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SOME PROPERTIES OF THE QUOTIENT SPACE $(L^1(T^d)\,/\,H^1(D^d))$

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1. Introduction

Let D be the unit disc of the complex plane and $\mathbf{T}$ the unit circle equipped with normalized Lebesgue measure. Let d be a positive integer. We denote by D^d and $\mathbf{T}^d$ respectively the products of d copies of D and $\mathbf{T}$. They are respectively the d-disc and d-torus. $\mathbf{T}^d$ is equipped with the product measure dm. Let $0 < p \leq \infty$. We denote by $H^p(D^d)$ the classical Hardy space in the polydisc D^d. If $p < \infty$, this is the space of the analytic functions f in D^d such that

$$\sup_{0 \leq r < 1} \int_{\mathbf{T}^d} |f(rz)|^p \, dm(z) < \infty,$$

where $r = (r_1, \ldots, r_d) \in [0, 1)^d$, $z = (z_1, \ldots, z_d) \in \mathbf{T}^d$ and $rz = (r_1 z_1, \ldots, r_d z_d) \in D^d$. If $p = \infty$, $H^\infty(D^d)$ is the space of the bounded analytic functions in D^d. Equipped with its natural norm or quasi-norm, $H^p(D^d)$ is a Banach space if $1 \leq p \leq \infty$ and a quasi-Banach space if $0 < p < 1$. It is well-known that every function in $H^p(D^d)$ admits a.e. radial limits on $\mathbf{T}^d$ and the function is uniquely determined by its boundary function on $\mathbf{T}^d$. Thus identifying functions in $H^p(D^d)$ with their boundary values, we may regard $H^p(D^d)$ as a closed subspace of $L^p(\mathbf{T}^d)$.

Let us recall that a Banach (or quasi-Banach) space X is of cotype 2 if there exists a constant C such that for all finite sequences $\{x_n\} \subset X$

$$\left(\sum \|x_n\|^2\right)^{1/2} \leq C \int \left\| \sum \varepsilon_n x_n \right\|,$$

where $\{\varepsilon_n\}$ is a Rademacher sequence. Recall also that a linear operator u: $X \to Y$ between two Banach spaces is called p-summing ($1 \leq p < \infty$) if there

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exists a constant C such that for all finite sequences $\{x_n\} \subset X$

$$\left(\sum \|ux_n\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum |\xi(x_n)|^p\right)^{1/p} : \xi \in X^*, \|\xi\| \leq 1\right\}.$$

The least such constant is denoted by $\pi_p(u)$. If every bounded operator from X into l^2 is 1-summing, X is called a GT space in Pisier's terminology. The reader is referred to [15] for more information about these notions.

The main result of this paper is the following

THEOREM 1. *$L^1(\mathbf{T}^d)/H^1(D^d)$ is a GT space of cotype 2.*

We can also formulate a dual version of Theorem 1. For this we need some notation. Let $C(\mathbf{T}^d)$ be the space of the continuous functions on $\mathbf{T}^d$, equipped with the uniform norm. Let $Q(\mathbf{T}^d)$ be the subspace of $C(\mathbf{T}^d)$ defined by

$$Q(\mathbf{T}^d) = \{f \in C(\mathbf{T}^d) : \hat{f}(n) = 0 \text{ if } n < 0\},$$

where $\hat{f}$ is the Fourier transform of f, and where $n = (n_1, \ldots, n_d) < 0$ means that $n_k < 0$ for all $1 \leq k \leq d$. Note that $Q(\mathbf{T})$ is just the disc algebra A (but $Q(\mathbf{T}^d)$ is not an algebra if $d > 1$).

THEOREM 2. *Let $2 < p < \infty$ and Y be a Banach space. Then for every 2-summing operator u: $Q(\mathbf{T}^d) \to Y$ we have*

$$\pi_p(u) \leq C_p \pi_2(u)^{2/p} \|u\|^{1-2/p}.$$

Consequently, every bounded operator from $Q(\mathbf{T}^d)$ into L^1 is 2-summing. More generally, if Y is of cotype 2, then every bounded operator from $Q(\mathbf{T}^d)$ into Y is 2-summing.

Remark. It is worth noting that if $d > 1$, A and $Q(\mathbf{T}^d)$ are not isomorphic, and neither are $L^1(\mathbf{T})/H^1(D)$ and $L^1(\mathbf{T}^d)/H^1(D^d)$ (see Section 5 below).

When $d = 1$, the above results are the famous theorems about Grothendieck's inequality for the disc algebra, proved by J. Bourgain in [3]. Very recently, Bourgain also obtained the ball version of the above results (cf. [2]).

To prove Theorems 1 and 2, we shall adapt a recent elegant argument of G. Pisier [13]. Pisier in [13] gave a simple proof of Bourgain's theorem for the disc algebra (i.e., the above theorems in the case $d = 1$). In this approach, the essential point is a result on interpolation between vector-valued Hardy spaces in the disc. For adapting this approach to our present situation, what we have to do is to prove the following result on interpolation between vector-valued Hardy spaces in the polydisc.

