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FUNDAMENTAL SOLUTIONS FOR POWERS OF THE HEISENBERG SUB-LAPLACIAN

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1. Introduction and statement of results

The *Heisenberg group* H_n of dimension $2n+1$ is given by

$$H_n := \mathbf{C}^n \times \mathbf{R} \tag{1.1}$$

with product

$$(z,t)(z',t') = \left(z+z', t+t' - \tfrac{1}{2}\operatorname{Im}(z\cdot\overline{z'})\right) \tag{1.2}$$

for $z, z' \in \mathbf{C}^n$, $t, t' \in \mathbf{R}$. Differentiation along the one-parameter subgroups

$$\{x_j(s) = (se_j, 0)\} \quad \text{and} \quad \left\{y_j(s) = \left(\sqrt{-1}\,se_j, 0\right)\right\},$$

where $\{e_j\}$ is the standard basis for $\mathbf{C}^n$, yields left invariant vector fields X_j and Y_j respectively. Letting $Z_j := X_j + \sqrt{-1}\,Y_j$ and $\overline{Z}_j := X_j - \sqrt{-1}\,Y_j$, one computes that

$$\begin{aligned} Z_j &= 2\frac{\partial}{\partial \overline{z}_j} + \frac{\sqrt{-1}}{2} z_j \frac{\partial}{\partial t}, \\ \overline{Z}_j &= 2\frac{\partial}{\partial z_j} - \frac{\sqrt{-1}}{2} \overline{z}_j \frac{\partial}{\partial t}. \end{aligned} \tag{1.3}$$

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The *Heisenberg sub-Laplacian* is the left invariant differential operator Δ_{H_n} on H_n given by

$$\Delta_{H_n} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right) = \frac{1}{2} \sum_{j=1}^{n} \left(Z_j \overline{Z}_j + \overline{Z}_j Z_j\right)$$
$$= 4 \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} + \sqrt{-1} \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \frac{\partial}{\partial t} - \sqrt{-1} \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial t} + \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2}. \tag{1.4}$$

$\Delta_{H_n} + \partial^2/\partial t^2$ is the Laplace-Beltrami operator for a left-invariant metric on H_n. The sub-Laplacian is homogeneous of degree 2 with respect to the *dilations* d_s given by

$$d_s(z, t) = (sz, s^2 t) \tag{1.5}$$

for $s \in \mathbf{R}^+$. That is, $\Delta_{H_n}(f \circ d_s) = s^2 \Delta_{H_n}(f) \circ d_s$ holds for smooth functions $f : H_n \to \mathbf{C}$.

In [Fol], G. Folland found that Δ_{H_n} has a fundamental solution F given by the formula

$$F = \frac{\Gamma\left(\frac{n}{2}\right)^2}{8\pi^{n+1}} r^{-n} \tag{1.6}$$

where

$$r = \left(\frac{|z|^4}{16} + t^2\right)^{1/2}. \tag{1.7}$$

The distribution F is tempered, given by a locally integrable function and has singular support $\{(0,0)\}$. Folland's result is motivated by the well known fact that a suitable multiple of $\|x\|^{2-n}$ is a fundamental solution for the usual Euclidean Laplace operator Δ on $\mathbf{R}^n$ for $n > 2$. (See e.g. [Hö].) The function $r(z, t)$ on H_n plays a role analogous to that of $\|x\|^2$ on $\mathbf{R}^n$. In particular, $r(z, t)$ is homogeneous of degree 2 with respect to the dilations given by Formula 1.5.

In this paper we consider the problem of finding fundamental solutions for (positive integral) powers $\Delta_{H_n}^p$ of Δ_{H_n}. Since $\Delta_{H_n}^p$ is homogeneous with respect to dilations, existence of a fundamental solution is equivalent to both local and global solvability [Ba]. The corresponding problem in $\mathbf{R}^n (n > 2)$ is easy. Since $\Delta(\|x\|^a) = a(a + n - 2)\|x\|^{a-2}$, we see that a multiple of $\|x\|^{2p-n}$ is a fundamental solution for Δ^p. The situation for H_n is more complicated.

Since $\Delta_{H_n}(r^a)$ is *not* a scalar multiple of r^{a-2}, we cannot use Folland's result to derive a fundamental solution for $\Delta_{H_n}^p$ in a simple fashion.

Let $\gamma := |z|^2/4 - it = re^{i\theta}$ where r is given by Formula 1.7 and $-\pi/2 < \theta \leq \pi/2$. Homogeneous functions of degree $2a$ on H_n can be written in (r, θ)-coordinates as $Q(\theta)r^a$. An exercise with the chain rule shows that Δ_{H_n} is given in (r, θ)-coordinates by the formula

$$\Delta_{H_n} = r\cos(\theta)\frac{\partial^2}{\partial r^2} + \frac{\cos(\theta)}{r}\frac{\partial^2}{\partial\theta^2} + (n+1)\cos(\theta)\frac{\partial}{\partial r} - \frac{n\sin(\theta)}{r}\frac{\partial}{\partial\theta}. \tag{1.8}$$

One has

$$\Delta_{H_n}(Q(\theta)r^a) = \left[\cos(\theta)Q''(\theta) - n\sin(\theta)Q'(\theta) + a(n+a)\cos(\theta)Q(\theta)\right]r^{a-1}. \tag{1.9}$$

We conclude that a fundamental solution for $\Delta_{H_n}^p$ should be expressible in the general form $Q(\theta)r^{p-n+1}$. When p is greater than 1, $Q(\theta)$ will *not* be a constant function.

Our main result is the following.

THEOREM A. *Let p be a fixed integer with $1 \leq p \leq n$ and let $\gamma := |z|^2/4 - it = re^{i\theta}$. For $0 < s < 1$ and $|\theta| < \pi/2$, define*

$$G_s(\theta) = e^{i(n-p+1)\theta}\int_0^s \frac{1}{s_n} \cdots \int_0^{s_3} \frac{1}{s_2} \times \int_0^{s_2} \frac{s_1^{n-1}}{\left(1-s_1^2\right)^{p-1}\left(s_1^2+e^{2i\theta}\right)^{n-p+1}} ds_1 \cdots ds_n.$$

Then, as $s \to 1^-$, $\mathrm{Re}(G_s(\theta)) \to \psi_p(\theta)$ uniformly on compact sets, where

(i) *$\psi_p(\theta)$ is smooth for $|\theta| < \pi/2$,*

(ii) $$\Psi_p(z,t) = \frac{2(-1)^p(n-p)!}{r^{n-p+1}}\psi_p(\theta)$$

extends to a function on H_n which is smooth away from $(0,0)$,

(iii) *Ψ_p is a tempered fundamental solution for $\Delta_{H_n}^p$ with singular support* $\{(0,0)\}$.

