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ON A THEOREM OF BURKHOLDER

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Let $\{r_k(t)\}_{k=0}^{\infty}$ be Rademacher functions defined as

$$r_0(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t < 1 \end{cases}$$

$$r_0(t+1) = r_0(t);$$

$$r_k(t) = r_0(2^k t).$$

E.M. Stein, in his important paper [3], applied the following result: Let E be any measurable subset of $[0,1]$ and $|E| > 0$, then there is an integer N and a constant A both depending only on E such that if $c, c_1, c_2, \dots$ are complex numbers and the series $\sum_{k=0}^{\infty} c_k r_k(t)$ converges almost everywhere, then

$$A\left(\sum_{k=N}^{\infty} |c_k|^2\right)^{1/2} \le \operatorname{esssup}\left\{\left|c + \sum_{k=0}^{\infty} c_k r_k(t)\right| : t \in E\right\}. \tag{1}$$

Rademacher functions are a sequence of independent random variables. D.L. Burkholder, in [1], extended (1) to other sequences of independent random variables satisfying certain conditions. In fact, Burkholder's result when specialized to Rademacher functions, is considerably stronger than (1). It is proved that there exist positive constants α and β so that for every set E, $|E| > 0$, there exists $N = N(E)$ so that

$$\left|\left\{t \in E : \beta\left(\sum_{k=N}^{\infty} |c_k|^2\right)^{1/2} \le \left|c + \sum_{k=0}^{\infty} c_k r_k(t)\right|\right\}\right| \ge \alpha |E|.$$

Using recently obtained norm inequalities for lacunary Walsh series [2] we extend Burkholder's theorem to q-lacunary Walsh series with $q > 1$. Since lacunary Walsh functions do not form an independent system of random variables, this case is not covered by Burkholder's theorem. Our proof is also

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valid for Rademacher series and in this context it provides an alternative, simple, real-variable proof.

Let $\{w_k(t)\}_{k=0}^{\infty}$ be Walsh functions in Paley's ordering defined as

$$w_0(t) = 1; w_k(t) = r_{a_1}(t) r_{a_2}(t) \cdots r_{a_m}(t),$$

where $k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}$ with integers $a_1 > a_2 > \cdots > a_m \geq 0$.

$\mathscr{K}$ will denote a strictly increasing subsequence $\{k_1, k_2, \ldots,\}$ of $\{1, 2, \ldots\}$. We say $\mathscr{K}$ is q-lacunary, if $k_{j+1}/k_j \geq q$, for all $j = 1, 2, \ldots$.

THEOREM 1. *There exist positive constants α and β so that for any measurable set $E \subset [0,1]$, $|E| \neq 0$, and any q-lacunary sequence $\mathscr{K}$, $q > 1$, there is an integer N which depends only on E and q such that for any real numbers $\{c_k\}_{k \in \mathscr{K}}$ with $\sum_{k \in \mathscr{K}} |c_k|^2 < \infty$, we have*

$$\inf_{c \in R} \left| \left\{ t \in E : \beta \left(\sum_{k \in \mathscr{K}_N} |c_k|^2 \right)^{1/2} \leq \left| c + \sum_{k \in \mathscr{K}} c_k w_k(t) \right| \right\} \right| \geq \alpha |E|, \quad (2)$$

where $\mathscr{K}_N = \{k \in \mathscr{K} : k \geq N\}$.

Proof. We show first that (2) holds for $E = [0, 1)$.

