^n} |R^{1+\beta} f(z)|^p (1 - |z|)^{p-1}\, dV(z).$$

When β is a positive integer, this definition is equivalent to the analogous space using $\mathscr{R}$ instead (see [BB]), and we will occasionally use $\mathscr{R}$ when it is convenient.

For functions in this space we show the existence of limits in certain non-isotropic tangential approach regions. The exceptional sets are shown to

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have singular measure zero. There is an explicit relationship among the various parameters: β, p, n, the order of tangency of the approach regions, and the singularity of the measure of the exceptional sets.

There are many papers dealing with the existence of tangential limits. Work has been done by Cargo [Ca], Kinney [K], Nagel, Rudin, and Shapiro [NRS], and others. In [AN] Ahern and Nagel published results for the upper half-space for both Besov and Sobolev spaces. In the setting of $\mathbf{C}^n$, Gowda [G] and Sueiro [S] have both studied tangential convergence for fractional Cauchy integrals of L^p functions. In the second section of this paper we use techniques from [AN] for Besov spaces in $\mathbf{C}^n$.

In [Co] and in [AC] Cohn, and Ahern and Cohn develop and apply techniques for completely characterizing exceptional sets of Sobolev functions in terms of non-isotropic Hausdorff measures. In the third section we apply these techniques to our Besov space setting. We show not only that every exceptional set has Hausdorff measure zero, using a Frostman-type result proved in [Co], but that every compact set of Hausdorff measure zero is an exceptional set for some Besov function.

This work forms part of the author's thesis. The author would like to thank Patrick Ahern for his help and advice.

2. A strong L^p estimate

We need to define the approach regions that we will be using. Note that in the following definitions, when we write $z = r\eta$ we intend for $0 \leq r < 1$, $\eta \in S$, so that $r\eta \in B^n$. Also, throughout this paper C will denote various positive constants which depend only on allowable constants and parameters. For $\zeta \in S$ and $\delta > 0$ let

$$B(\zeta, \delta) = \{\eta \in S : |1 - \langle \eta, \zeta \rangle| < \delta\}$$

be the Koranyi ball. Define approach regions at $\zeta \in S$ by

$$D_\alpha(\zeta) = \{z = r\eta : |1 - \langle \eta, \zeta \rangle| < \alpha(1 - r)\}.$$

Note that these are equivalent to the usual admissible approach regions

$$\left\{z = r\eta : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - r^2)\right\}.$$

Our definition of $D_\alpha(\zeta)$ will simplify later computations. These approach regions are tangential in some directions and non-tangential in others [R]. We next define approach regions at $\zeta \in S$ which are tangential in all directions. For $\tau > 1$ set

$$\Omega_{\alpha,\tau}(\zeta) = \{z = r\eta : |1 - \langle \eta, \zeta \rangle|^\tau < \alpha(1-r)\}.$$

Thus, τ is the order of tangency of the approach regions.

A standard way to prove the existence of limits in approach regions is to prove an L^p estimate on a maximal function. Set $Mf(\zeta) = \sup_{z \in \Omega_{\alpha,\tau}(\zeta)} |f(z)|$. For $0 < m \leq n$, let ν be a singular measure on S with $\nu(B(\zeta, \delta)) \leq C\,\delta^m$. We will prove the following.

THEOREM 1. *If $m/\tau = n - \beta p$ there is a constant C such that*

$$\int_S Mf(\zeta)^p \, d\nu(\zeta) \leq C \|f\|_{p,\beta}^p.$$

The proof will be quite similar to that of Theorem 7.1 in [AN]. First we will need two technical lemmas. We define a non-isotropic polydisc in B^n as follows. If $z = r\eta$ choose $\eta_2, \eta_3, \ldots, \eta_n$ so that $\{\eta, \eta_2, \eta_3, \ldots, \eta_n\}$ is an orthonormal basis for $\mathbf{C}^n$. Set

$$P(z; \delta_r, \delta_t) = \left\{ w = r\eta + \lambda\eta + \sum_{j=2}^{n} \lambda_j \eta_j : |\lambda| < \delta_r(1-r), \right.$$
$$\left. |\lambda_j| < \sqrt{\delta_t(1-r)},\, j = 2, 3, \ldots, n \right\}.$$