Theorem A shows that for $s < 1$, G_s can be expressed in terms of iterated antiderivatives of elementary functions. G_s is determined by the differential equation

$$\left(s\frac{d}{ds}\right)^p G_s = \frac{e^{i(n-p+1)\theta}s^n}{(1-s^2)^{p-1}(s^2+e^{2i\theta})^{n-p+1}} \tag{1.10}$$

together with the initial conditions

$$\left(s\frac{d}{ds}\right)^j G_s\bigg|_{s=0} = 0 \quad \text{for } j = 0, 1, \dots, p-1.$$

One can recover Folland's Formula 1.6 for the fundamental solution Ψ_1 of Δ_{H_n} from Theorem A by showing that $\psi_1(\theta)$ is a constant function. In the case $p = 2$, we have been able to express the general fundamental solution in closed form. We consider the cases $p = 1$ and $p = 2$ below in Section 3. One can also use Theorem A to derive various series representations for Ψ_p. In particular, we prove the following.

THEOREM B. *Let $\gamma = re^{i\theta}$ as in Theorem A. The series*

$$\frac{(-1)^p 2(n-p)!}{r^{n-p+1}} \sum_{m=0}^{\infty} \frac{1}{(2m+n)^p} \times \sum_{k+l=m} (-1)^l \binom{p+k-2}{k}\binom{n-p+l}{l} \cos((n-p+1+2l)\theta).$$

converges weakly to Ψ_p.

The series in Theorem B diverges pointwise. One must integrate term-wise against a test function before summing the series. In this sense, Theorem B is a weaker result than Theorem A.

The unitary group $U(n)$ acts on H_n via

$$k \cdot (z,t) = (kz, t) \quad \text{for } k \in U(n), (z,t) \in H_n. \tag{1.11}$$

The operator $\Delta_{H_n}^p$ is invariant under the $U(n)$-action. The key idea in our proof of Theorem A is to exploit this invariance by expanding Ψ_p in terms of $U(n)$-spherical functions $\phi_{\lambda,m}$ on H_n. (See Equation 2.5.) Each $\phi_{\lambda,m}$ satisfies $\phi_{\lambda,m}(0,0) = 1$ and is an eigenfunction for Δ_{H_n} and its powers. In fact,

$$\Delta_{H_n}^p(\phi_{\lambda,m}) = (-1)^p |\lambda|^p (2m+n)^p \phi_{\lambda,m}. \tag{1.12}$$

The set $\{\phi_{\lambda,m} : \lambda \in \mathbf{R}\setminus\{0\},\ \mathrm{m} \in \mathbf{Z}^+ \cup \{0\}\}$ has full measure in the space of positive definite $U(n)$-spherical functions. Reasoning *formally* using

Godement's Plancherel Theorem [Go], one is led to a decomposition

$$\Psi_p = \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \binom{m+n-1}{m} \langle \Psi_p, \phi_{\lambda,m} \rangle \phi_{\lambda,m} |\lambda|^n \, d\lambda \tag{1.13}$$

for the fundamental solution Ψ_p of $\Delta_{H_n}^p$. Moreover,

$$\begin{aligned} \langle \Psi_p, \phi_{\lambda,m} \rangle &= \frac{(-1)^p}{|\lambda|^p (2m+n)^p} \left\langle \Psi_p, \Delta_{H_n}^p(\phi_{\lambda,m}) \right\rangle \\ &= \frac{(-1)^p}{|\lambda|^p (2m+n)^p} \left\langle \Delta_{H_n}^p(\Psi_p), \phi_{\lambda,m} \right\rangle \\ &= \frac{(-1)^p}{|\lambda|^p (2m+n)^p} \left\langle \delta_{(0,0)}, \phi_{\lambda,m} \right\rangle \\ &= \frac{(-1)^p}{|\lambda|^p (2m+n)^p} \phi_{\lambda,m}(0,0) = \frac{(-1)^p}{|\lambda|^p (2m+n)^p}. \end{aligned}$$

Thus we obtain a *formal* expansion

$$\Psi_p = (-1)^p \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2m+n)^p} \binom{m+n-1}{m} \phi_{\lambda,m} |\lambda|^{n-p} \, d\lambda \tag{1.14}$$

for Ψ_p.

Ψ_p is the weak limit of tempered distributions P_s as $s \to 1$ defined by

$$P_s = (-1)^p \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2m+n}}{(2m+n)^p} \binom{m+n-1}{m} \phi_{\lambda,m} |\lambda|^{n-p} \, d\lambda. \tag{1.15}$$

We show that for $s < 1$, P_s is given by

$$\frac{2(-1)^p (n-p)!}{r^{n-p+1}} \operatorname{Re}(G_s)$$

where G_s is defined in the statement of Theorem A, and that the limit distribution Ψ_p is a smooth function away from $(0,0)$.

Section 2 of this paper contains the proofs of Theorems A and B. Some of the detailed analysis parallels that found in [MR1] (see also [MR2]) which was a source of inspiration for the present work. In Section 3 we recover Folland's formula for $p = 1$ and consider the case $p = 2$ in more detail producing an explicit closed formula for this case. This answers a question of Koranýi (personal communication). Section 4 addresses the scope of our methods and describes some directions for further research. We expect that our methods can be used to find fundamental solutions for other differential operators (on H_n and on certain solvable groups) which satisfy strong invariance conditions. Throughout, p, n denote fixed positive integers with $1 \leq p \leq n$.

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2. Proofs of Theorems A and B

We begin by reviewing some standard facts about the representation theory for H_n. The infinite dimensional irreducible unitary representations π_λ of H_n are parametrized by non-zero real numbers λ. For $\lambda > 0$, π_λ can be realized in the *Fock space* $\mathcal{F}_\lambda$ of entire functions $f(w)$ on $\mathbf{C}^n$ which are square integrable with respect to $(\lambda/2\pi)^n e^{-\lambda|w|^2/2}\, dw\, d\bar{w}$ [Br]. The holomorphic polynomials $\mathbf{C}[w_1, \ldots, w_n]$ form a dense subspace of each $\mathcal{F}_\lambda$ and the scaled monomials $\{u_{\lambda,\alpha} : \alpha \in (\mathbf{Z}^+)^n\}$ given by

$$u_{\lambda,\alpha}(w) = \frac{|\lambda|^{|\alpha|/2} w^\alpha}{\left(q^{|\alpha|}\alpha!\right)^{1/2}} \tag{2.1}$$

provide an orthonormal basis for $(\mathcal{F}_\lambda, \langle\ ,\ \rangle_\lambda)$. Here we adopt the usual multi-index conventions $w^\alpha := w_1^{\alpha_1} \cdots w_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$. One has for $\lambda > 0$,

$$\pi_\lambda(z,t) f(w) = \exp\left(i\lambda t - \tfrac{1}{2}\lambda w \cdot \bar{z} - \tfrac{1}{4}\lambda |z|^2\right) f(w+z). \tag{2.2}$$

For $\lambda < 0$, one defines $\mathcal{F}_\lambda = \overline{\mathcal{F}_{|\lambda|}}$ and $\pi_\lambda = \overline{\pi}_{|\lambda|}$. Using Formula 2.2, one computes that

$$\pi_\lambda(\Delta_{H_n}) u_{\lambda,\alpha} = -|\lambda|(2|\alpha| + n) u_{\lambda,\alpha}. \tag{2.3}$$

We also require a lemma that appears in [MR1].

