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RELATION OF THE VAN EST SPECTRAL SEQUENCE TO K-THEORY AND CYCLIC HOMOLOGY

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1. Introduction

In this paper we study how the smooth cohomology of the infinite general linear group GLA for a Banach algebra A relates to cyclic cohomology, Lie algebra cohomology and Dennis trace. Our main result is as follows.

THEOREM A. *The following diagram is commutative*:

$$\begin{array}{ccc} HH_c^*A & \xrightarrow{B} & HC_c^{*-1}A \\ {\scriptstyle D_{sm}}\downarrow & & \downarrow{\scriptstyle \mathscr{A}} \\ H_{sm}^*GLA & \xrightarrow{\lambda} & H_{Lie}^*\mathfrak{gla} \end{array}$$

Here B is the boundary map in Connes' long exact sequence relating continuous cyclic cohomology to continuous Hochschild cohomology [C]. $\mathscr{A}$ denotes the dual of the alternation operation that induces an isomorphism between the primitive elements in the Lie algebra homology of $\mathfrak{gla} = MA$ and the cyclic homology of A [LQ] [T]. λ is the classical map from the smooth cohomology of a group to its Lie algebra cohomology, which can be identified with one of the edge homomorphisms in the van Est spectral sequence. The definition of D_{sm} will rest on the observation that the dual of the Dennis trace map factors through the smooth group cohomology of GLA.

We incorporate the above diagram into a bigger commutative diagram to show its relation with the van Est spectral sequence and the various other well-known cohomology groups associated with a topological group.

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Diagram B.

$$\begin{array}{ccccccc}
HC_c^*A & \xrightarrow{I} & HH_c^*A & \xrightarrow{B} & HC_c^{*-1}A & \xrightarrow{S} & HC_c^{*+1}A \\
 & & \downarrow{\scriptstyle D_{sm}} & & \downarrow{\scriptstyle \mathscr{A}} & & \\
H^*BGLA & \longrightarrow & H^*_{sm}GLA & \xrightarrow{\lambda} & H^*_{Lie}\mathfrak{gl}\mathfrak{a} & \longrightarrow & H^*GLA \\
\| & & \downarrow & & \downarrow & & \| \\
H^*BGLA & \longrightarrow & H^*(BGLA^\delta) & \longrightarrow & H^*(GLA/GLA^\delta) & \longrightarrow & H^*GLA
\end{array}$$

The top row is Connes' exact sequence. The second row is associated to the van Est spectral sequence. While the bottom row we consider as the cohomology sequence induced by the fibration

$$(GLA/GLA^\delta)^+ \to (BGLA^\delta)^+ \to BGLA.$$

(A^δ denotes the algebra A with discrete topology. The definition of the quotient space GLA/GLA^δ will be made precise as a quotient of two simplicial sets. $+$ denotes Quillen's plus construction which does not alter the cohomology of a space.) Its homotopy sequence is by definition the long exact sequence of relative, Quillen's algebraic, and periodic K-theory groups

$$\cdots \to K_*^{rel}A \to K_*^{alg}A \to K_*^{top}A \to \cdots .$$

The columns in Diagram B find their interpretation as follows. The left one is dual to the Dennis trace map $D: K_*^{alg}A \to H_*(BGLA^\delta) \to HH_*A$. In particular, the dual factors through the smooth cohomology of GLA. This is shown at the end of Section 2. The right column is the dual of Karoubi's relative Chern character

$$ch^{rel}: K_*^{rel}A \to H_*(GLA/GLA^\delta) \to HC_{*-1}^cA$$

which was shown to factor through $H_*^{Lie}\mathfrak{gl}\mathfrak{a}$ in [Ti].

Diagram B can thus be interpreted as the dual of diagram [K2, III] where $H^*_{sm}GLA$ takes the role of multiplicative K-theory MK_*A. (See also [K3, Théorème 7.11].) Hence, Theorem A may be considered as a step towards verifying a conjecture by Karoubi that $H^*_{sm}GLA$ is the dual of MK_*A [K5]. This should not be too surprising as both groups yield secondary characteristic classes as for example those of foliations [K4] [D]. While Karoubi's point of view is very much rooted in differential geometry, our treatment here relies on purely algebraic and topological methods.