Let $\mathscr{K}$ be a lacunary sequence with $q > 1$ and let

$$f(t) = c + \sum_{k \in \mathscr{K}} c_k w_k(t),$$

where $\sum_{k \in \mathscr{K}} |c_k|^2 < \infty$. In [2] it is proved that for any $0 < p < \infty$,

$$A(p,q)\left(c^2 + \sum_{k \in \mathscr{K}} |c_k|^2\right)^{1/2} \leq \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p} \leq B(p,q)\left(c^2 + \sum_{k \in \mathscr{K}} |c_k|^2\right)^{1/2}.$$

Using the equivalence of all L^p norms we can apply a classical theorem of Paley and Zygmund (see [4], p. 216): Suppose that $g \geq 0$ is defined on a set E, $|E| > 0$, and that

$$\frac{1}{|E|} \int_E g \, dt \geq A > 0 \quad \text{and} \quad \frac{1}{|E|} \int_E g^2 \, dt \leq B^2.$$

Then for any $0 < \delta < 1$,

$$|\{t \in E : g(t) > \delta A\}| \geq (1 - \delta)^2 \left(\frac{A}{B}\right)^2 |E|.$$

Let $A = A(1,q)$ and $B = B(2,q)$. Then it follows that for any $0 < \delta < 1$ and any $c \in R$,

$$\begin{aligned}\left|\left\{t \in [0,1): \delta A\left(\sum_{k\in\mathscr{K}} |c_k|^2\right)^{1/2} \le |c + f(t)|\right\}\right| \\ \ge \left|\left\{t \in [0,1): \delta A\left(c^2 + \sum_{k\in\mathscr{K}} |c_k|^2\right)^{1/2} \le |c + f(t)|\right\}\right| \\ \ge (1-\delta)^2\left(\frac{A}{B}\right)^2.\end{aligned}$$

Therefore, (2) holds for $[0,1)$ with $\beta = A/2$, $\alpha = A^2/4B^2$, and $N = 1$.

We next show that (2) holds for $E = I$, where I is a dyadic interval. Let $|I| = 2^{-n}$. Note that if $l < n$ then $r_l(t)$ is a constant on I. Therefore, if $k < 2^n$ then $k = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_s}$ with $0 \le n_0 < n_1 < \cdots < n_s < n$ so that $w_k(t) = r_{n_0}(t) \cdots r_{n_s}(t)$ is identically 1 or -1 on I. If $k \ge 2^n$ then $k = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_s}$ with $0 \le n_0 < n_1 < \cdots < n_s$ and $n_s \ge n$. Assume $n_{j-1} < n$, $n_j \ge n$, and let $k' = 2^{n_j} + \cdots + 2^{n_s}$. Then $c_k r_{n_0}(t) \cdots r_{n_{j-i}}(t)$ is identically equal to c_k or $-c_k$ on I. Denote it by c'_k. We have $c_k w_k(t) = c'_k w_{k'}(t)$, $t \in I$, $|c'_k| = |c_k|$. Let $N > 2^n$ be a number which we choose later. A simple change of variable then gives us that for any $c \in R$,

$$\begin{aligned}\left|\left\{t \in I: \beta\left(\sum_{k\in\mathscr{K}_N} |c_k|^2\right)^{1/2} \le \left|c + \sum_{k\in\mathscr{K}} c_k w_k(t)\right|\right\}\right| \\ = 2^{-n}\left|\left\{t \in [0,1): \beta\left(\sum_{k\in\mathscr{K}_N} |c'_k|^2\right)^{1/2} \le \left|c' + \sum_{k\in\mathscr{K}_{2^n}} c'_k w_{k'}(t)\right|\right\}\right|.\end{aligned}$$

Let $q' > 0$ be such that $1 < q' < q$. Define

$$N = (2^n - 1)/(q - q').$$

We show that $\{k' \colon k \in \mathscr{K},\ k \ge N\}$ is a q'-lacunary sequence. Let k'_j and k'_{j+1} be two consecutive numbers in $\mathscr{K}'$ so that $k'_{j+1} \ge k'_j \ge N$. Note that $k_j - (2^n - 1) \le k'_j \le k_j$ for any $k_j \in \mathscr{K}$. We have

$$\begin{aligned}\frac{k'_{j+1}}{k'_j} \ge \frac{k_{j+1} - (2^n - 1)}{k_j} \ge \frac{k_{j+1}}{k_j} - \frac{2^n - 1}{k_j} \\ \ge q - \frac{2^n - 1}{N} \ge q'.\end{aligned}$$

We may assume that $N > 2^n$. For $k \in \mathscr{K}$ with $2^n \le k < N$, k' may repeat. Assume that

$$\{k' \colon k \in \mathscr{K}, k < N\} = \{l_1, l_2, \dots, l_m\},$$

where $l_1 < l_2 \cdots < l_m$. Then $\{l_1, l_2, \dots, l_m\} \cup \{k'\colon\ k \in \mathscr{K},\ k \ge N\}$ is a lacunary sequence with ratio $q'' = \min\{q', l_{j+1}/l_j\colon 1 \le j \le m\} > 1$.