LEMMA 2.4 (Müller-Ricci). *Let $f \in S(H_n)$ and $N \in \mathbf{N}$ be given. There is a constant c_N for which*

$$\left|\langle \pi_\lambda(f) u_{\lambda,\alpha}, u_{\lambda,\alpha}\rangle_\lambda\right| \leq \frac{c_N}{(1+|\lambda|)^N (1+2|\alpha|)^N}.$$

A smooth $U(n)$-invariant function $\phi : H_n \to \mathbf{C}$ is said to be *$U(n)$-spherical* if $\phi(0,0) = 1$ and ϕ is an eigenfunction for both Δ_{H_n} and $\partial/\partial t$. The *bounded* $U(n)$-spherical functions have been computed by many authors [BJR2], [Fa],

[HR], [Ko], [St], [Str]. The *generic* bounded $U(n)$-spherical functions are given by

$$\phi_{\lambda,m}(z,t) = e^{i\lambda t}e^{-|\lambda||z|^2/4}L_m^{(n-1)}\big(|\lambda||z|^2/2\big) \tag{2.5}$$

where $\lambda \in \mathbf{R}\setminus\{0\}$, $m \in \mathbf{Z}^+\cup\{0\}$ and $L_m^{(n-1)}$ is the generalized Laguerre polynomial of degree m and order $(n-1)$ normalized to have value 1 at 0. Explicitly,

$$L_m^{(n-1)}(x) = (n-1)!\sum_{j=0}^{m}\binom{m}{j}\frac{(-x)^j}{(j+n-1)!}. \tag{2.6}$$

The remaining bounded $U(n)$-spherical functions do not depend on the variable t and can be expressed in terms of Bessel functions. These play no role in the subsequent analysis.

The spherical function $\phi_{\lambda,m}$ is related to π_λ by

$$\begin{aligned}\binom{m+n-1}{m}\phi_{\lambda,m}(z,t) &= \operatorname{tr}\big(\pi_\lambda(z,t)|_{\mathscr{P}_m}\big)\\ &= \sum_{|\alpha|=m}\langle \pi_\lambda(z,t)u_{\lambda,\alpha}, u_{\lambda,\alpha}\rangle_\lambda \end{aligned}\tag{2.7}$$

where $\mathscr{P}_m \subset \mathscr{F}_\lambda$ denotes the homogeneous polynomials of degree m. Note that $\binom{m+n-1}{m}$ is the dimension of $\mathscr{P}_m$. It follows from Formula 2.3 and Proposition 3.20 of [BJR2] (or by direct computation) that

$$\Delta_{H_n}\phi_{\lambda,m} = -|\lambda|(2m+n)\phi_{\lambda,m}. \tag{2.8}$$

For each $0 < s \le 1$, formally define $\langle P_s, f\rangle$ for $f \in S(H_n)$ by

$$\begin{aligned}\langle P_s, f\rangle = (-1)^p\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{s^{2m+n}}{(2m+n)^p}\binom{m+n-1}{m}&\\ \times\int_{H_n}\phi_{\lambda,m}(z,t)f(z,t)\,dz\,dt|\lambda|^{n-p}\,d\lambda.&\end{aligned}\tag{2.9}$$

LEMMA 2.10. (1) *P_s is a tempered distribution for each $0 < s \le 1$ and*

$$P_s = (-1)^p 2\operatorname{Re}\int_0^{\infty}\sum_{m=0}^{\infty}\frac{s^{2m+n}}{(2m+n)^p}\binom{m+n-1}{m}L_m^{(n-1)}\left(\frac{\lambda|z|^2}{2}\right)e^{-\lambda\gamma}\lambda^{n-p}\,d\lambda$$

in $S'(H_n)$ where $\gamma = |z|^2/4 - it$.

(2) $\lim_{s\to 1^-} P_s = P_1$ *in $S'(H_n)$*

(3) $\Psi_p = P_1$ *is a fundamental solution for $\Delta_{H_n}^p$ on H_n.*

Proof. In view of the relation between $\phi_{\lambda,m}$ and π_λ,

$$\langle P_s, f\rangle = (-1)^p \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{s^{2m+n}}{(2m+n)^p} \operatorname{tr}(\pi_\lambda(f)|_{\mathscr{P}_m})|\lambda|^{n-p}\, d\lambda$$
$$= (-1)^p \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{s^{2m+n}}{(2m+n)^p} \langle \pi_\lambda(f) u_{\lambda,\alpha}, u_{\lambda,\alpha}\rangle_\lambda |\lambda|^{n-p}\, d\lambda. \tag{2.11}$$

We will show that this converges absolutely. Indeed, by Lemma 2.4,

$$\int_{-\infty}^{\infty} \sum_\alpha \frac{|\langle \pi_\lambda(f) u_{\lambda,\alpha}, u_{\lambda,\alpha}\rangle_\lambda|}{(2|\alpha|+n)^p} |\lambda|^{n-p}\, d\lambda$$
$$\le \int_{-\infty}^{\infty} \sum_\alpha \frac{c_N}{(2|\alpha|+n)^p (1+2|\alpha|)^N (1+|\lambda|)^N} |\lambda|^{n-p}\, d\lambda$$
$$\le c_N \int_{-\infty}^{\infty} \sum_\alpha \frac{1}{(1+2|\alpha|)^N} \frac{|\lambda|^{n-p}}{(1+|\lambda|)^N}\, d\lambda.$$

Here

$$\sum_\alpha \frac{1}{(1+2|\alpha|)^N} = \sum_{m=0}^{\infty} \binom{m+n-1}{m} \frac{1}{(1+2m)^N}$$

converges for $N > n$ since

$$\binom{m+n-1}{m} \sim \frac{m^{n-1}}{(n-1)!} \quad \text{as } m \to \infty.$$

Also, since $p \le n$,

$$\int_{-\infty}^{\infty} \frac{|\lambda|^{n-p}}{(1+|\lambda|)^N}\, d\lambda \le 2\int_0^{\infty} (1+\lambda)^{n-p-N}\, d\lambda$$

converges for $N > n$.