In its topological outlook, our approach is closer related to some widely circulated but unpublished notes by Graeme Segal [S3], in which he defines a functor $\mathscr{K}_*A$ as the homotopy groups of the fiber $\mathscr{F}$ of the Chern character as a map in the category of spaces with two topologies. This fibration is closely related to another fibration that gives rise to the van Est spectral sequence. Indeed, the cohomology of the fiber space $\mathscr{F}$ in this category is identical to the continuous cohomology $H_c^* GLA$.

Contents. Section 2 is devoted to a categorical approach in which the smooth Dennis trace map D_{sm} finds its natural setting. This leads us to the construction of a free module over a topological ring A with basis X, a topological space. The construction relies on the Dold-Thom free abelian group functor [DT]. The resulting space is identified in 2.13 with the infinite loop space associated to the spectrum

$$\{X \wedge BA, X \wedge B^2A, X \wedge B^3A, \dots\}.$$

We also prove in 2.6 that the continuous cyclic cohomology of a topological group G is isomorphic to the continuous cohomology of G tensored with the cyclic cohomology of a point.

Section 3 proves Theorem A. Commutativity of the remaining squares in Diagram B is shown in Section 4. Section 5 contains some final remarks.

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2. Continuous cohomology and Dennis trace map

With the possible exception of Theorem 2.6 and the factorization of the Dennis trace map, this section is of purely expository nature. Its main purpose is to establish notation and give most of the definitions needed later.

Recall from [C] [G], a cyclic object in a category $\mathscr{C}$ is a simplicial object $X_\cdot = (X_n, \partial_i, s_i)$ with an action of the cyclic group $\mathbf{Z}/(n+1) = \langle t_{n+1} \rangle$ on X_n, for each n, satisfying the following relations:

$$\partial_i t_{n+1} = \begin{cases} t_n \partial_{i-1}, & 0 < i \le n, \\ \partial_n, & i = 0, \end{cases}$$

$$s_i t_{n+1} = \begin{cases} t_{n+2} s_{i-1}, & 0 < i \le n, \\ t_{n+2}^2 s_n, & i = 0. \end{cases}$$

Cyclic spaces. Let $\mathscr{T}$ denote the category of compactly generated spaces in the sense of [St]. For $Y \in \mathscr{T}$ and A a locally contractible, abelian group, let Maps(Y, A) denote the space of continuous maps from Y to A. Given a cyclic object $X_.$ in $\mathcal{T}$, Maps($X_.$, A) forms a cocyclic space. The associated cochain complex has total boundary

$$b = \sum_{i=0}^{n} (-1)^i \partial_i^*.$$

The cyclic structure defines an action T on the cochain complex, such that for $f \in$ Maps(X_n, A),

$$T(f)(x) = (-1)^n f(t_{n+1}x).$$

It is easy to check that the set of T-invariant maps Maps$(X_., A)^T$ form a subcomplex. We define,

$$H^*_{\mathcal{T}}(X_., A) := H_*(\mathrm{Maps}(X_., A), b)$$
$$HC^*_{\mathcal{T}}(X_., A) := H_*\left(\mathrm{Maps}(X_., A)^T, b\right)$$

The first cohomology theory is of course the continuous cohomology of $X_.$ with coefficients in A (see for example [BS]). The second cohomology theory is called the continuous cyclic cohomology of $X_.$ with coefficients in A. If $A = \mathbf{C}$, we will subpress the second argument.