Let $0 < p < \infty$ and $0 < q \leq \infty$. Set

$$L^p(l^q;\mathbf{T}^d) = \left\{\{f_n\}_{n\geq 0} \subset L^p(\mathbf{T}^d): \int_{\mathbf{T}^d}\left(\sum_{n\geq 0}|f_n|^q\right)^{p/q} dm < \infty\right\},$$

$$H^p(l^q;D^d) = \left\{\{f_n\}_{n\geq 0} \subset H^p(\mathbf{D}^d): \int_{\mathbf{T}^d}\left(\sum_{n\geq 0}|f_n|^q\right)^{p/q} dm < \infty\right\}.$$

These spaces are equipped with their natural norms or quasi-norms. We also need the so-called K-functional from the interpolation theory (cf. [1]). Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces. Then for any $t > 0$ and $x \in X_0 + X_1$,

$$K_t(x; X_0, X_1) = \inf\left\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}: x = x_0 + x_1,\, x_j \in X_j,\, j = 0,1\right\}.$$

THEOREM 3. *There exists a constant C (depending only on d) such that*

$$\forall t > 0, \quad \forall f \in H^1(l^2; D^d),$$
$$K_t(f; H^1(l^1; D^d), H^1(l^2; D^d)) \leq CK_t(f; L^1(l^1;\mathbf{T}^d), L^1(l^2;\mathbf{T}^d)).$$

In the language of Pisier [14], Theorem 3 says that $(H^1(l^1, D^d), H^1(l^2; D^d))$ is K-closed relative to $(L^1(l^1;\mathbf{T}^d), L^1(l^2;\mathbf{T}^d))$. More generally, let (X_0, X_1) be an interpolation couple and $S_j \subset X_j$ $(j = 0,1)$ a closed subspace. Following Pisier, (S_0, S_1) is said to be K-closed relative to (X_0, X_1) if there exists a constant C such that for any $t > 0$ and any $x \in S_0 + S_1$,

$$K_t(x; S_0, S_1) \leq CK_t(x; X_0, X_1).$$

Using Theorem 3, we can deduce Theorems 1 and 2 as in [13]. The details are left to the reader.

The remaining part of this paper is mainly devoted to the proof of Theorem 3. We shall give two different proofs. The first one is via tent spaces introduced by R. Coifman, Y. Meyer and E.M. Stein [5]. The second one, pointed out to us by J. Bourgain and S.V. Kisliakov, uses unconditional bases in $H^1(D^d)$. The existence of such bases was first proved by B. Maurey [11]. The two proofs are respectively presented in the following two sections (in the case $d = 2$). The reason that we give two different proofs in that each of them has its advantage over the other. The proof of Theorem 3 in the case of d bigger than 2 is sketched in Section 4. Section 5 contains some more results and remarks.

2. The first proof of Theorem 3: the bidisc

By conformal transformation it suffices to show Theorem 3 (in the case $d = 2$) in the bi-upper-half-plane. Hence we shall consider the Hardy space

$$H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2) \text{ in } \mathbf{R}_+^2 \times \mathbf{R}_+^2 \quad (0 < p \leq \infty).$$

If $0 < p < \infty$, $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ is the space of the analytic functions f in $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ satisfying

$$\sup_{\substack{y_1>0\\ y_2>0}} \int_{\mathbf{R}^2} |f(z_1, z_2)|^p \, dx_1 \, dx_2 < \infty, \quad z_k = x_k + iy_k, \quad k = 1,2.$$

$H^\infty(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ is the space of all bounded analytic functions in $\mathbf{R}_+^2 \times \mathbf{R}_+^2$. The conformal transformation mapping isometrically $H^1(D^2)$ onto $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ is defined by

$$f \mapsto \frac{1}{\pi^2 (z_1 + i)^2 (z_2 + i)^2} f(w(z_1), w(z_2)), \quad f \in H^1(D^2),$$

where $w(z) = (i - z)/(i + z)$ is the conformal transformation mapping $\mathbf{R}_+^2$ onto D (cf. [6]). As in the bidisc case, $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ is also considered as a closed subspace of $L^p(\mathbf{R}^2)$. Similarly, we define the vector-valued spaces $H^p(l^q; \mathbf{R}_+^2 \times \mathbf{R}_+^2)$ and $L^p(l^q; \mathbf{R}^2)$.

Therefore, to prove Theorem 3 is equivalent to show that the couple

$$\left(H^1(l^1; \mathbf{R}_+^2 \times \mathbf{R}_+^2), H^1(l^2; \mathbf{R}_+^2 \times \mathbf{R}_+^2)\right)$$

is K-closed relative to $(L^1(l^1; \mathbf{R}^2), L^1(l^2; \mathbf{R}^2))$. The main ingredients of the following proof for this last statement are tent spaces and interpolation results between vector-valued Hardy spaces in the disc. The idea is to consider Hardy spaces in the bidisc as Hardy spaces of functions in the disc with values in Hardy spaces in the disc.

Now let us introduce tent spaces. Let $0 < p$, $q < \infty$. Let $f\colon \mathbf{R}_+^2 \to \mathbf{C}$ be a measurable function. Define

$$A_q(f)(x) = \left(\int_{|y-x|<t} |f(y,t)|^q \frac{dy\,dt}{t^2} \right)^{1/q}, \quad x \in \mathbf{R},$$

and

$$T_q^p(\mathbf{R}_+^2) = T_q^p = \{ f\colon A_q(f) \in L^p(\mathbf{R}) \}.$$

T_q^p is equipped with the quasi-norm $\|A_q(f)\|_p$. It is easy to see that T_q^p is a quasi-Banach lattice (a Banach lattice if $1 \le p, q < \infty$). It is also clear that if $1 < p, q < \infty$, T_q^p is a UMD space (cf. [4] for the definition of UMD spaces). Let $\alpha > 0$. The α-convexification $(T_q^p)^{(\alpha)}$ of T_q^p is isometrically equal to $T_{\alpha q}^{\alpha p}$ (the reader is referred to [10] for the notion of α-convexification). Therefore, for any $p, q \in (0, \infty)$, there exists $\alpha > 0$ such that $(T_q^p)^{(\alpha)}$ is a UMD space. If X is a Banach space, we define similarly the vector-valued tent space $T_q^p(X; \mathbf{R}_+^2) = T_q^p(X)$ of functions on $\mathbf{R}_+^2$ with values in X as above by replacing the absolute value by the norm of X. Then $T_q^p(X)$ is a UMD space if $1 < p, q < \infty$ and X is UMD. If additionally X is a Banach lattice, then $T_q^p(X)$ is a quasi-Banach lattice, and in that case $(T_q^p(X))^{(\alpha)} = T_{\alpha q}^{\alpha p}(X^{(\alpha)})$ for any $\alpha > 0$.