The formula for P_s given in (1) results from substituting Formula 2.5 for $\phi_{\lambda,m}$ and manipulating. Here, equality means weak convergence in the space of tempered distributions.

Let

$$g_s(\lambda) = (-1)^p |\lambda|^{n-p} \sum_{m=0}^{\infty} \frac{s^{2m+n}}{(2m+n)^p} \binom{m+n-1}{m}$$
$$\times \int_{H_n} \phi_{\lambda,m}(z,t) f(z,t)\, dz\, dt.$$

In the proof for (1) we saw that $|g_s(\lambda)|$ is integrable. Since $g_s(\lambda) \to g_1(\lambda)$ as $s \to 1$ and $|g_s(\lambda)| \leq |g_1(\lambda)|$, the Lebesgue Dominated Convergence Theorem shows that

$$\langle P_s, f\rangle = \int_{-\infty}^{\infty} g_s(\lambda)\, d\lambda \to \int_{-\infty}^{\infty} g_1(\lambda)\, d\lambda = \langle P_1, f\rangle \quad \text{as } s \to 1.$$

This shows that $\lim_{s\to 1^-} P_s = P_1$ in $S'(H_n)$. That P_1 is tempered follows from the w^*-completeness of $S'(H_n)$.

The distribution T_D given by a left invariant differential operator D on H_n is defined by

$$\langle T_D, f\rangle := (Df)(0,0) \quad \text{for } f \in \mathscr{E}(H_n). \tag{2.12}$$

The assertion that $\Psi_p = P_1$ is a fundamental solution for $\Delta_{H_n}^p$ means $\Psi_p * \check{T}_{\Delta_{H_n}^p} = \delta_0$. That is, we must show that for $f \in \mathscr{D}(H_n)$,

$$\left\langle \Psi_p, f * T_{\Delta_{H_n}^p} \right\rangle = f(0,0). \tag{2.13}$$

Using Formula 2.11 we see that

$$\left\langle \Psi_p, f * T_{\Delta_{H_n}^p} \right\rangle = \int_{-\infty}^{\infty} \sum_{\alpha} \frac{\left\langle \pi_\lambda\left(f * T_{\Delta_{H_n}^p}\right) u_{\lambda,\alpha}, u_{\lambda,\alpha}\right\rangle_\lambda}{(-(2|\alpha| + n))^p |\lambda|^p} |\lambda|^n\, d\lambda.$$

Since $\pi_\lambda(f * T_{\Delta_{H_n}^p}) = \pi_\lambda(f)\pi_\lambda(\Delta_{H_n}^p)$ and

$$\pi_\lambda\left(\Delta_{H_n}^p\right) u_{\lambda,\alpha} = (-(2|\alpha| + n))^p |\lambda|^p u_{\lambda,\alpha} \quad \text{(by Formula 2.3)}$$

we obtain

$$\begin{aligned}\left\langle \Psi_p, f * T_{\Delta_{H_n}^p} \right\rangle &= \int_{-\infty}^{\infty} \sum_{\alpha} \langle \pi_\lambda(f) u_{\lambda,\alpha}, u_{\lambda,\alpha}\rangle_\lambda |\lambda|^n\, d\lambda \\ &= \int_{-\infty}^{\infty} \operatorname{tr}(\pi_\lambda(f)) |\lambda|^n\, d\lambda \\ &= f(0,0)\end{aligned}$$

by the Plancherel formula for H_n. (See e.g. [Fo2].) ∎

LEMMA 2.14. (1)

$$\sum_{m=0}^{\infty} \frac{s^{2m+n}}{(2m+n)^p} \binom{m+n-1}{m} L_m^{(n-1)}\left(\frac{\lambda |z|^2}{2}\right)$$

converges absolutely and uniformly for all z and $0 \le s < 1$ to a Schwartz function $F_s(z, \lambda)$ characterized by

$$\left(s\frac{d}{ds}\right)^p F_s(z, \lambda) = s^n \left(\frac{s}{1-s^2}\right)^n \exp\left(\frac{-s^2}{1-s^2}\left(\frac{\lambda |z|^2}{2}\right)\right)$$

and

$$\left(\frac{d}{ds}\right)^j F_s \bigg|_{s=0} = 0 \quad \textit{for } j = 0, 1, \ldots, p-1.$$

(2) *$P_s = (-1)^p 2\,\mathrm{Re} \int_0^\infty F_s(z, \lambda)\, e^{-\lambda\gamma} \lambda^{n-p}\, d\lambda$ where $\gamma = |z|^2/4 - it$. The integral converges uniformly for $0 \le s < 1$ and z bounded away from 0. In particular, P_s is a smooth function for $z \ne 0$.*

Proof. Part (1) follows from the classical fact that

$$\sum_{m=0}^{\infty} \binom{m+n-1}{m} L_m^{(n-1)}(w) t^m = \frac{1}{(1-t)^n} \exp\left(\frac{-tw}{1-t}\right) \tag{2.15}$$

where convergence is absolute and uniform in t for $0 < t < 1$. (See e.g. [Fa].) Since $F_s(z, \lambda)$ is divisible by s^n and $p \le n$, the appropriate initial conditions are as claimed.

Part (1) of Lemma 2.10 shows that $P_s = (-1)^p 2\,\mathrm{Re} \int_0^\infty F_s(z, \lambda) e^{-\lambda\gamma} \lambda^{n-p}\, d\lambda$ as distributions. On the other hand, the differential equation for $F_s(z, \lambda)$ given above shows that for $0 < s < 1$ and $|z| \ne 0$, $F_s(z, \lambda)$ decays exponentially as $\lambda \to \infty$. This proves part (2). ■