This is a polydisc of radius $\delta_r(1-r)$ in the radial direction and radius $\sqrt{\delta_t(1-r)}$ in the $n-1$ tangential directions. Its size is proportional to the distance $1-r$ from the center of the polydisc to the boundary of the ball in the radial direction, and proportional to the square root of that distance in the tangential directions. The constants δ_r and δ_t may be different. If $\delta_r = \delta_t = \delta$ say that $P(z; \delta, \delta) = P(z; \delta)$. Note that the volume of this polydisc is

$$C(\delta_r(1-r))^2 \sqrt{\delta_t(1-r)}^{\,2(n-1)} = C\,\delta_t^{n-1}\,\delta_r^2 (1-r)^{n+1}.$$

LEMMA 2. *For each α, γ with $1 < \alpha < \gamma$ there is a $\delta = \delta(\alpha, \gamma, n)$ such that if $\zeta \in S$ and $z = r\eta \in D_\alpha(\zeta)$, $1 > r > \frac{1}{2}$, then the polydisc $P(z; \delta) \subset D_\gamma(\zeta)$.*

The proof, essentially that of Lemma 3.5 in [AB], is omitted.

LEMMA 3. *Let $\tau > 1$, $\gamma > 0$, and $0 < \rho, \delta < 1$. For δ and $\rho - \frac{1}{2}$ sufficiently small there is an $\varepsilon = \varepsilon(\tau, \gamma, \rho, \delta, n)$ such that if $z \in \Omega_{\alpha/A,\tau}(\zeta)$, $|z| > \frac{1}{2}$, and $\alpha, A > 0$, but $z \notin D_\gamma(\zeta)$ then $P(z; \rho, \delta) \subset \Omega_{\alpha/A\varepsilon,\tau}(\zeta)$.*

Proof. The proof is similar to that of Lemma 7.1 in [AN]. Write $z = r\eta$. Let $w = s\xi \in P(z; \rho, \delta)$. It is easy to show that $|1 - \langle \xi, \eta \rangle| < C(1 - r)$ and

$$s^2 = w^2 < r^2 + 2r\rho(1 - r) + \rho^2(1 - r)^2 + (n - 1)\,\delta(1 - r)$$

so

$$1 - s^2 \geq (1 - r)\left(\frac{3}{2} - 2\rho - \frac{1}{2}\rho^2 - (n - 1)\,\delta\right).$$

If δ is small and ρ is close to $\frac{1}{2}$ we obtain

$$1 - r \leq \frac{1 - s^2}{C} \leq C(1 - s).$$

Thus,

$$\begin{aligned}
|1 - \langle \xi, \zeta \rangle|^\tau &\leq \left(|1 - \langle \xi, \eta \rangle|^{1/2} + |1 - \langle \eta, \zeta \rangle|^{1/2}\right)^{2\tau} \\
&\leq \left(C^{1/2}(1 - r)^{1/2} + |1 - \langle \eta, \zeta \rangle|^{1/2}\right)^{2\tau} \\
&\leq \left(C^{1/2}\frac{|1 - \langle \eta, \zeta \rangle|^{1/2}}{\gamma^{1/2}} + |1 - \langle \eta, \zeta \rangle|^{1/2}\right)^{2\tau} \\
&\leq C|1 - \langle \eta, \zeta \rangle|^\tau \\
&< C\frac{\alpha}{A}(1 - r) \\
&\leq C\frac{\alpha}{A}(1 - s).
\end{aligned}$$

Let ε be the reciprocal of this constant C. ■

Proof of Theorem 1. Suppose $z \in \Omega_{\alpha,\tau}(\zeta)$ and $|z| > \frac{1}{2}$. An easy calculation shows that

$$f(z) = C\int_0^1 \left(\log\frac{1}{t}\right)^\beta (R^{\beta+1}f(tz))\,dt.$$

Apply the mean value property to $R^{\beta+1}f(tz)$ on the polydisc $P(tz;\delta)$ defined earlier, where δ will be chosen later.

$$|R^{\beta+1}f(tz)| \leq |P(tz;\delta)|^{-1}\int_{P(tz;\delta)}|R^{\beta+1}f(w)|dV(w)$$

$$\leq \left(\int_{P(tz;\tau)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}|P(tz;\delta)|^{-1/p}$$

by Hölder's inequality

$$= C\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}(\delta(1-tr))^{-(n+1)/p}.$$