We also need Hardy spaces of harmonic functions. Let $H_h^p(\mathbf{R}_+^2) = H_h^p$ $(0 < p < \infty)$ be the Hardy space of the complex harmonic functions in $\mathbf{R}_+^2$ whose nontangential maximal functions belong to $L^p(\mathbf{R})$. Similarly, we define $H_h^p(l^2; \mathbf{R}_+^2) = H_h^p(l^2)$ as the Hardy space of harmonic functions with values in l^2.

The relation between tent spaces and Hardy spaces is expressed by the following important result (cf. [5]).

LEMMA 4. *There exist bounded linear operators*

$$\sigma: H_h^1 \to T_2^1 \quad \text{and} \quad \tau: T_2^1 \to H_h^1$$

such that $\tau\sigma = \mathrm{id}_{H_h^1}$. *Moreover,* σ *and* τ *extend in the natural way (coordinates by coordinates) to bounded operators from* $H_h^1(l^2)$ *to* $T_2^1(l^2)$ *and from* $T_2^1(l^2)$ *to* $H_h^1(l^2)$.

Remark. The desired operators σ and τ can be defined as follows. For σ we set

$$\sigma f(x, t) = t \frac{\partial f}{\partial t}(x, t).$$

Then it is well known that $f \in H_h^1$ iff $\sigma f \in T_2^1$. In order to define τ we take a function $\varphi \in C^\infty(\mathbf{R})$ with compact support satisfying

$$\int_{\mathbf{R}} \varphi(x)\, dx = 0 \quad \text{and} \quad -2\pi \int_0^\infty \hat{\varphi}(\xi t) e^{-2\pi|\xi| t}\, dt = 1, \quad \forall \xi \neq 0,$$

where $\hat{\varphi}$ is the Fourier transform of φ. Write $\varphi_t(x) = (1/t)\varphi(x/t)$. If $f \in T_2^1$, $\tau(f)$ is defined as the Poisson integral in $\mathbf{R}_+^2$ of the following function in $\mathbf{R}$

$$\int_0^\infty f(\cdot, t) * \varphi_t \frac{dt}{t}.$$

Then by [5] $\tau\colon T_2^1 \to H_h^1$ is bounded and $\tau\sigma = \mathrm{id}_{H_h^1}$.

Remark. The last part of Lemma 4 concerning l^2-valued functions is not explicitly stated in [5]. But the proof there works without any change for this vector-valued case. The extended operators will be still denoted by σ and τ.

Now we consider the couple $(l^1(T_2^1), T_2^1(l^2))$. Since $(l^1(T_2^1))^{(2)} = l^2(T_4^2)$ and $(T_2^1(l^2))^{(2)} = T_4^2(l^4)$ are UMD lattices, by [9] Theorem 5.3 (in the present situation Lemma 5 can be also deduced from arguments in [14]), we have:

LEMMA 5.

$$\big(H^1\big(l^1\big(T_2^1\big); \mathbf{R}_+^2\big), H^1\big(T_2^1\big(l^2\big); \mathbf{R}_+^2\big)\big)$$

is K-closed relative to

$$\big(L^1\big(l^1\big(T_2^1\big); \mathbf{R}\big), L^1\big(T_2^1\big(l^2\big); \mathbf{R}\big)\big).$$

By Lemmas 4 and 5 we easily show:

LEMMA 6.

$$\big(H^1\big(l^1\big(H_h^1\big); \mathbf{R}_+^2\big), H^1\big(H_h^1\big(l^2\big); \mathbf{R}_+^2\big)\big)$$

is K-closed relative to

$$\big(L^1\big(l^1\big(H_h^1\big)\mathbf{R}\big), L^1\big(H_h^1\big(l^2\big); \mathbf{R}\big)\big).$$

Proof. Let $t > 0$ and $f \in H^1(H_h^1(l^2); \mathbf{R}_+^2)$ be such that

$$K_t\big(f; L^1\big(l^1\big(H_h^1\big); \mathbf{R}\big), L^1\big(H_h^1\big(l^2\big); \mathbf{R}\big)\big) < 1.$$

Choose $f_0 \in L^1(l^1(H_h^1); \mathbf{R})$ and $f_1 \in L^1(H_h^1(l^2); \mathbf{R})$ such that $f = f_0 + f_1$ and

$$\|f_0\|_{L^1(l^1(H_h^1); \mathbf{R})} + t\|f_1\|_{L^1(H_h^1(l^2); \mathbf{R})} < 1.$$

Then it follows from Lemma 4 that $\sigma f_0 \in L^1(l^1(T_2^1); \mathbf{R})$, $\sigma f_1 \in L^1(T_2^1(l^2); \mathbf{R})$ and

$$\|\sigma f_0\|_{L^1(l^1(T_2^1); \mathbf{R})} + t\|\sigma f_1\|_{L^1(T_2^1(l^2); \mathbf{R})} \leq C;$$

so

$$K_t\big(\sigma f; L^1\big(l^1\big(T_2^1\big);\mathbf{R}\big), L^1\big(T_2^1(l^2);\mathbf{R}\big)\big) \le C.$$