Proof of Theorem A. Let

$$\widetilde{P_s}(z, t) = \int_0^\infty F_s(z, \lambda) e^{-\lambda\gamma} \lambda^{n-p}\, d\lambda$$

for $s < 1$ where $F_s(z, \lambda)$ is as in Lemma 2.14. The lemma shows that

$$\begin{aligned} \left(\frac{d}{ds}\right)^j \widetilde{P_s}(z, t) \bigg|_{s=0} &= \int_0^\infty \left(\frac{d}{ds}\right)^j F_s(z, \lambda) \bigg|_{s=0} e^{-\lambda\gamma} \lambda^{n-p}\, d\lambda \\ &= 0 \quad \text{for } j = 0, 1, \ldots, p-1 \end{aligned}$$

and that

$$\left(s\frac{d}{ds}\right)^p \widetilde{P_s}(z,t) = \int_0^\infty \left(s\frac{d}{ds}\right)^p F_s(z,\lambda)e^{-\lambda\gamma}\lambda^{n-p}\,d\lambda$$
$$= \int_0^\infty \left(\frac{s}{1-s^2}\right)^n \exp\left(\frac{-s^2}{1-s^2}\left(\frac{\lambda|z|^2}{2}\right)\right)e^{-\lambda\gamma}\lambda^{n-p}\,d\lambda$$
$$= \frac{s^n}{(1-s^2)^n}\int_0^\infty e^{-\lambda\alpha}\lambda^{n-p}\,d\lambda$$

where

$$\alpha = \frac{s^2}{1-s^2}\frac{|z|^2}{2} + \gamma.$$

Since $p \le n$, the above integral converges to yield

$$\left(s\frac{d}{ds}\right)^p \widetilde{P_s}(z,t) = \frac{s^n}{(1-s^2)^n}\frac{(n-p)!}{\alpha^{n-p+1}}. \tag{2.16}$$

Using the formula for γ one obtains

$$\alpha = \frac{s^2\bar{\gamma}+\gamma}{1-s^2}. \tag{2.17}$$

Substituting Formula 2.17 in 2.16 and manipulating yields

$$P_s = \frac{(-1)^p 2(n-p)!}{r^{n-p+1}}\operatorname{Re}(G_s(\theta))$$

where

$$\left(s\frac{d}{ds}\right)^p G_s(\theta) = \frac{e^{i(n-p+1)\theta}s^n}{(1-s^2)^{p-1}(s^2+e^{2i\theta})^{n-p+1}} \tag{2.18}$$

as claimed.

Lemmas 2.10 and 2.14 show that the *weak* limit Ψ_p as $s \to 1^-$ of the functions

$$\frac{2(-1)^p(n-p)!}{r^{n-p+1}}\operatorname{Re}(G_s)$$

gives a tempered fundamental solution for $\Delta_{H_n}^p$. It remains to prove that we also have pointwise convergence to a smooth function away from $\{(0,0)\}$ as asserted in the statement of Theorem A.

The integrand

$$R_s(\theta) = \frac{e^{i(n-p+1)\theta} s^n}{(1-s^2)^{p-1}(s^2+e^{2i\theta})^{n-p+1}}$$

is a sum of terms, each with a singularity at one of the points $s = 1, -1, ie^{i\theta}$, or $-ie^{i\theta}$ and coefficients which are smooth functions of θ in the domain $|\theta| < \pi/2$. As $s \to 1^-$, we need only consider the terms with singularities at 1 to check for uniform convergence on compacta. The degree of the singularity for such a term is at most $p-1$. After $p-2$ integrations, the "worst terms" are of the form

$$\frac{s}{s-1} = \sum_{j=1}^{\infty} s^j.$$

After two more divisions and integrations, this becomes $\Sigma_{j=1}^{\infty} s^n/n^2$, which is absolutely convergent as $s \to 1^-$. Hence, $\psi_p(\theta)$ is a smooth function for $|\theta| < \pi/2$.

When $\theta = \pi/2$, $R_s(\theta)$ becomes

$$\frac{(-1)^{n-p+1} s^n \, e^{i(n-p+1)\pi/2}}{(1-s^2)^n}$$

and the above analysis fails. On the other hand, we know that $\Delta_{H_n}\Psi_2 = \Psi_1$, where Ψ_1 is given by Folland's Formula 1.6. Since Ψ_1 is smooth away from $(0,0)$ and Δ_{H_n} is hypo-elliptic (see [FS]), we conclude that Ψ_2 is smooth away from $(0,0)$. Similarly, the singular support of each Ψ_p is $\{(0,0)\}$. We conclude that the smooth function Ψ_p for $r \neq 0$ and $|\theta| < \pi/2$ must extend continuously to a smooth function for $r \neq 0$. That is, the apparent singularities at $\theta = \pi/2$ are an artifact of our choice of coordinates (r, θ). ■

Proof of Theorem B. Theorem A together with the binomial expansions

$$\frac{1}{(1-s^2)^{p-1}} = \sum_{k=0}^{\infty} \binom{p+k-2}{k} s^{2k}$$

and

$$\frac{1}{(s^2+e^{2i\theta})^{n-p+1}} = (e^{-2i\theta})^{n-p+1} \sum_{l=0}^{\infty} \binom{n-p+l}{l} (-e^{-2i\theta})^l s^{2l}$$

yield

$$\left(s\frac{d}{ds}\right)^p G_s$$

$$= e^{-i(n-p+1)\theta} s^n \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{p+k-2}{k} \binom{n-p+l}{l} \left(-e^{-2i\theta}\right)^l s^{2(k+l)}$$

$$= e^{-i(n-p+1)\theta} \sum_{m=0}^{\infty} \left(\sum_{k+l=m} \binom{p+k-2}{k} \binom{n-p+l}{l} (-1)^l e^{-i2l\theta} \right) s^{2m+n}.$$

This fact together with the initial conditions on G_s at $s = 0$ gives

$$G_s = e^{-i(n-p+1)\theta} \sum_{m=0}^{\infty} \frac{1}{(2m+n)^p}$$

$$\times \left(\sum_{k+l=m} \binom{p+k-2}{k} \binom{n-p+l}{l} (-1)^l e^{-i2l\theta} \right) s^{2m+n}. \quad (2.19)$$

Applying Formula 2.19 to a test function and setting $s = 1$, Theorem A now shows

$$\Psi_p = \frac{(-1)^p 2(n-p)!}{r^{n-p+1}}$$

$$\times \operatorname{Re}\left[\sum_{m=0}^{\infty} \frac{1}{(2m+n)^p} \sum_{k+l=m} (-1)^l \binom{p+k-2}{k} \binom{n-p+l}{l} e^{-(n-p+1+2l)i\theta} \right]$$

$$= \frac{(-1)^p 2(n-p)!}{r^{n-p+1}} \sum_{m=0}^{\infty} \frac{1}{(2m+n)^p}$$

$$\times \sum_{k+l=m} (-1)^l \binom{p+k-2}{k} \binom{n-p+l}{l} \cos((n-p+1+2l)\theta).$$

where the series converges to Ψ_p in $S'(H_n)$.