So

$$|f(z)| \leq C\int_0^1\left|\log\frac{1}{t}\right|^{\beta}\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}(\delta(1-tr))^{-(n+1)/p}\,dt.$$

Use the estimate $\log(1/t) \sim 1-t$ to get

$$|f(z)| \leq C\int_0^1(1-t)^{\beta}\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}(\delta(1-tr))^{-(n+1)/p}\,dt.$$

Let $t_0, 0 < t_0 \leq 1$, be the number such that $t_0 z \in \partial D_\alpha(\zeta)$. That is, for $z = r\eta$ we have $|1 - \langle\eta,\zeta\rangle| = \alpha(1 - rt_0)$. If $z \in D_\alpha(\zeta)$ then let $t_0 = 1$.

We write

$$|f(z)| \leq C\left(\int_0^{t_0} + \int_{t_0}^1\right)(1-t)^{\beta}\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}$$
$$\times(\delta(1-tr))^{-(n+1)/p}\,dt$$
$$= A + B.$$

First look at part A.

$$A = C\,\delta^{-(n+1)/p}\int_0^{t_0}(1-t)^{\beta}(1-tr)^{-(n+1)/p}$$
$$\times\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}dt$$
$$\leq C\,\delta^{-(n+1)/p}\int_0^{t_0}(1-t)^{a}(1-t)^{b}(1-tr)^{-(n+1)/p}$$
$$\times\left(\int_{P(tz;\delta)}|R^{\beta+1}f(w)|^p\,dV(w)\right)^{1/p}dt$$

where $a + b = \beta$.

Consider $w \in P(tz;\delta)$. One can show that

$$1 - |w|^2 \geq (1 - tr)(1 - \delta(n+2)) \quad \text{and} \quad 1 - |w| \leq 4(1 - tr).$$

For small values of δ we obtain

$$1 - t \leq 1 - tr \leq 4(1 - |w|) \leq 16(1 - tr).$$

By these calculations,

$$|A| \leq \int_0^{t_0} (1-t)^a \delta^{-(n+1)/p} \times \left(\int_{P(tz;\delta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{bp-n-1} \, dV(w) \right)^{1/p} dt$$

as long as $b > 0$.

We have $0 < t < t_0$ so, applying Lemma 2 with $tz \in D_\alpha(\zeta)$, we obtain $P(tz;\delta) \subset D_\gamma(\zeta)$ for $\gamma > \alpha$ and appropriate δ, independent of tz. Fix this δ. Now

$$\begin{aligned} |A| &\leq C \int_0^{t_0} (1-t)^a \delta^{-(n+1)/p} \\ &\quad \times \left(\int_{D_\gamma(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{bp-n-1} \, dV(w) \right)^{1/p} dt \\ &= C \left(\int_0^{t_0} (1-t)^a \, dt \right) \delta^{-(n+1)/p} \\ &\quad \times \left(\int_{D_\gamma(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{bp-n-1} \, dV(w) \right)^{1/p} \\ &= C \left(\int_{D_\gamma(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{bp-n-1} \, dV(w) \right)^{1/p} \end{aligned}$$

if $a > -1$.

This last integral is now independent of z. Raise it to the p^{th} power, take the supremum over all $z \in \Omega_\tau(\zeta)$, and integrate over the sphere S with respect to the measure ν to get

$$\begin{aligned}
&\int_S \sup_{z\in\Omega_\tau(\zeta)} |A|^p \, d\nu(\zeta) \\
&\quad \le \int_S \int_{B^n} \chi_{D_\gamma(\zeta)}(w) |R^{\beta+1} f(w)|^p (1-|w|)^{bp-n-1} \, dV(w)\, d\nu(\zeta) \\
&\quad \le \int_{B^n} |R^{\beta+1} f(w)|^p (1-|w|)^{bp-n-1} \int_S \chi_{D_\gamma(\zeta)}(w)\, d\nu(\zeta)\, dV(w)
\end{aligned}$$

by Fubini. Note that

$$\begin{aligned}
D_\gamma(\zeta) &= \{w + s\xi : |1 - \langle \xi, \zeta\rangle| < \gamma(1-s)\} \\
&= \{w = s\xi : \zeta \in B(\xi, \gamma(1-s))\},
\end{aligned}$$