Remark 2.1. Alternatively, following [LQ] or [G], one can define $HC^*_{\mathcal{T}}(X_., A)$ as the total homology of the following double complex C^{**}:

$$\begin{array}{ccccc}
\uparrow b & & \uparrow b' & & \\
\mathrm{Maps}(X_{n+1}, A) & \xrightarrow{1-T} & \mathrm{Maps}(X_{n+1}, A) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow b' & & \\
\mathrm{Maps}(X_n, A) & \xrightarrow{1-T} & \mathrm{Maps}(X_n, A) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow b' & &
\end{array} \tag{2.2}$$

where $b' = \sum_{i=0}^{n-1}(-1)^i\partial_i^*$ and $N = 1 + T + T^2 + \cdots T^n$. If A is a topological vector space over $\mathbf{R}$ or $\mathbf{C}$ then these two definitions coincide. (The proof in the algebraic case [LQ] carries over straight away.) Consider the shift operator $S: C^{**} \to C^{*+2,*}$. The cokernel of S are the first two columns. Since the odd columns are acyclic, its homology is the homology of the first column. This yields Connes' long exact sequence

$$\to HC^*_{\mathcal{T}}(X_.) \xrightarrow{I} H^*_{\mathcal{T}}(X_.) \xrightarrow{B} HC^{*-1}_{\mathcal{T}}(X_.) \xrightarrow{S} HC^{*+1}_{\mathcal{T}}(X_.) \to \tag{2.3}$$

Example 2.4. Let $X_\cdot = (\mathrm{pt})$ be the trivial cyclic space of a point. Then

$$H^*_{\mathcal{T}}(\mathrm{pt}, A) = H_*\left(A \xrightarrow[0]{} A \xrightarrow[1]{} A \cdots\right) = \begin{cases} A, & * = 0, \\ 0, & * > 0, \end{cases}$$

$$HC^*_{\mathcal{T}}(\mathrm{pt}, A) = H_*(A \to 0 \to A \to 0 \cdots) = \begin{cases} A, & * \text{ even}, \\ 0, & * \text{ odd}. \end{cases}$$

Example 2.5. Let G be a topological group. Then its bar construction is the cyclic space $E_\cdot G$ with $E_n G = G^{n+1}$ and

$$\partial_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots g_n)$$
$$t_{n+1}(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1})$$

The diagonal left action of G on $E_\cdot G$ is compatible with the cyclic action. The quotient space $E_\cdot G/G$ can be identified with the cyclic space $B_\cdot G$ where $B_n G = G^n$ and

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n), & i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & 0 < i < n, \\ (g_1, \ldots, g_{n-1}), & i = n, \end{cases}$$

$$t_{n+1}(g_1, \ldots, g_n) = \left((g_1 g_2 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}\right)$$

This identification is induced by the map

$$\pi : (g_0, \ldots, g_n) \to \left((g_0)^{-1} g_1, \ldots, (g_{n-1})^{-1} g_n\right).$$

THEOREM 2.6.

$$HC^*_{\mathcal{T}}(B_\cdot G) = H^*_{\mathcal{T}}(B_\cdot G) \otimes HC^*_{\mathcal{T}}(\mathrm{pt})$$

In particular, Connes' boundary operator B is zero in cohomology.

This is a generalization of Karoubi's result [K1, Théorème II]. For the proof we need to introduce the definition of continuous cohomology of G with coefficients in a (non-trivial) G-module A [HM] [S1]. The standard homogeneous resolution of A is just

$$\tilde{F}^*(G, A) := (\mathrm{Maps}(E_\cdot G, A), b)$$

Thus, in particular, both the cohomology and the cyclic cohomology of $E_\cdot G$

are that of a point. The G-action on $\tilde{F}^*(G, A)$ is given by

$$(g.f)(g_0, \ldots, g_n) = gf(g^{-1}g_0, \ldots, g^{-1}g_n).$$

π induces an isomorphism of $\tilde{F}^*(G, A)^G$ with the non-homogeneous cochains

$$F^*(G, A) := (\mathrm{Maps}(B_{\cdot}G, A), b)$$

where the action of G on A twists the boundary b. The continuous cohomology of G with coefficients in A is defined as

$$\begin{aligned} H_c^*(G, A) &:= H_*\left(\tilde{F}^*(G, A)^G, b\right) \\ &= H_*(F^*(G, A), b) \end{aligned}$$

By definition then, $H^*_{\mathscr{T}}(B_{\cdot}G) = H_c^*(G)$.