3. The cases $p = 1$ and $p = 2$

One can recover Folland's Formula 1.6 for the fundamental solution Ψ_1 of Δ_{H_n} from Theorem A by showing that $\psi_1(\theta)$ is a constant function. Note that when $p = 1$, the integrand in the definition of $G_s(\theta)$ has no singularity at

$s = 1$. Thus, we can write

$$\psi_1(\theta) = \mathrm{Re}(G_1(\theta)) \quad \text{where } G_1(\theta) = \int_0^1 \frac{e^{in\theta}s^{n-1}}{(s^2 + e^{2i\theta})^n}\, ds.$$

One computes,

$$\begin{aligned}
\frac{d}{d\theta} G_1(\theta) &= \frac{d}{d\theta} \int_0^1 \frac{e^{in\theta}s^{n-1}}{(s^2 + e^{2i\theta})^n}\, ds \\
&= ine^{in\theta} \int_0^1 \frac{s^{n-1}(s^2 - e^{2i\theta})}{(s^2 + e^{2i\theta})^{n+1}}\, ds \\
&= ie^{in\theta} \int_0^1 \frac{\partial}{\partial s}\left[\frac{-s^n}{(s^2 + e^{2i\theta})^n}\right] ds \\
&= \frac{-ie^{in\theta}}{(1 + e^{2i\theta})^n} = -i\left(\frac{1}{2\cos(\theta)}\right)^n.
\end{aligned}$$

We see that $(d/d\theta)G_1(\theta)$ is pure imaginary and hence $\psi_1(\theta)$ is constant.

Theorem B yields a weak power series representation for Ψ_p. We will describe an alternative approach to deriving a formula for Ψ_p from Theorem A. Rather than expanding $1/(1 - s^2)^{p-1}$ and $1/(s^2 + e^{2i\theta})^{n-p+1}$ in power series, one can form a (finite) partial fractions decomposition for

$$\frac{s^n}{(1 - s^2)^{p-1}(s^2 + e^{2i\theta})^{n-p+1}}.$$

In principle, this can be done for all n and p. Unfortunately, integration in s will yield a log term which one must expand in power series to carry out successive integrations. Here we will illustrate the procedure for $p = 2$ and $n = 2N$ even.

Let $\gamma = re^{i\theta}$ as before and write $\beta = e^{i\theta}$. We must solve

$$\left(s\frac{d}{ds}\right)^2 G_s = \frac{\beta^{n-1}s^n}{(1 - s^2)(s^2 + \beta^2)^{n-1}} \tag{3.1}$$

subject to

$$G_0 = 0 = \frac{d}{ds}(G_s)|_{s=0}.$$

Letting $u = s^2$ and $a = \beta^2$, Formula 3.1 becomes

$$\frac{4}{\beta^{n-1}}\left(u\frac{d}{du}\right)^2 G_u = \frac{u^N}{(1-u)(u+a)^{2N-1}}. \tag{3.2}$$

Dividing

$$\frac{u^N}{(1-u)(u+a)^{2N-1}}$$

by u and expanding in partial fractions yields

$$\begin{aligned}
&\frac{u^{N-1}}{(1-u)(u+a)^{2N-1}}\\
&\quad= \frac{1}{(1+a)^{2N-1}}\frac{1}{(1-u)} + \frac{1}{(1+a)^N}\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{-a}{1+a}\right)^k\\
&\qquad\times\sum_{j=0}^{N+k-1}\frac{(1+a)^j}{(u+a)^{j+1}}.
\end{aligned}$$

Integrating in u, taking initial conditions into account, gives

$$\begin{aligned}
&\frac{1}{(1+a)^{2N-1}}\left[\log\left(1+\frac{u}{a}\right) - \log(1-u)\right]\\
&\quad-\frac{1}{(1+a)^N}\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{-a}{1+a}\right)^k\\
&\quad\times\left[\sum_{j=1}^{N+k-1}\frac{(1+a)^j}{j(u+a)^j} - \sum_{j=1}^{N+k-1}\frac{(1+a)^j}{ja^j}\right].
\end{aligned} \tag{3.3}$$

We divide Expression 3.3 by u, expand the log terms in power series and re-write the inner sums using partial fractions. This gives

$$\begin{aligned}
&\frac{1}{(1+a)^{2N-1}}\sum_{k=1}^{\infty}\left[\frac{(-u)^{k-1}}{ka^k} + \frac{u^{k-1}}{k}\right]\\
&\quad+\frac{1}{(1+a)^N}\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{-a}{1+a}\right)^k\sum_{j=1}^{N+k-1}\frac{(1+a)^j}{ja^j}\sum_{m=0}^{j-1}\frac{a^m}{(u+a)^{m+1}}.
\end{aligned}$$

Integrating with respect to u, taking initial conditions into account and letting $u \to 1^-$ yields

$$\frac{4}{\beta^{n-1}} G_1 = \frac{1}{(1+\beta^2)^{2N-1}} \sum_{k=1}^{\infty} \left[-\frac{(-1)^k}{k^2 \beta^{2k}} + \frac{1}{k^2} \right] + \frac{1}{(1+\beta^2)^N} \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{-\beta^2}{1+\beta^2} \right)^k \sum_{j=1}^{N+k-1} \frac{(1+\beta^2)^j}{j\beta^{2j}} \times \left[\log(1+1/\beta^2) - \sum_{m=1}^{j-1} \left(\frac{\beta^{2m}}{m(1+\beta^2)^m} - \frac{1}{m} \right) \right],$$

or equivalently

$$4G_1 = \frac{1}{(2\cos\theta)^{2N-1}} \sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - (-1)^k \bar{\beta}^{2k}\right) + \frac{1}{(2\cos\theta)^N} \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{-1}{2\cos\theta} \right)^k \sum_{j=1}^{N+k-1} \frac{(2\cos\theta)^j}{j} \times \left[\beta^{N+k-j-1} \log(2\bar{\beta}\cos\theta) - \sum_{m=1}^{j-1} \frac{1}{m} \left(\frac{\beta^{N-1+k-j+m}}{(2\cos\theta)^m} - \beta^{N-1+k-j} \right) \right].$$

Finally, taking the real part of G_1 and multiplying by 2 gives the fundamental solution for $\Delta_{H_n}^2$:

$$\Psi_2 = \frac{(n-2)!}{2r^{n-1}} Q_1(\theta) \tag{3.4}$$

where

$$Q_1(\theta) = \sum_{k=1}^{\infty} \frac{1 + (-1)^{k-1}\cos(2k\theta)}{k^2(2\cos\theta)^{n-1}} + \frac{1}{(2\cos\theta)^N} \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{-1}{2\cos\theta} \right)^k \sum_{j=1}^{N+k-1} \frac{(2\cos\theta)^j}{j} \times \Bigg[\cos((N-1+k-j)\theta) \log(2\cos\theta) + \theta \sin((N-1+k-j)\theta) - \sum_{m=1}^{j-1} \frac{1}{m} \left(\frac{\cos((N-1+k-j+m)\theta)}{(2\cos\theta)^m} - \cos((N-1+k-j)\theta) \right) \Bigg],$$

and $n = 2N$.