and so

$$\begin{aligned}
\int_S \chi_{D_\gamma(\zeta)}(w)\, d\nu(\zeta) &= \nu(B(\xi, \gamma(1-|w|))) \\
&= C\gamma^m (1-|w|)^m.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_S \sup_{z\in\Omega_\tau(\zeta)} |A|^p\, d\nu(\zeta) &\le C \int_{B^n} |R^{\beta+1} f(w)|^p (1-|w|)^{bp-n-1+m} \, dV(w) \\
&= C\|f\|_{p,\beta}^p \quad \text{if } bp - n - 1 + m = p - 1.
\end{aligned}$$

Easy calculations show that if $bp - n - 1 + m = p - 1$, $a + b = \beta$, and $m/\tau = n - \beta p$, then

$$b = \frac{1}{p}(p + n - m) \text{ and } a = \frac{m}{p}\left(1 - \frac{1}{\tau}\right) - 1.$$

Thus, $b > 0$ since $m \le n$, and $a > -1$ since $\tau > 1$.

Now we turn to integral B. Break up the interval $[t_0, 1]$ as follows. There is a positive integer N such that $1 - t_0 r > 2^N(1-r) \ge \frac{1}{2}(1 - t_0 r)$. Find $t_0 < t_1 < \dots < t_{N+1} = 1$ so that $1 - t_j r = 2^{N-j+1}(1-r)$. For these choices of t_j

the distance from t_jz to the boundary of the ball is twice the distance from $t_{j+1}z$ to the boundary, t_0z is on $\partial D_\alpha(\zeta)$, and $t_{N+1}z = z$.

We claim that $P(tz;\delta) = P(tz;\delta,\delta) \subset P(t_jz;(1+\delta)/2,\delta)$ if $t \in [t_j, t_{j+1}]$. Suppose $w \in P(tz;\delta,\delta)$. Then

$$\begin{aligned} w &= rt\eta + \lambda\eta + \textstyle\sum \lambda_j\eta_j \quad \text{where } |\lambda| < \delta(1-tr), |\lambda_j| < \sqrt{\delta(1-tr)} \\ &= rt_j\eta + (rt - rt_j + \lambda)\eta + \textstyle\sum \lambda_j\eta_j. \end{aligned}$$

We must show that

$$|rt - rt_j + \lambda| < \left(\frac{1+\delta}{2}\right)(1 - t_jr) \quad \text{and} \quad |\lambda_j| < \sqrt{\delta(1-t_jr)}\,.$$

Clearly $|\lambda_j| < \sqrt{\delta(1-tr)} \le \sqrt{\delta(1-t_jr)}$ since $t \ge t_j$. Also,

$$\begin{aligned} |rt - rt_j + \lambda| &\le rt - rt_j + \delta(1-tr) \\ &= 1 - t_jr - (1-tr) + \delta(1-tr) \\ &= 1 - t_jr - (1-\delta)(1-tr) \\ &\le 1 - t_jr - (1-\delta)(1-t_{j+1}r) \quad \text{since } t \le t_{j+1} \\ &\le 1 - t_jr - \frac{1}{2}(1-\delta)(1-t_jr) \quad \text{since } 1 - t_{j+1}r \ge \frac{1}{2}(1-t_jr) \\ &= (1-t_jr)\left(1 - \frac{1-\delta}{2}\right) \\ &= (1-t_jr)\left(\frac{1+\delta}{2}\right) \end{aligned}$$

so we have proved the claim.

Write

$$\Delta_j = P\left(t_jz; \frac{1+\delta}{2}, \delta\right).$$

We now have

$$\begin{aligned} |B| &\le \sum_{j=0}^{N} C\int_{t_j}^{t_{j+1}} (1-t)^\beta (1-tr)^{-(n+1)/p} \left(\int_{\Delta_j} |R^{\beta+1}f(w)|^p\, dV(w)\right)^{1/p} dt \\ &\le \sum_{j=0}^{N} C\int_{t_j}^{t_{j+1}} (1-t_j)^\beta (1-t_{j+1}r)^{-(n+1)/p} \left(\int_{\Delta_j} |R^{\beta+1}f(w)|^p\, dV(w)\right)^{1/p} dt \\ &= \sum_{j=0}^{N} C(t_{j+1} - t_j)(1-t_j)^\beta 2^{(n+1)/p}(1-t_jr)^{-(n+1)/p} \\ &\quad \times \left(\int_{\Delta_j} |R^{\beta+1}f(w)|^p\, dV(w)\right)^{1/p}. \end{aligned}$$