For $j = 1, 2, \dots, m$, let d_{l_j} be such that

$$\sum_{k \in \mathscr{K}_{2^n}\colon k < N} c'_k w_{k'}(t) = \sum_{1 \le j \le m} d_{l_j} w_{l_j}(t).$$

From the result for $E = [0, 1)$ we have

$$\left|\left\{t \in [0,1)\colon \beta\left(\sum_{1 \le j \le m} |d_{l_j}|^2 + \sum_{k \in \mathscr{K}_N} |c'_k|^2\right)^{1/2} \le \left|c' + \sum_{k \in \mathscr{K}_{2^n}} c'_k w_{k'}(t)\right|\right\}\right| \ge \alpha.$$

It follows that

$$\left|\left\{t \in [0,1)\colon \beta\left(\sum_{k \in \mathscr{K}_N} |c'_k|^2\right)^{1/2} \le \left|c' + \sum_{k \in \mathscr{K}_{2^n}} c'_k w_{k'}(t)\right|\right\}\right| \ge \alpha$$

and that (2) holds for $E = I$.

Let E be a union of finitely many disjoint dyadic intervals I_j, $E = \bigcup_{j=1}^m I_j$. We may assume $|I_j| = 2^{-n}$ for all j. Let N be the number stated as above. Note that N depends only on the length of I. We have

$$\begin{aligned}
&\inf_{c \in R}\left|\left\{t \in E\colon \beta\left(\sum_{k \in \mathscr{K}_N} |c_k|^2\right)^{1/2} \le \left|c + \sum_{k \in \mathscr{K}} c_k w_k(t)\right|\right\}\right| \\
&\quad = \inf_{c \in R} \sum_{j=1}^m \left|\left\{t \in I_j\colon \beta\left(\sum_{k \in \mathscr{K}_N} |c_k|^2\right)^{1/2} \le \left|c + \sum_{k \in \mathscr{K}} c_k w_k(t)\right|\right\}\right| \\
&\quad \ge \sum_{j=1}^m \inf_{c \in R}\left|\left\{t \in I_j\colon \beta\left(\sum_{k \in \mathscr{K}_N} |c_k|^2\right)^{1/2} \le \left|c + \sum_{k \in \mathscr{K}} c_k w_k(t)\right|\right\}\right| \\
&\quad \ge \sum_{j=1}^m \alpha |I_j| = \alpha |E|
\end{aligned}$$

Let E be any measurable set with $|E| \ne 0$. For $j = 0, 1, \dots;\ i = 1, 2, \dots, 2^j$, let $I_{i,j}$ be the dyadic intervals $(i-1)2^{-j} \le t < i2^{-j}$.

We will show that for any $\varepsilon > 0$, there exists a set $G = \bigcup I_{i,j}$, a finite union of disjoint dyadic intervals such that $|E \cap (\bigcup I_{i,j})| > (1/2)|E|$ and $|I_{i,j} \cap E^c| < \varepsilon |I_{i,j} \cap E|$. We use Calderon-Zygmund decomposition of χ_E to construct such G. Let $\eta = 1/(1+\varepsilon)$. Given E, we begin with $j = 0$. If $|E \cap I_{1,0}| = |E| \geq \eta |I_{1,0}|$, we choose $I_{1,0}$ and stop. If $|E \cap I_{1,0}| < \eta |I_{1,0}|$, consider $E \cap I_{1,1}$ and $E \cap I_{2,1}$. If $|E \cap I_{1,1}| \geq \eta |I_{1,1}|$, choose $I_{1,1}$. Note that necessarily $|E \cap I_{2,1}| < \eta |I_{2,1}|$. We proceed to the next level, $j = 2$, disregarding all subsequent divisions of $I_{1,1}$. In this manner we obtain a set F which is the union of countably many non-overlapping dyadic intervals. F obviously contains all points of density of E, and $\eta |F| \leq |E|$. If $F = \bigcup_{j=0}^{\infty} I_j$, let $G = \bigcup_{j=0}^{m} I_j$ be such that $|G| > (1/2)|F|$. We have