Note that $\sigma f \in H^1(T_2^1(l^2); \mathbf{R}_+^2)$; and we find, by Lemma 5,

$$g_0 \in H^1\big(l^1\big(T_2^1\big);\mathbf{R}_+^2\big), \quad g_1 \in H^1\big(T_2^1(l^2);\mathbf{R}_+^2\big)$$

such that $\sigma f = g_0 + g_1$ and

$$\|g_0\|_{H^1(l^1(T_2^1);\mathbf{R}_+^2)} + t\|g_1\|_{H^1(T_2^1(l^2);\mathbf{R}_+^2)} \le C.$$

Now by Lemma 4 once more,

$$f = \tau\sigma f = \tau g_0 + \tau g_1,$$
$$\tau g_0 \in H^1\big(l^1\big(H_h^1\big);\mathbf{R}_+^2\big), \tau g_1 \in H^1\big(H_h^1(l^2);\mathbf{R}_+^2\big);$$

moreover,

$$\|\tau g_0\|_{H^1(l^1(H_h^1);\mathbf{R}_+^2)} + t\|\tau g_1\|_{H^1(H_h^1(l^2);\mathbf{R}_+^2)} \le C.$$

This shows Lemma 6 by homogeneity.

Now consider

$$H^1\big(\mathbf{R}_+^2\big) \xrightarrow{i} H_h^1\big(\mathbf{R}_+^2\big) \xrightarrow{R} H^1\big(\mathbf{R}_+^2\big),$$

where i is the natural inclusion and R the Riesz projection. Then from these two elementary operators and Lemma 6 we deduce as above the following result.

LEMMA 7.

$$\big(H^1\big(l^1\big(H^1\big(\mathbf{R}_+^2\big);\mathbf{R}_+^2\big)\big); H^1\big(H^1\big(l^2;\mathbf{R}_+^2\big);\mathbf{R}_+^2\big)\big)$$

is K-closed relative to

$$\big(L^1\big(l^1\big(H^1\big(\mathbf{R}_+^2\big)\big);\mathbf{R}\big), L^1\big(H^1\big(l^2;\mathbf{R}_+^2\big);\mathbf{R}\big)\big).$$

Now we can easily finish the proof of Theorem 3 (in the case $d = 2$) as follows. By the Fubini theorem

$$L^1\big(l^1\big(H^1\big(\mathbf{R}_+^2\big)\big);\mathbf{R}\big)\big) = H^1\big(L^1\big(l^1;\mathbf{R}\big);\mathbf{R}_+^2\big)$$
$$L^1\big(H^1\big(l^2;\mathbf{R}_+^2\big);\mathbf{R}\big) = H^1\big(L^1\big(l^2;\mathbf{R}\big);\mathbf{R}_+^2\big).$$

On the other hand by [9], Theorem 5.3 (cf. also [14]),

$$\left(H^1\left(L^1(l^1;\mathbf{R});\mathbf{R}^2_+\right), H^1\left(L^1(l^2;\mathbf{R});\mathbf{R}^2_+\right)\right)$$

is K-closed relative to

$$\left(L^1\left(L^1(l^1;\mathbf{R});\mathbf{R}\right), L^1\left(L^1(l^2;\mathbf{R});\mathbf{R}\right)\right).$$

This latter couple is exactly equal to $(L^1(l^1;\mathbf{R}^2), L^1(l^2;\mathbf{R}^2))$. Hence by Lemma 7,

$$\left(H^1\left(l^1;\mathbf{R}^2_+\times \mathbf{R}^2_+\right), H^1\left(l^2;\mathbf{R}^2_+\times \mathbf{R}^2_+\right)\right)$$

is K-closed relative to

$$\left(L^1(l^1;\mathbf{R}^2), L^1(l^2;\mathbf{R}^2)\right),$$

which is what we want to show.

3. The second proof Theorem 3: the bidisc

We shall use unconditional bases of H^1 spaces instead of tent spaces. The existence of unconditional bases in $H^1(D)$ was first proved by B. Maurey [11]. So let $\{\chi_n\}_{n\geq 0}$ be such a basis. Then by the classical Khintchine inequality we have the following characterization of $H^1(D)$ in terms of the "square function". A function $f = \sum_{n\geq 0}a_n\chi_n$ ($a_n \in \mathbf{C}$, $n \geq 0$) belongs to $H^1(D)$ iff $(\sum_{n\geq 0}|a_n|^2|\chi_n|^2)^{1/2} \in L^1(\mathbf{T})$. Moreover,

$$\|f\|_{H^1(D)} \sim \left\|\left(\sum_{n\geq 0}|a_n|^2|\chi_n|^2\right)^{1/2}\right\|_{L^1(\mathbf{T})},$$

where the equivalence constants are independent of f. Now we consider $H^1(l^1; D)$ and $H^1(l^2; D)$. Functions in these spaces are sequences of functions in $H^1(D)$. Let $\{e_n\}_{n\geq 0}$ be the canonical basis of l^1, that is,

$$e_n = \Big(\underbrace{0,\dots,0}_{n\text{ times}}, 1, 0, 0, \dots\Big).$$

Then every function f in $H^1(l^1; D)$ or $H^1(l^2; D)$ can be written as

$$f = \sum_{\substack{n\geq 0\\ k\geq 0}} a_{nk}\chi_n e_k, \quad a_{nk} \in \mathbf{C}, n, k \geq 0.$$