Proof. Let C^{**} denote the double complex (2.2) with $X_{\cdot} = E_{\cdot}G$ and $A = \mathbf{C}$. Via the map π, its G-invariant subspace $(C^{**})^G$ can canonically be identified with the double complex for the cyclic space $B_{\cdot}G$, the homology of which we seek to calculate. For this consider the triple complex

$$D^{rst} := F^r(G, C^{st}).$$

If $\partial_1(= b), \partial_2(= 1 - T \text{ or } N), \partial_3(= b \text{ or } b')$ denote the respective differentials in the r, s, t directions, then $d = \partial_1 + (-1)^r\partial_2 + (-1)^{r+s}\partial_3$ is the differential of the total complex Tot D^{***}, the homology of which we abbreviate by H^*D.

To prove the theorem, we compare two spectral sequences, both converging to H^*D. The first one is the (first) spectral sequence associated with the double complex

$$E^{pq} := F^p(G, (\mathrm{Tot}\, C^{**})^q).$$

Since each $C^{st} = \mathrm{Maps}(G^{s+1}, \mathbf{C})$ is G-injective,[2]

$$E_1^{pq} = H^0(G, (\mathrm{Tot}\, C^{**})^q) = \left(\mathrm{Tot}(C^{**})^G\right)^q$$

for $p = 0$, and zero otherwise. Thus the spectral sequence collapses at the

[2]In [HM] only locally compact groups are considered. However, all the relevant results in §2 are valid for the larger class of compactly generated groups such as GLA. Alternatively, C^{st} is clearly of the form Maps(G, A) where A is contractible, i.e. C^{st} is soft in the terminology of [S1] and hence G-injective.

E_2-level with

$$E_2^{0q} = E_\infty^{0q} = HC^q_{\mathcal{G}}(B_{.}G).$$

As $E_\infty^{pq} = 0$ for $p > 0$, there are no extension problems, and hence canonically

$$H^*D = HC^*_{\mathcal{G}}(B_{.}G).$$

Consider now the (second) spectral sequence associated to the double complex

$$E^{pq} := \sum_{s+t=p} F^s(G, C^{tq})$$

For a fixed even t, $C^{t*} = \tilde{F}^*(G)$ is homotopy equivalent to its cohomology $H^*(\tilde{F}^*(G)) = \mathbf{C}$. The inclusion of the constant functions $\mathbf{C} \to \tilde{F}^0(G)$ and the evaluation at the identity $\tilde{F}^*(G) \to \mathbf{C}$ give such a homotopy equivalence. Thus C^{t*} is split.[3] Similarly, for a fixed odd t, C^{t*} is split with zero homology. Therefore,

$$E_2^{pq} = \sum_{s+t=p} F^s(G, H^q(C^{t*})).$$

As $H^q(C^{t*}) = \mathbf{C}$ when $q = 0$ and t even, and zero otherwise, again the spectral sequence collapses to yield

$$E_2^{p0} = E_\infty^{p0} = H^p(\mathrm{Tot}\, F^*(G, \mathbf{C}^*)) = \left(H^*_{\mathcal{G}}(B_{.}G) \otimes \mathbf{C}^*\right)^p$$

with $\mathbf{C}^* = \mathbf{C} \to 0 \to \mathbf{C} \to 0 \to \cdots = HC^*_{\mathcal{G}}(\mathrm{pt})$. As before, H^*D can canonically be identified with E_∞^{*0}, and hence

$$HC^*_{\mathcal{G}}(B_{.}G) = H^*_{\mathcal{G}}(B_{.}G) \otimes HC^*_{\mathcal{G}}(\mathrm{pt}).$$

To prove the second assertion, consider the double complex $(C^{**})^G$ and filter it by columns. Then the associated spectral sequence gives

$$E_1^{**} = H^*_{\mathcal{G}}(B_{.}G) \otimes \mathbf{C}^*$$

and thus collapses at this level. In particular, $d_1 = B = 0$. (Compare [LQ, Theorem 1.9] and [G, II.2.4]). □

[3]See [B, Lemma 3.1]. A cochain complex C^* is split if $C^q = B^q \oplus H^q(C^*) \oplus \tilde{B}^{q+1}$ with $B^q \subset \ker(\partial_q)$ and $\tilde{B}^{q+1}$ isomorphic to B^{q+1}.