We now apply Lemma 3 to $P(t_j z;(1+\delta)/2,\delta)$. In the lemma take $A = 2^{N-j+1}$ and $t_j z$ in place of z. Note that we can take δ small and $(1+\delta)/2$ close to $\frac{1}{2}$ as the lemma requires. Then Lemma 3 says that $\Delta_j \subset \Omega_{\alpha/A\varepsilon}(\zeta) = \Omega_j(\zeta)$. From the proof of Lemma 3 and similarly to part A we obtain

$$\begin{aligned} 1 - t_j r &\le C(1 - |w|), \\ t_{j+1} - t_j \le 1 - t_j \le 1 - t_j r &\le C(1 - |w|), \\ 1 - t_j r &\ge C(1 - |w|) \end{aligned}$$

so that now

$$\begin{aligned} |B| &\le C \sum_{j=0}^{N} \left(\int_{\Delta_j} |R^{\beta+1} f(w)|^p (1 - |w|)^{\beta p + p - n - 1} \, dV(w) \right)^{1/p} \\ &\le C \sum_{j=0}^{N} \left(\int_{\Omega_j(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{\beta p + p - n - 1} \, dV(w) \right)^{1/p}. \end{aligned}$$

Now if we take the supremum, integrate, and use Minkowski and Fubini we obtain

$$\begin{aligned} &\int_S \sup_{z \in \Omega_\tau(\zeta)} |B|^p \, d\nu(\zeta) \\ &\quad \le C \int_S \left[\sum_{j=0}^{N} \left(\int_{\Omega_j(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{\beta p + p - n - 1} \, dV(w) \right)^{1/p} \right]^p d\nu(\zeta) \\ &\quad \le C \left[\sum_{j=0}^{N} \left(\int_S \int_{\Omega_j(\zeta)} |R^{\beta+1} f(w)|^p (1 - |w|)^{\beta p + p - n - 1} \, dV(w) \, d\nu(\zeta) \right)^{1/p} \right]^p \\ &\quad \le C \left[\sum_{j=0}^{N} \left(\int_{B^n} |R^{\beta+1} f(w)|^p (1 - |w|)^{\beta p + p - n - 1} \right. \right. \\ &\qquad\qquad \left. \left. \times \int_S \chi_{\Omega_j(\zeta)}(w) \, d\nu(\zeta) \, dV(w) \right)^{1/p} \right]^p. \end{aligned}$$

We know that

$$\begin{aligned} \Omega_j(\zeta) &= \left\{ w = s\xi : |1 - \langle \xi, \zeta \rangle|^\tau \le \frac{\alpha}{A\varepsilon}(1 - s) \right\} \\ &= \left\{ w = s\xi : \zeta \in B\left(\xi, \left(\frac{\alpha}{A\varepsilon}(1 - s) \right)^{1/\tau} \right) \right\} \end{aligned}$$

and so

$$\int_S \chi_{\Omega_j(\zeta)}(w)\, d\nu(\zeta) = \nu\left(B\left(\xi, \left(\frac{\alpha}{A\varepsilon}(1-|w|)\right)^{1/\tau}\right)\right)$$
$$\leq C\left(\frac{\alpha}{A\varepsilon}\right)^m (1-|w|)^{m/\tau}.$$

Finally we have

$$\int_S \sup_{z\in\Omega_\tau(\zeta)} |B|^p\, d\nu(\zeta)$$
$$\leq C\left[\sum_{j=0}^N \left(\frac{1}{A}\right)^{m/p}\left(\int_{B^n} |R^{\beta+1}f(w)|^p(1-|w|)^{\beta p+p-n-1+m/\tau}\, dV(w)\right)^{1/p}\right]^p$$
$$= C\left[\sum_{j=0}^N \left(\frac{1}{A}\right)^{m/p}\left(\int_{B^n} |R^{\beta+1}f(w)|^p(1-|w|)^{p-1}\, dV(w)\right)^{1/p}\right]^p$$
$$\leq C\|f\|_{p,\beta}^p\left(\sum_{j=0}^N (2^{-N+j-1})^{m/p}\right)^p$$
$$= C\|f\|_{p,\beta}^p\left(\sum_{k=1}^{N+1} 2^{-km/p}\right)^p$$
$$\leq C\|f\|_{p,\beta}^p\left(\sum_{k=1}^{\infty} 2^{-km/p}\right)^p$$
$$\leq C\|f\|_{p,\beta}^p. \qquad \blacksquare$$