Therefore, by the above characterization of $H^1(D)$ in terms of the square function, we have

$$\begin{aligned}
\|f\|_{H^1(l^1;D)} &= \int_{\mathbf{T}} \sum_{k\geq 0} \Big| \sum_{n\geq 0} a_{nk}\chi_n \Big| \\
&= \sum_{k\geq 0} \int_{\mathbf{T}} \Big| \sum_{n\geq 0} a_{nk}\chi_n \Big| \sim \sum_{k\geq 0} \int_{\mathbf{T}} \Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2} \\
&= \left\| \left\{ \Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2} \right\}_{k\geq 0} \right\|_{L^1(l^1;\mathbf{T})}
\end{aligned}$$

Hence identifying f with the double sequence $\{a_{nk}\}$ of its coefficients, we may regard $H^1(l^1;D)$ as a space of double sequences as follows:

$$H^1(l^1;D) = \left\{ \{a_{nk}\} \subset \mathbf{C}\colon \left\{ \Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2} \right\}_{k\geq 0} \in L^1(l^1;\mathbf{T}) \right\}.$$

Then equipped with the norm

$$\int_{\mathbf{T}} \sum_{k\geq 0} \Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2},$$

$H^1(l^1;D)$ clearly becomes a Banach lattice of double sequences.

Similarly, $H^1(l^2;D)$ can be also viewed as a Banach lattice of double sequences. Indeed, let $\{\varepsilon_n\}_{n\geq 0}$ be a Rademacher sequence on a probability space (Ω, P) and $\{\varepsilon'_n\}_{n\geq 0}$ an independent copy of $\{\varepsilon_n\}_{n\geq 0}$. Then for every function $f = \sum_{n,k} a_{nk}\chi_n e_k$ as above, we have by Khintchine's inequality

$$\begin{aligned}
\|f\|_{H^1(l^2;D)} &= \int_{\mathbf{T}} \Big(\sum_{k\geq 0} \Big| \sum_{n\geq 0} a_{nk}\chi_n \Big|^2 \Big)^{1/2} \\
&\sim \int_{\mathbf{T}} \int_{\Omega} \Big| \sum_{k\geq 0} \Big(\sum_{n\geq 0} a_{nk}\chi_n \Big) \varepsilon_k \Big| \\
&= \int_{\Omega} \int_{\mathbf{T}} \Big| \sum_{n\geq 0} \Big(\sum_{k\geq 0} a_{nk}\varepsilon_k \Big) \chi_n \Big| \\
&\sim \int_{\Omega} \int_{\mathbf{T}} \int_{\Omega} \Big| \sum_{n\geq 0} \Big(\sum_{k\geq 0} a_{nk}\varepsilon_k \Big) \chi_n \varepsilon'_n \Big| \\
&\sim \int_{\mathbf{T}} \Big(\sum_{k\geq 0} \sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2} \\
&= \left\| \left\{ \Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2 \Big)^{1/2} \right\}_{k\geq 0} \right\|_{L^1(l^2;\mathbf{T})}
\end{aligned}$$

Therefore, we get as above

$$H^1(l^2;D) = \left\{\{a_{nk}\} \subset \mathbf{C}: \left\{\left(\sum_{n\geq 0} |a_{nk}|^2|\chi_n|^2\right)^{1/2}\right\}_{k\geq 0} \in L^1(l^2;\mathbf{T})\right\}.$$

In this way, $H^1(l^2; D)$ also becomes a Banach lattice of double sequences, equipped with the obvious norm.

Then we calculate the 2-convexifications of these lattices $H^1(l^1; D)$ and $H^1(l^2; D)$. By definition (cf. [10]), the 2-convexification $E^{(2)}$ of a Banach lattice E of measurable functions on a measure space is the following Banach lattice

$$E^{(2)} = \{f \text{ measurable}: |f|^2 \in E\}$$

and

$$\|f\|_{E^{(2)}} = \left\||f|^2\right\|_E^{1/2}.$$

Now let $\{a_{nk}\}$ be a double sequence in the 2-convexification $(H^1(l^1; D))^{(2)}$ of $H^1(l^1; D)$. Then

$$\begin{aligned}\|\{a_{nk}\}\|_{(H^1(l^1;D))^{(2)}} &= \left\|\{|a_{nk}|^2\}\right\|_{H^1(l^1;D)}^{1/2} \\ &= \left[\int_{\mathbf{T}} \sum_{k\geq 0}\left(\sum_{n\geq 0} |a_{nk}|^4|\chi_n|^2\right)^{1/2}\right]^{1/2} \\ &= \left[\int_{\mathbf{T}}\left(\sum_{k\geq 0}\left(\sum_{n\geq 0} |a_{nk}|^4|\chi_n|^2\right)^{(1/4)\cdot 2}\right)^{(1/2)\cdot 2}\right]^{1/2} \\ &= \left\|\left\{\left(\sum_{n\geq 0} |a_{nk}|^4|\chi_n|^2\right)^{1/4}\right\}_{k\geq 0}\right\|_{L^2(l^2;\mathbf{T})}.\end{aligned}$$

Hence

$$(H^1(l^1;D))^{(2)} = \left\{\{a_{nk}\} \subset \mathbf{C}: \left\{\left(\sum_{n\geq 0} |a_{nk}|^4|\chi_n|^2\right)^{1/4}\right\}_{k\geq 0} \in L^2(l^2;\mathbf{T})\right\}.$$

Similarly,

$$(H^1(l^2;D))^{(2)} = \left\{\{a_{nk}\} \subset \mathbf{C}: \left\{\left(\sum_{n\geq 0} |a_{nk}|^4|\chi_n|^2\right)^{1/4}\right\}_{k\geq 0} \in L^2(l^4;\mathbf{T})\right\}.$$

It then is easy to see from the above two equalities that $(H^1(l^1; D))^{(2)}$ and $(H^1(l^2; D))^{(2)}$ are UMD lattices.