Example 2.7. As above, let G be a topological group. Its cyclic bar construction is the cyclic space $W_{\cdot}G$ with $W_nG = G^{n+1}$ and

$$\partial_i(g_0,\dots,g_n) = \begin{cases} (g_0,\dots,g_ig_{i+1},\dots,g_n), & 0 \le i < n, \\ (g_ng_0,g_1,\dots,g_{n-1}), & i = n, \end{cases}$$

$$t_{n+1}(g_0,\dots,g_n) = (g_n,g_0,\dots,g_{n-1}).$$

Define $a : B_{\cdot}G \to W_{\cdot}G$ via $(g_1,\dots,g_n) \to ((g_1\cdots g_n)^{-1}, g_1,\dots,g_n)$. It is easy to check that this is a map of cyclic spaces and hence induces maps in cohomology

$$H^*_{\mathscr{T}}(W_{\cdot}G) \to H^*_{\mathscr{T}}(B_{\cdot}G)$$

and cyclic cohomology

$$HC^*_{\mathscr{T}}(W_{\cdot}G) \to HC^*_{\mathscr{T}}(B_{\cdot}G) = H^*_cG \otimes HC^*_{\mathscr{T}}(\mathrm{pt}).$$

Remark 2.8. If $X_{\cdot}$ is actually a simplicial manifold we may replace the continuous functions by the smooth functions $\mathrm{Maps}^{\infty}(X_{\cdot}, \mathbf{C})$. Though in most cases of interest the smooth cohomology groups are the same as the continuous ones described above, for example see [HM, Theorem 5.1], this might not be true in general [Mo, Theorem 8.3].

Cyclic vector spaces. Let $\mathscr{V}$ denote the category of topological vector spaces over $\mathbf{C}$, and let V and W be elements in $\mathscr{V}$. Denote by $\mathrm{Hom}(V,W)$ the set of continuous $\mathbf{C}$-linear maps from V to W. For a cyclic object $E_{\cdot}$ in $\mathscr{V}$, we define

$$H^*_{\mathscr{V}}(E_{\cdot},W) := H_*(\mathrm{Hom}(E_{\cdot},W),b)$$
$$HC^*_{\mathscr{V}}(E_{\cdot},W) := H_*\big(\mathrm{Hom}(E_{\cdot},W)^T,b\big)$$

Equivalently, we could have defined the continuous cyclic cohomology of $E_{\cdot}$ as the total homology of a double complex similar to (2.2). We are only interested when $W = \mathbf{C}$ and subpress the second argument.

Example 2.9. For every topological algebra A over $\mathbf{C}$, define a cyclic vector space $N_{\cdot}A$ with $N_nA = A^{\otimes n+1}$ and

$$\partial_i(a_0,\dots,a_n) = \begin{cases} (a_0,\dots,a_ia_{i+1},\dots,a_n), & 0 \le i < n, \\ (a_na_0,a_1,\dots,a_{n-1}), & i = n, \end{cases}$$
$$t_{n+1}(a_0,\dots,a_n) = (a_n,a_0,\dots,a_n).$$

The associated complex $C^*(A) = (\mathrm{Hom}(N_{.}A, \mathbf{C}); b)$ is just the continuous dual of the Hochschild complex, and its invariant subcomplex $C^*(A)^T$ is by definition Connes' cyclic complex for the algebra A. Hence,

$$(2.10) \qquad \begin{aligned} H^*_{\mathscr{V}}(N_{.}A) &= HH^*_c A \\ HC^*_{\mathscr{V}}(N_{.}A) &= HC^*_c A. \end{aligned}$$

Let $M_q A$ denote the algebra of $q \times q$ matrices over A. Matrix multiplication and the usual trace define a map

$$\mathrm{tr} : (M_q A)^{\otimes n+1} \to M_q(A^{\otimes n+1}) \to A^{\otimes n+1}.$$