COROLLARY 4. *Suppose that ν is a positive measure on S and C is a constant satisfying $\nu(B(\zeta,\delta)) \leq C\,\delta^m$ for $\zeta \in S$, $\delta > 0$. Then if $f \in B_\beta^p(B^n)$,* $\lim f(z)$ *exists as $z \to \zeta_0$, $z \in \Omega_{\alpha,\tau}(\zeta_0)$, for all ζ_0 with the exception of a set E with $\nu(E) = 0$.*

The proof is a standard one, which we omit.

3. Characterization using Hausdorff capacity

We now look more closely at the exceptional sets for these holomorphic Besov functions. Define the exceptional set of a function $f \in B_\beta^p(B^n)$ as

$$E(f) = \left\{\zeta \in S:\ \lim_{\substack{z\to\zeta,\\ z\in\Omega_{\alpha,\tau}(\zeta)}} f(z) \text{ does not exist for some } \alpha\right\}.$$

Then Corollary 4 states that $\nu(E(f)) = 0$ for any m-dimensional positive measure ν. We will further characterize these exceptional sets using Hausdorff capacity.

For $m > 0$ and E a compact subset of S, let H_m be the m-dimensional Hausdorff capacity defined by

$$H_m(E) = \inf\left\{\sum \delta_k^m : E \subset \bigcup B(\zeta_k, \delta_k)\right\}.$$

This is non-isotropic because we are using the non-isotropic balls $B(\zeta_k, \delta_k)$ on S. In [Co], Cohn proves a Frostman-type result which says that for a compact set $E \subset S$, $H_m(E) > 0$ if and only if E contains the support of a positive measure ν satisfying $\nu(B(\zeta, \delta)) \leq C\delta^m$. This yields the following immediate extension of Corollary 4.

COROLLARY 5. *If $m/\tau = n - \beta p$ and E is a compact subset of S with $E \subset E(f)$ for some $f \in B_\beta^p(B^n)$, then $H_m(E) = 0$.*

This corollary says that compact exceptional sets have Hausdorff capacity 0. We now show that the compact exceptional sets are precisely those with Hausdorff capacity 0. We will prove the following.

THEOREM 6. *Let $m/\tau = n - \beta p$ with $\tau > 1$, $0 < m \leq n$, $\beta > 0$, $p > 1$, and let E be a compact subset of S with $H_m(E) = 0$. Then there is a function $f \in B_\beta^p(B^n)$ such that $E = E(f)$.*

The proof of Theorem 6 will follow that of Theorem 1.2 in [AC]. We require a sequence of lemmas which construct the desired function f. Lemma 7 is obtained by using Cauchy's formula on a polydisc, so we omit the proof. The proof of Lemma 9 is almost identical to that of Lemma 1.6 in [AC]. The proof of Lemma 8 is included. The proof of Theorem 6 is included for the sake of completeness.

LEMMA 7. *If f is holomorphic in B^n, $z \in B^n$, $|z| \geq \frac{1}{2}$, $q > 1$, k is a positive integer, and $P(z; \delta)$ is a polydisc as defined earlier, then there is a constant $C = C(k, n, \delta, q)$ such that*

$$|\mathscr{R}^k g(z)|^q \leq \frac{C}{(1 - |z|)^{qk+n+1}} \int_{P(z;\delta)} |g(w)|^q \, dV(w).$$

LEMMA 8. *Suppose $\{B(\zeta_j, \delta_j)\}$ is a finite collection of pairwise disjoint balls in S, with $\delta_j \leq \frac{1}{2}$. Set $z_j = (1 - \delta_j)\zeta_j$ and*

$$F(z) = \sum_j \delta_j^{n+p(k-\beta)+l} \langle z_j, z_j\rangle^{-l} \langle z, z_j\rangle^l \left(1 - \langle z, z_j\rangle\right)^{-(n+p(k-\beta)+l)}$$

where k is an integer greater than β and $l > 0$. Then $\|F\|_{p,\beta}^p \leq C \sum_j \delta_j^{n-\beta p}$.