Let us summarize the preceding discussions in the following lemma.

LEMMA 8. *With the identification between functions in $H^1(l^1; D)$, $H^1(l^2; D)$ and the double sequences of their coefficients, $H^1(l^1; D)$ and $H^1(l^2; D)$ are Banach lattices of double sequences; moreover, their 2-convexifications are UMD lattices.*

Now using Lemma 8 and [9], Theorem 5.3, we can easily prove Theorem 3 (in the case $d = 2$). Indeed, considering $(H^1(l^1; D), H^1(l^2; D))$ as a couple of Banach lattices as in Lemma 8, we deduce from Lemma 8 and [9] Theorem 5.3 that

$$\left(H^1\left(H^1(l^1; D); D\right), H^1\left(H^1(l^2; D); D\right)\right)$$

is K-closed relative to

$$\left(L^1\left(H^1(l^1; D); \mathbf{T}\right), L^1\left(H^1(l^2; D); \mathbf{T}\right)\right).$$

Then using the Fubini Theorem and [9], Theorem 5.3, once more as in Section 2, we show that

$$\left(H^1(l^1; D^2), H^1(l^2; D^2)\right)$$

is K-closed relative to

$$\left(L^1(l^1; \mathbf{T}^2), L^1(l^2; \mathbf{T}^2)\right).$$

4. Proof of Theorem 3: the polydisc

The polydisc case can be similarly treated as the bidisc one. Both proofs presented in the preceding sections extend to polydiscs. The proof via tent spaces uses this time tent spaces in product domains (cf. [7]). The second one by unconditional bases now employs unconditional bases for H^1 in polydiscs (cf. [11]). We give here the second one only since it is a straight generalization of the previous proof for the bidisc.

We shall show Theorem 3 only in the case of $d = 3$. The other cases can be similarly proved by induction. So we shall consider the couple $(H^1(l^1; D^3), H^1(l^2; D^3))$. As in the bidisc case we write again $H^1(l^1; D^3) = H^1(H^1(l^1; D^2); D)$; similarly, for $H^1(l^2, D^3)$. By [11], $H^1(D^2)$ has an unconditional basis. Thus Theorem 3 in the case $d = 3$ will follow from Theorem 3 in the case $d = 2$, which was already proved previously, and the following lemmas.

LEMMA 9. *Let (Ω, μ) be an arbitrary measure space. Let $X \subset L^1(\mu)$ be a closed subspace. If X has an unconditional basis, then the couple*

$$(H^1(X(l^1)), H^1(X(l^2)))$$

is K-closed relative to

$$(L^1(X(l^1); \mu), L^1(X(l^2); \mu))).$$

Here, for a Banach space Y we have denoted by $H^1(Y)$ the closure in $L^1(Y; \mu)$ of all the complex polynomials with coefficients in Y.

$X(l^p)$ is defined as a space of sequences of functions in X in the natural way.

We may prove Lemma 9 by the arguments in Section 3, just replacing H^1 by X.

LEMMA 10. *Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces and $S_j \subset X_j$ $(j = 0, 1)$ a closed subspace. Let $0 < p_0,\ p_1 \leq \infty$ and (Ω, μ) be an arbitrary measure space. If (S_0, S_1) is K-closed relative to (X_0, X_1), then*

$$(L^{p_0}(S_0; \Omega), L^{p_1}(S_1; \Omega))$$

is K-closed relative to

$$(L^{p_0}(X_0; \Omega), L^{p_1}(X_1; \Omega)).$$

Proof. To prove the lemma, first note that (S_0, S_1) is K-closed relative to (X_0, X_1) iff there exists a constant C such that whenever $x \in S_0 + S_1$ can be written as $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$, then it can be written as $x = s_0 + s_1$ with $s_0 \in S_0$ and $s_1 \in S_1$ such that $\|s_0\| \leq C\|x_0\|$ and $\|s_1\| \leq C\|x_1\|$. Then we need only apply this remark pointwise.

5. More results and remarks

Let us first combine the reasoning of this paper and that of Pisier [13] to obtain the following general result.

PROPOSITION 11. *Let (Ω, μ) be an arbitrary measure space. Let $X \subset L^1(\mu)$ be a closed subspace. If $(X(l^1), X(l^p))$ is K-closed relative to $(L^1(l^1; \mu), L^1(l^p; \mu))$ for some $p \in (1, \infty)$, then $L^1(\mu)/X$ is a GT space of cotype 2.*

Proof. Let $Y \subset L^\infty(\mu)$ be the annihilator of X. Then the K-closedness of $(X(l^1), X(l^p))$ in $(L^1(l^1;\mu), L^1(l^p;\mu))$ implies the K-closedness of $(Y(l^q), Y(l^\infty))$ in $(L^\infty(l^q;\mu), L^\infty(l^\infty;\mu))$, where $1/p + 1/q = 1$ (see [14]). Thus by interpolation and the reasoning of [13], we see that any bounded operator from Y to L^1 is 2-summing. Hence, by duality, $L^1(\mu)/X$ is a GT space.

Now note that by Lemma 10,

$$\left(L^1(X(l^1)), L^1(X(l^p))\right)$$

is K-closed relative to

$$\left(L^1(L^1(l^1;\mu)), L^1(L^1(l^p;\mu))\right).$$

Then applying the above arguments to $L^1(X)$ and $L^1(L^1(\mu))$ instead of X and $L^1(\mu)$ respectively, we deduce that $L^1(L^1(\mu))/L^1(X)$ is a GT space, which implies the cotype 2 of $L^1(\mu)/X$. We thus conclude the proof of the proposition.