Moreover,

$$\sum_{k=1}^{\infty} \frac{1+(-1)^{k+1}\cos(2k\theta)}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(2k\theta)$$
$$= \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(2k\theta)$$
$$= \left(\frac{\pi}{2} - \theta\right)\left(\frac{\pi}{2} + \theta\right). \tag{3.5}$$

The last equality is a nice exercise using the fact that the 2π-periodic extension of $\theta(\pi - \theta)$ on $[0, \pi]$ has Fourier series

$$\frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2}\cos(2k\theta).$$

Thus we obtain a *finite* expression for the fundamental solution (3.4) where

$$\begin{aligned} Q_1(\theta) = {} & \frac{1}{(2\cos\theta)^{n-1}}\left(\frac{\pi^2}{4} - \theta^2\right) \\ & + \frac{1}{(2\cos\theta)^N}\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{-1}{2\cos\theta}\right)^k \sum_{j=1}^{N+k-1}\frac{(2\cos\theta)^j}{j} \\ & \times\Bigg[\cos((N-1+k-j)\theta)\log(2\cos\theta) + \theta\sin((N-1+k-j)\theta) \\ & \quad - \sum_{m=1}^{j-1}\frac{1}{m}\left(\frac{\cos((N-1+k-j+m)\theta)}{(2\cos\theta)^m} \right. \\ & \qquad \left. -\cos((N-1+k-j)\theta)\right)\Bigg], \end{aligned}$$

and $n = 2N$.

A similar analysis may be carried out for the case where n is odd, $n = 2N + 1$. In this case, we obtain (after considerable labour) the closed expression

$$\Psi_2 = \frac{-(n-2)!}{r^{n-1}}Q_1(\theta), \tag{3.6}$$

where

$$Q_1(\theta) = \frac{\pi^2}{2^{2N+3}(\cos\theta)^{2N}} + \frac{1}{(2\cos\theta)^N} \sum_{j=0}^{N} \binom{N}{j} \left(\frac{-1}{2\cos\theta}\right)^j \sum_{k=0}^{N+j-1} (\cos\theta)^k$$
$$\times \Bigg[\binom{2k}{k} \frac{1}{2^k} \left\{ \frac{\pi\theta}{4} \sin(N+j-k-1)\theta - \psi(\theta)\cos(N+j-k-1)\theta \right\}$$
$$+ \sum_{l=0}^{k-1} \binom{k+l}{l} \frac{1}{k-l} \frac{1}{2^l}$$
$$\times \Bigg\{ \frac{\pi}{4} \cos(N+j-k-1)\theta + \frac{1}{2} \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$$
$$\times \sin(N+j-k-1)\theta - \sum_{m=1}^{k-l-1} \frac{1}{m(2\cos\theta)^m}$$
$$\times \sum_{q=1}^{[(m+1)/2]} \binom{m}{2q-1} (-1)^q \cos(N+j-k+m-2q)\theta \Bigg\} \Bigg].$$

Here

$$\psi(\theta) = \frac{1}{2} \int_0^{\pi/2-\theta} \log\left(\tan\frac{\varphi}{2}\right) d\varphi, \quad |\theta| < \frac{\pi}{2},$$

is the function with Fourier series

$$-\sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)\theta}{(2k+1)^2}.$$

(See Formula 1.32 in [Ob].)

For low values of n, some of the sums in the formulae for Ψ_2 collapse and one can obtain further simplifications by using multiple angle identities. We used the Mathematica system on a computer to obtain the following forms for Ψ_2 with $n = 2$, 3, 4 and 5.

$$\Psi_2 = \begin{cases} \dfrac{\pi^2 - 4\theta^2}{8r\cos\theta}, & \text{for } n = 2 \\[2ex] \dfrac{\pi(2\theta\sin\theta + 2\cos\theta - \pi)}{32r^2\cos^2\theta}, & \text{for } n = 3 \\[2ex] \dfrac{\pi^2 - 4\theta^2 - 4\cos^2\theta - 8\theta\cos\theta\sin\theta}{32r^3\cos^3\theta}, & \text{for } n = 4 \\[2ex] \dfrac{\pi(6\theta\sin\theta + 3\theta\cos^2\theta\sin\theta + \cos^3\theta + 6\cos\theta - 3\pi)}{64r^4\cos^4\theta}, & \text{for } n = 5. \end{cases} \tag{3.7}$$

We used the computer to check that $\Delta^2_{H_n}(\Psi_2) = 0$ for $r \neq 0$ in each case by applying Formula 1.8. Using l'Hospital's rule, one can check that there are no singularities at $\theta = \pi/2$ as asserted in part (ii) of Theorem A.

4. Concluding remarks

In this section we will place some of the preceding discussion in a more general setting. Let K be a compact subgroup of $U(n)$. We say that (K, H_n) is a *Gelfand pair* if the convolution algebra $L^1_K(H_n)$ of K-invariant L^1-functions is commutative. This condition is equivalent to the action of K on the space $\mathscr{P}(\mathbf{C}^n)$ of holomorphic polynomials on $\mathbf{C}^n$ being multiplicity free [Ca], [BJR1], [BJR2]. It is well known that $(U(n), H_n)$ is a Gelfand pair but one also obtains Gelfand pairs using many proper subgroups K of $U(n)$ [BJR2]. Suppose below that (K, H_n) is a Gelfand pair and let

$$\mathscr{P}(\mathbf{C}^n) = \sum_{\alpha \in \Lambda} V_\alpha \tag{4.1}$$

denote the multiplicity free decomposition of $\mathscr{P}(\mathbf{C}^n)$ into irreducible K-modules.

Let $\mathbf{D}_K(H_n)$ denote the algebra of left-H_n-invariant differential operators on H_n that are also invariant under the action of K. For example, $\mathbf{D}_{U(n)}(H_n) = \mathbf{C}[\Delta_{H_n}, \partial/\partial t]$. In [BJR2] it is shown that in general

$$\mathbf{D}_K(H_n) = \mathbf{C}\left[D_1, D_2, \ldots, D_r, \frac{\partial}{\partial t}\right] \tag{4.2}$$

where $D_1, \ldots, D_r$ are certain "fundamental" homogeneous differential operators of even degree (with respect to the dilations given in Formula 1.5). When K acts irreducibly on $\mathbf{C}^n$, Δ_{H_n} will be one of the generators $D_1, \ldots, D_r$ and the remaining generators each have degree at least 4. Below, we consider the problem of finding a fundamental solution for a given differential operator $D \in \mathbf{D}_K(H_n)$.