Proof. It is well known that an equivalent formulation of the Besov norm is

$$\|F\|_{p,\beta}^{p} = \int_{B^n} |R^k F|^p (1 - |z|)^{(k-\beta)p-1}\, dV(z)$$

where k is any integer greater than β. We use this with $\mathscr{R}$ in place of R. We must show that $\mathscr{R}^k F \in L^p((1 - |z|)^{(k-\beta)p-1}\, dV)$ with the correct bound on the norm. We will use a duality argument. Let

$$g \in L^q\big((1 - |z|)^{(k-\beta)p-1}\, dV\big) \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1,$$

and the norm of g is ≤ 1.

Compute that

$$\begin{aligned}
&\int_{B^n} g(z)\overline{\mathscr{R}^k F(z)}(1 - |z|)^{(k-\beta)p-1}\, dV(z) \\
&= \int_{B^n} \overline{F(z)}\mathscr{R}^k g(z)(1 - |z|)^{(k-\beta)p-1}\, dV(z) \\
&= \sum_j \delta^{n+(k-\beta)p+l} |z_j|^{-2l} \\
&\quad \times \int_{B^n} \frac{\langle z_j, z\rangle^l \mathscr{R}^k g(z)}{\big(1 - \langle z_j, z\rangle\big)^{n+(k-\beta)p+l}} (1 - |z|)^{(k-\beta)p-1}\, dV(z) \\
&= C \sum_j \delta_j^{n+(k-\beta)p+l} |z_j|^{-2l} \int_{B^n} \frac{\langle z_j, z\rangle^l \mathscr{R}^k g(z)}{\big(1 - \langle z_j, z\rangle\big)^{n+l+1}} dV(z) \\
&= C \sum_j \delta_j^{n+(k-\beta)p+l} |z_j|^{-2l} \mathscr{R}^{l+k} g(z_j).
\end{aligned}$$

It follows that

$$\|\mathscr{R}F\|_{p,\beta} \le C \sum_j \delta_j^{n+(k-\beta)p+l} |\mathscr{R}^{l+k} g(z_j)|$$

so, by Hölder and Lemma 7,

$$\begin{aligned}\|\mathscr{R}F\|_{p,\beta}^p &\le C\Big(\sum_j \delta_j^{n-\beta p}\Big)\Big(\sum_j \delta_j^{n+\beta q+pq(k-\beta)+lq}|\mathscr{R}^{k+l}g(z_j)|^q\Big)^{p/q}\\ &\le C\Big(\sum_j \delta_j^{n-\beta p}\Big)\Big(\sum_j \delta_j^{n+p(k-\beta)+qk+lq}\,\delta_j^{-q(k+l)-n-1}\\ &\qquad\qquad \times\int_{P(z_j;\alpha)}|g(w)|^q\,dV(w)\Big)^{p/q}\\ &= C\Big(\sum_j \delta_j^{n-\beta p}\Big)\Big(\sum_j \delta_j^{(k-\beta)p-1}\int_{P(z_j;\alpha)}|g(w)|^q\,dV(w)\Big)^{p/q}\end{aligned}$$

where α is chosen so that $\{P(z_j;\alpha)\}$ are disjoint. Since $\delta_j \doteq 1-|w|$ we have

$$\begin{aligned}\|\mathscr{R}F\|_{p,\beta}^p &\le C\Big(\sum_j \delta_j^{n-\beta p}\Big)\Big(\sum_j\int_{P(z_j;\alpha)}|g(w)|^q(1-|w|)^{(k-\beta)p-1}\,dV(w)\Big)^{p/q}\\ &\le C\sum_j \delta_j^{n-\beta p}\Big(\int_{B^n}|g(w)|^q(1-|w|)^{(k-\beta)p-1}\,dV(w)\Big)^{p/q}\end{aligned}$$

since the $\{P(z_j;\alpha)\}$ are disjoint. Finally we see that

$$\begin{aligned}\|\mathscr{R}^kF\|_{p,\beta}^p &\le C\sum_j \delta_j^{n-\beta p}\|g\|_{L^q((1-|z|)^{(k-\beta)p-1}dV)}^p\\ &\le C\sum_j \delta_j^{n-\beta p}\end{aligned}$$

since $\|g\|\le 1$. ■

LEMMA 9. *For $N > n + p(k-\beta)$ there exist constants $C, t > 0$ such that for a disjoint collection $\{B(\zeta_k,\delta_k)\}$, $\delta_k \le \frac{1}{2}$, there is a function $F \in H^\infty(B^n)$ with*