The following remarks were pointed out to us by the referee.

Remarks. (i) It is well-known that if X is a reflexive subspace of $L^1(\mu)$, then $L^1(\mu)/X$ is a GT space of cotype 2 (cf. [15]). This result can be recovered by Proposition 11. Indeed, by a well-known theorem of Resenthal [16], $X(l^p) = l^p(X)$ (with equivalent norms) for some $p \in (1,\infty)$. On the other hand, $X(l^1) = l^1(X)$ (with equal norms). It then follows that $(X(l^1), X(l^p))$ is K-closed relative to $(L^1(l^1;\mu), L^1(l^p;\mu))$. Therefore, it remains to apply Proposition 11.

(ii) Let X be as in (i). Using the fact that $H^1(X(l^p)) = H^1(l^p(X))$, we may show that $(H^1(X(l^1)), H^1(X(l^p)))$ is K-closed relative to $(L^1(L^1(l^1;\mu)), L^1(L^1(l^p;\mu)))$. Then by Proposition 11, $L^1(L^1(\mu))/H^1(X)$ is a GT space of cotype 2.

Now we turn to generalize Theorem 1 to $L^p(\mathbf{T}^d)/H^p(D^d)$ with $p < 1$.

PROPOSITION 12. *Let $0 < p < 1$. Then $L^p(\mathbf{T}^d)/H^p(D^d)$ is of cotype 2.*

If $d = 1$, Proposition 12 was already proved by G. Pisier [13]. As for Theorems 1 and 2, we shall prove Proposition 12 again following arguments of [13]. So, like the case $p = 1$, what we have to do is to show the following (generalization of Theorem 3 to $p < 1$).

LEMMA 13. *Let $0 < p < 1$. Then*

$$\left(H^p(l^p; D^d), H^p(l^2; D^d)\right)$$

is K-closed relative to

$$(L^p(l^p;\mathbf{T}^d), L^p(l^2;\mathbf{T}^d)).$$

Proof. It can be done by either tent spaces or unconditional bases. We give here the proof by unconditional bases and for $d = 2$ only. It was shown by P. Wojtaszczyk [17] that $H^p(D)$ possesses unconditional bases. Let $\{\chi_n\}_{n\geq 0}$ be such a basis. Then as in the case of $p = 1$, functions in $H^p(l^p; D)$ and $H^p(l^2; D)$ may be regarded as double sequences of complex numbers. An easy application of Khintchine's inequality shows

$$H^p(l^p;D) = \left\{\{a_{nk}\} \subset \mathbf{C}: \left\{\Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2\Big)^{1/2}\right\}_{k\geq 0} \in L^p(l^p;\mathbf{T})\right\},$$

$$H^p(l^2;D) = \left\{\{a_{nk}\} \subset \mathbf{C}: \left\{\Big(\sum_{n\geq 0} |a_{nk}|^2 |\chi_n|^2\Big)^{1/2}\right\}_{k\geq 0} \in L^p(l^2;\mathbf{T})\right\}.$$

Therefore, these spaces may be viewed as quasi-Banach lattices of double sequences. Then we may show that their $(2/p)$-convexifications are UMD lattices; so the couple

$$(H^p(l^p;D), H^p(l^2;D))$$

satisfies the conditions of [9], Theorem 5.3. By that theorem,

$$(H^p(H^p(l^p;D);D), H^p(H^p(l^2;D);D))$$

is K-closed relative to

$$(L^p(H^p(l^p;D);\mathbf{T}), L^p(H^p(l^2;D);\mathbf{T})).$$

Then we may finish the proof as in Section 3. The remainder is omitted.

Remark. From Lemma 13 we can deduce the following stronger result than Proposition 12. Let L^p be an arbitrary L^p-space. Let

$$u: c_0 \to L^p(L^p(\mathbf{T}^d)/H^p(D^d))$$

be a bounded operator. Then for any finite sequence $\{x_n\} \subset c_0$,

$$\|\{u(x_n)\}\|_{L^p(L^p(l^2;\mathbf{T}^d)/H^p(l^2;D^d))} \leq C\|u\| \, \|\{x_n\}\|_{c_0(l^2)},$$

where C is a constant depending on d and p only. In particular, u is

2-summing. We also deduce that if $\{\varepsilon_n\}$ is a Rademacher sequence, then for every finite sequence $\{f_n\} \subset L^p(\mathbf{T}^d)/H^p(D^d)$ there exists $\{\tilde{f}_n\} \subset L^p(\mathbf{T}^d)$ such that

$$q(\tilde{f}_n) = f_n, \quad \forall n \quad \big(q\colon L^p(\mathbf{T}^d) \to L^p(\mathbf{T}^d)/H^p(D^d)$$

$$\text{being the canonical quotient map}\big),$$

$$\int \Big\| \sum \varepsilon_n \tilde{f}_n \Big\|_{L^p(\mathbf{T}^d)} \le C \int \Big\| \sum \varepsilon_n f_n \Big\|_{L^p(\mathbf{T}^d)/H^p(D^d)}.$$

This remark also applies to $H^1(D^d)$.