A smooth function $\phi : H_n \to \mathbf{C}$ is said to be *K-spherical* if $\phi(0,0) = 1$ and ϕ is a simultaneous eigenfunction for all operators $E \in \mathbf{D}_K(H_n)$. A K-spherical function is bounded if and only if it is positive definite [BJR1]. Let $\Delta(K, H_n)$ denote the set of bounded K-spherical functions. The *spherical transform* $S(f) : \Delta(K, H_n) \to \mathbf{C}$ of an integrable function $f : H_n \to \mathbf{C}$ is

$$S(f)(\phi) = \int_{H_n} f(n) \check{\phi}(n)\, dn \tag{4.3}$$

where $\check{\phi}(n) := \phi(n^{-1})$. Godement's Plancherel Theorem [Go] shows that there is a measure ν on $\Delta(K, H_n)$ for which the formula

$$f(n) = \int_{\Delta(K,N)} S(f)(\phi)\phi(n)\, d\nu(\phi) \tag{4.4}$$

holds for all positive definite functions $f \in L^1_K(H_n)$.

It is shown in [BJR2] that a set of full measure in $\Delta(K, H_n)$ is parametrized by pairs

$$(\lambda, \alpha) \in (\mathbf{R}\backslash\{0\}) \times \Lambda)$$

using the formula

$$\phi_{\lambda,\alpha}(z,t) = \frac{1}{\dim(V_\alpha)} \sum_{i=1}^{\dim(V_\alpha)} \langle \pi_\lambda(z,t)v_i, v_i\rangle_\lambda \tag{4.5}$$

where $\{v_i\}$ is any orthonormal basis for $V_\alpha \subset \mathscr{P}(\mathbf{C}^n) \subset \mathscr{F}_\lambda$. $\phi_{\lambda,\alpha}$ has the general form

$$\phi_{\lambda,\alpha}(z,t) = q_\alpha\left(\sqrt{|\lambda|}\, z\right)e^{-|\lambda|\,|z|^2/4}e^{i\lambda t} \tag{4.6}$$

where q_α is a homogeneous polynomial of even degree in $(z, \bar{z})$. Note that the identity $\check{\phi}_{\lambda,\alpha} = \overline{\phi}_{\lambda,\alpha}$ follows. The polynomials $\{q_\alpha : \alpha \in \Lambda\}$ are, in principle, computable for given K.

Formula 4.4 is related to the usual Plancherel Formula for H_n and in terms of our parametrization one has

$$d\nu(\phi_{\lambda,\alpha}) = \dim(V_\alpha)|\lambda|^n\, d\lambda. \tag{4.7}$$

(Recall that $d\mu(\pi_\lambda) = |\lambda|^n\, d\lambda$ is Plancherel measure on $\hat{H}_n$ [Fo2].) We rewrite Formula 4.4 as

$$f(n) = \int_{-\infty}^{\infty} \sum_{\alpha\in\Lambda} \dim(V_\alpha)S(f)\left(\check{\phi}_{\lambda,\alpha}\right)\check{\phi}_{\lambda,\alpha}(n)|\lambda|^n\, d\lambda. \tag{4.8}$$

Suppose now that we are given $D \in \mathbf{D}_K(H_n)$ and that D is positive definite. Otherwise, $(D^*) * P$ will be a fundamental solution for D if P is a fundamental solution for $D * D^*$. If D has a fundamental solution then K-averaging will yield a K-invariant fundamental solution P. This is clear since both D and δ_0 are K-invariant. *Formally* we can use Formula 4.4 to expand P in terms of K-spherical functions provided we can compute the coefficients $S(P)(\check{\phi}_{\lambda,\alpha}) = \langle P, \phi_{\lambda,\alpha}\rangle$.

Recall that V_α can be regarded as a subspace of the representation space $\mathscr{F}_\lambda$ for π_λ. Since D is K-invariant, $\pi_\lambda(D)$ must preserve V_α and commute with the action of K. Schur's Lemma implies that $\pi_\lambda(D)|_{V_\alpha}$ is a scalar operator $\chi_{\lambda,\alpha}(D)I_{V_\alpha}$ say. In fact, $\chi_{\lambda,\alpha}(D)$ is the $\phi_{\lambda,\alpha}$-eigenvalue for D [BJR2],

$$D(\phi_{\lambda,\alpha}) = \chi_{\lambda,\alpha}(D)\phi_{\lambda,\alpha}. \tag{4.9}$$

Similar reasoning shows that for $f \in L^1_K(H_n)$, one has

$$\pi_\lambda(f)|_{V_\alpha} = \langle f, \phi_{\lambda,\alpha}\rangle I_{V_\alpha}. \tag{4.10}$$

Since $\pi_\lambda(D)\pi_\lambda(P) = \pi_\lambda(\delta_0) = I$, we see that

$$\pi_\lambda(P)|_{V\alpha} = \frac{1}{\chi_{\lambda,\alpha}(D)} I_{P_\alpha}$$

and conclude formally from Formula 4.10 that

$$\langle P, \phi_{\lambda,\alpha}\rangle = \frac{1}{\chi_{\lambda,\alpha}(D)}. \tag{4.11}$$

Combining Formulas 4.6, 4.8 and 4.11 produces a formal expression for a fundamental solution for D.

$$\begin{aligned} P(z,t) &= \int_{-\infty}^{\infty} \sum_{\alpha\in\Lambda} \frac{\dim(V_\alpha)}{\chi_{\lambda,\alpha}(D)} \cdot \overline{\chi_{\lambda,\alpha}(z,t)}|\lambda|^n \, d\lambda \\ &= \int_{-\infty}^{\infty} \sum_{\alpha\in\Lambda} \frac{\dim(V_\alpha)}{\chi_{\lambda,\alpha}(D)} e^{-i\lambda t} q_\alpha\left(\sqrt{|\lambda|}\, z\right) e^{-|\lambda||z|^2/4} |\lambda|^n \, d\lambda. \end{aligned} \tag{4.12}$$

One is left with the problem of determining whether or not this expression yields a well defined distribution. Some problems related to this were studied in [BaDo].

Finally we mention that the formal method described above carries over to certain more general solvable Lie groups G. Suppose that G is connected, simply connected and solvable, $K \subset \mathrm{Aut}(G)$ is compact and (K, G) is a Gelfand pair. That is, the convolution algebra $L^1_K(G)$ is commutative. This situation is studied in [BJR1]. One can hope to apply the techniques here to find fundamental solutions for left G- and K-invariant differential operators on G.

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