(i) $\|F\|_{p,\beta}^p \le C\sum\delta_k^{n-\beta p}$,
(ii) $F((1-\delta_k t^{-1})\zeta_k) \ge \frac{1}{4}$, *and*
(iii) $|F(z)| \le C(1-|z|)^{-N}\sum\delta_k^{n-\beta p}$.

Proof of Theorem 6. We are given a compact set $E\subset S$ with $H_m(E)=0$. Inductively define numbers $m_k>0$ and families $\{B(\zeta_{kl},\delta_{kl})\}$ so that for each fixed k the balls $\{B(\zeta_{kl},\delta_{kl})\}$ are disjoint, and $E\subset\bigcup_l B(\zeta_{kl}, C\,\delta_{kl}^{1/\tau})$. If F_k is associated with $\{B(\zeta_{kl}, \delta_{kl})\}$ as in Lemma 9 and $z_{kl} = (1 - \delta_{kl} t^{-1})\zeta_{kl}$, then we further require

(i) $$m_k \geq m_{k-1} + 1,$$

(ii) $$m_k > 8 \sum_{j<k} m_j \|F_j\|_\infty,$$

(iii) $$m_k \|F_k\|_{p,\beta} \leq m_k C \sum_l \delta_{kl}^{n-\beta p} \leq 2^{-k},$$

(iv) $$m_k |F_k(z_{jl})| < \frac{1}{10(2^k)} \quad \text{for } j < k.$$

If this has been done $k-1$ times, first choose m_k so that (i) and (ii) hold. Then, using Lemma 9(ii) and (iii) and the fact that $H_m(E) = 0$ where $m/\tau = n - \beta p$, we can choose $\{B(\zeta_{kl}, \delta_{kl})\}$ so that (iii) and (iv) hold.

Let $F = \sum m_k F_k$. From (iii), $F \in B_\beta^p(B^n)$.

Suppose $\zeta \in E$. Since, for each k, $E \subset \bigcup_l B(\zeta_{kl}, C\delta_{kl}^{1/\tau})$, there is an l with

$$\zeta \in B(\zeta_{kl}, C\delta_{kl}^{1/\tau}).$$

That is,

$$|1 - \langle \zeta, \zeta_{kl} \rangle| \leq C\delta_{kl}^{1/\tau} = Ct^{1/\tau}(1 - |z_{kl}|)^{1/\tau},$$
$$|1 - \langle \zeta, \zeta_{kl} \rangle|^\tau \leq C^\tau t(1 - |z_{kl}|),$$

so $z_{kl} \in \Omega_{\tau,\alpha}(\zeta)$ with $\alpha = C^\tau t$. But

$$\begin{aligned}|F(z_{kl})| &\geq m_k |F_k(z_{kl})| - \sum_{j<k} m_j \|F_j\|_\infty - \sum_{j>k} m_j |F_j(z_{kl})| \\ &\geq \frac{m_k}{4} - \frac{m_k}{8} - \frac{1}{10} \\ &\geq \frac{m_k}{10} \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

So $MF(\zeta) \equiv \infty$ for $\zeta \in E$.

Suppose $z \in \overline{B} \setminus E$. Then

$$\begin{aligned}|F_k(z)| &\leq C|1 - \langle z, z_{kl} \rangle|^{-N} \sum_l \varepsilon_{kl}^N \\ &\leq C \sum_l \delta_{kl}^{n-\beta p}\end{aligned}$$

where C depends only on fixed constants and the distance from z to E. But from (iii) we have

$$m_k C \sum_l \delta_{kl}^{n-\beta p} \leq C 2^{-k},$$

so the series for F converges uniformly on compact subsets of $\bar{B} \setminus E$. Thus $E(F) = E$. ■

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