Remark. It was provided in [17] that for any $0 < p_0 < \infty$ there exists a sequence $\{\chi_n\}_{n \ge 0}$ which is an unconditional basis in $H^p(D)$ for every $p \in [p_0, \infty)$. Using this result and the previous arguments we may show the following

PROPOSITION 14. *Let $0 < p_0, p_1 < \infty$. Then*

$$\big(H^{p_0}(D^d), H^{p_1}(D^d)\big)$$

is K-closed relative to

$$\big(L^{p_0}(\mathbf{T}^d), L^{p_1}(\mathbf{T}^d)\big).$$

Remark. In other words, Proposition 14 says that for any $f \in H^{p_0}(D^d) + H^{p_1}(D^d)$ and any $t > 0$

$$K_t\big(f; H^{p_0}(D^d), H^{p_1}(D^d)\big) \sim K_t\big(f; L^{p_0}(\mathbf{T}^d), L^{p_1}(\mathbf{T}^d)\big).$$

It seems that this statement is new. At this point, let us recall the previously known results about the K-functional for the couple $(H^{p_0}(D^d), H^{p_1}(D^d))$. It was shown in [8] that for any $f \in H^{p_0}(D^d) + H^{p_1}(D^d)$ and $t > 0$,

$$\begin{aligned} K_t\big(f; H^{p_0}(D^d), H^{p_1}(D^d)\big) &\sim K_t\big(Sf; L^{p_0}(\mathbf{T}^d), L^{p_1}(\mathbf{T}^d)\big) \\ &\sim K_t\big(Mf; L^{p_0}(\mathbf{T}^d), L^{p_1}(\mathbf{T}^d)\big), \end{aligned}$$

where Sf and Mf are respectively the square and nontangential maximal functions of f. Note that the norms of these two functions in L^p $(0 < p \le 1)$ are generally larger than that of f itself.

Let us end this section by some more properties of $L^1(\mathbf{T}^d)/H^1(D^d)$ and $Q(\mathbf{T}^d)$.

PROPOSITION 15. *Let $d \geq 2$. Then*:

(i) *$Q(\mathbf{T}^d)$ is not isomorphic to the disc algebra A*;

(ii) *$L^1(\mathbf{T}^d)/H^1(D^d)$ is not isomorphic to $L^1(\mathbf{T})/H^1(D)$*;

(iii) *the dual space of $Q(\mathbf{T}^d)$ can be written as*

$$\left(Q(\mathbf{T}^d)\right)^* = \frac{L^1(\mathbf{T}^d)}{H_0^1(D^d)} \underset{1}{\oplus} M_s,$$

where $H_0^1(D^d) = \{f \in H^1(D^d): \hat{f}(n) = 0$ if one of the coordinates of n is not positive$\}$, and where M_s is the subspace in $(C(\mathbf{T}^d))^*$ of all the measures singular with respect to Lebesgue measure on $\mathbf{T}^d$.

Remark. The statement (i) above is known to specialists. It was pointed out to us by S.V. Kisliakov in January 1991.

(iii) is the generalization to $Q(\mathbf{T}^d)$ of the well-known classical F. and M. Riesz theorem for the disc algebra (corresponding to $d = 1$).

Proof of Proposition 15. (i) This result may be proved in a similar way as the proof of the non-isomorphism between the polydisc and disc algebras given in [12] p. 84.

(ii) If the two quotient spaces in question were isomorphic, then their dual spaces would be isomorphic as well. These dual ones are respectively

$$\left(H^1(D^d)\right)^{\perp} = \left\{f \in L^\infty(\mathbf{T}^d): \hat{f}(n) = 0 \text{ if } n \geq 0\right\},$$

$$\left(H^1(D)\right)^{\perp} = \left\{f \in L^\infty(\mathbf{T}): \hat{f}(n) = 0 \text{ if } n \geq 0\right\}.$$

Note that $(H^1(D))^\perp$ has the (i_p, π_p)-property and the relevant constant is of order $p^2/(p-1)$ $(1 < p < \infty)$ (cf. [12]). Therefore, $(H^1(D^d))^\perp$ would also possess the same property. Now consider the natural inclusion

$$\left(H^1(D^d)\right)^{\perp} \xrightarrow{I_p} S_p,$$

where S_p is the closure of $(H^1(D^d))^\perp$ in $L^p(\mathbf{T}^d)$ $(1 < p < \infty)$. I_p is clearly p-summing and $\pi_p(I_p) = 1$; so it would be p-integral and $i_p(I_p) \leq C[p^2/(p-1)]$. Then the same would be true for the restriction of I_p to the subspace of the continuous functions. But using the argument in [12] again, we know that this last statement about the restriction of I_p is impossible. Hence (ii) is proved.

(iii) We first calculate the annihilator $(Q(\mathbf{T}^d))^\perp$ of $Q(\mathbf{T}^d)$. We claim that $(Q(\mathbf{T}^d))^\perp = H_0^1(D^d)$. Indeed, let μ be a measure from $(Q(\mathbf{T}^d))^\perp$. Then

$$\int_{\mathbf{T}^d} f\,d\mu = 0, \quad \forall f \in Q(\mathbf{T}^d).$$

It follows that $\hat{\mu}(n) = 0$ if one of the coordinates of n is less or equal to zero. We then deduce that the Poisson integral $P_r * \mu$ of μ is an analytic function in D^d. By Jensen's inequality

$$\sup_{0 \le r < 1} \int_{\mathbf{T}^d} |P_r(z) * \mu|\,dm(z) \le \|\mu\|.$$

Therefore, $P_r * \mu \in H^1(D^d)$; and so its boundary function is an L^1-function on $\mathbf{T}^d$. This boundary function is just μ. Hence $\mu \in L^1(\mathbf{T}^d)$, proving our claim. Now (iii) is obvious from this claim. Thus we conclude the proof of Proposition 15.

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