$$|E \cap G| \geq |E \cap F| - |F \setminus G| \geq |E| - (1/2)|E| = (1/2)|E|.$$

Let I_j be any dyadic interval in G. From the construction of F we have $|I_j \cap E| > \eta |I_j|$. It follows that $|I_j \cap E^c| = |I_j| - |I_j \cap E| \leq \varepsilon |I_j \cap E|$.

Let $G = \bigcup_{\text{finite}} I_j$. We may assume $n = -\log_2 |I_j|$ for all $I_j \in G$. Let

$$S_1 = \left\{ t \in G : \beta \left(\sum_{k \in \mathcal{K}_N} |c_k|^2 \right)^{1/2} \leq \left| c + \sum_{k \in \mathcal{K}} c_k w_k(t) \right| \right\},$$

and

$$S_2 = G \cap E^c,$$

where N is as stated above.

It is clear that

$$\left\{ t \in E : \beta \left(\sum_{k \in \mathcal{K}_N} |c_k|^2 \right)^{1/2} \leq \left| c + \sum_{k \in \mathcal{K}} c_k w_k(t) \right| \right\} \supset S_1 \setminus S_2.$$

We have

$$|S_1| \geq \alpha |G| \geq \alpha |G \cap E|.$$

We also have

$$|G \cap E^c| = \sum |I_j \cap E^c| \leq \varepsilon \sum |I_j \cap E| = \varepsilon |G \cap E|.$$

Taking $\varepsilon = \alpha/2$ we have $|S_1| - |S_2| \geq (\alpha - \varepsilon)|G \cap E| \geq (\alpha/4)|E|$. □

COROLLARY 2. *There exists a constant $A > 0$ so that for any measurable set $E \subset [0,1]$, $|E| \neq 0$, and any q-lacunary sequence $\mathcal{K}$, $q > 1$, there is an* *integer N which depends only on E and q such that for any real numbers $\{c_k\}_{k\in\mathscr{K}}$ with $\sum_{k\in\mathscr{K}}|c_k|^2<\infty$, we have*

$$A\left(\sum_{k\in\mathscr{K}_N}|c_k|^2\right)^{1/2}\le \inf_{c\in R}\left(\frac{1}{|E|}\int_E\left|c+\sum_{k\in\mathscr{K}}c_kw_k(t)\right|^p dt\right)^{1/p}.$$

Proof. Let N be the number stated in Theorem 1. Let

$$E_1=\left|\left\{t\in E:\beta\left(\sum_{k\in\mathscr{K}_N}|c_k|^2\right)^{1/2}\le\left|c+\sum_{k\in\mathscr{K}}c_kw_k(t)\right|\right\}\right|.$$

We have

$$\begin{aligned}\frac{1}{|E|}\int_E&\left|c+\sum_{k\in\mathscr{K}}c_kw_k(t)\right|^p dt\\ &\ge\frac{|E_1|}{|E|}\cdot\frac{1}{|E_1|}\int_{E_1}\left|c+\sum_{k\in\mathscr{K}}c_kw_k(t)\right|^p dt\\ &\ge\alpha\beta^p\cdot\frac{1}{|E_1|}\int_{E_1}\left(\sum_{k\in\mathscr{K}_N}|c_k|^2\right)^{p/2}dt=\alpha\beta^p\left(\sum_{k\in\mathscr{K}_N}|c_k|^2\right)^{p/2}.\end{aligned}$$

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