Multiplication here means $(m_{ij}) \otimes (\tilde{m}_{ij}) \to (\Sigma_k m_{ik} \otimes \tilde{m}_{kj})$. It is easy to check that tr defines a map of cyclic vector spaces $\mathrm{tr} : N_{.}M_q A \to N_{.}A$. As in the algebraic case [DI], this map induces Morita equivalence so that

$$(2.11) \qquad \begin{aligned} HH^*_c M_q A &= HH^*_c A \\ HC^*_c M_q A &= HC^*_c A. \end{aligned}$$

Topological vector space functor. We will now construct a natural functor from $\mathscr{T}$ to $\mathscr{V}$ that will take cyclic spaces to cyclic vector spaces. It will be the continuous version of the free abelian group functor that takes a set S to $\mathbf{Z}[S]$ inducing the Hurewicz map on simplicial sets. More precisely, it associates to a space X the set of finite linear combinations $\mathbf{C}[X]$, topologized in such a way that it becomes a topological vector space over $\mathbf{C}$ and the natural inclusion $X \to \mathbf{C}[X]$ is continuous. For the definition of the topology we are guided by the following two extreme cases:

(1) If $X = X^\delta$ is discrete then a natural topology for $\mathbf{C}[X]$ is the inductive limit topology, which, for a topologist, is the fine topology with respect to the finite dimensional linear subspaces. It is the finest locally convex topology on $\mathbf{C}[X]$. In particular, $\mathrm{Hom}(\mathbf{C}[X], \mathbf{C}) = \mathrm{Maps}(X, \mathbf{C})$.

(2) Instead of the base X, the coefficients may be discrete. Dold and Thom's [DT] construction of a model for $\mathbf{N}[X]$ is well known. Recall, for a based space $(Y, *)$, $SP(Y) = \varinjlim SP^q(Y)$ where $SP^q(Y) = Y^q/\Sigma_q$ has the quotient topology. The inclusions are given by $(y_1, \ldots, y_q) \to (y_1, \ldots, y_q, *)$. Let X_+ denote X with an adjoint base point. Then $\mathbf{N}[X] := SP(X_+)$.

The definition of the topology on $\mathbf{C}[X]$ is part of a more general setting which to explain we will take a small detour. Let A be a locally contractible, abelian group.

DEFINITION 2.12.

$$SP_A(Y) := SP(A \wedge Y)/\sim$$

where the equivalence relation is generated by $(a, y) + (a', y) \sim (a + a', y)$. $SP_A(Y)$ is equipped with the quotient topology.

An abelian group is the simplest example of a Γ-space as considered by Segal [S2]. The spectrum it generates is the Eilenberg-MacLane spectrum

$$\mathbf{B}A := \{A, BA, B^2A, \ldots\}$$

with $\pi_*\mathbf{B}A = \pi_*A$. With the notation of [S2], note that

$$SP_A(X) = \coprod_{n\geq 0} A^n \times X^n/\Gamma.$$

The equivalence relation Γ identifies $(\theta^*a, x) \in A^m \times X^m$ with $(a, \theta_* x) \in A^n \times X^n$ for all maps θ from $\{1, \ldots, m\}$ to the power set of $\{1, \ldots, n\}$ such that $\theta(i)$ is disjoint from $\theta(j)$ whenever $i \neq j$. Here

$$\theta^*(a_1, \ldots, a_n) = (b_1, \ldots, b_m) \quad \text{with } b_i = \sum_{k\in\theta(i)} a_k$$

and

$$\theta_*(x_1, \ldots, x_m) = (y_1, \ldots, y_n) \quad \text{with } y_k = x_i \text{ if } k \in \theta(i).$$

The right hand side of above formula represents the infinite loop space associated to the spectrum

$$\mathbf{B}A \wedge X = \{A \wedge X, BA \wedge X, B^2A \wedge X, \ldots\}.$$

Thus $SP_A(X) = \Omega^\infty(\mathbf{B}A \wedge X)$ and we have the Dold-Thom theorem for $SP_A(X)$:

COROLLARY 2.13. *The homotopy functor $\pi_* SP_A$ from topological spaces to graded abelian groups is the homology functor associated to the spectrum $\mathbf{B}A$. For $X \in \mathcal{T}$,*

$$\pi_n SP_A(X) = \bigoplus_k H_{n-k}(X; \pi_k A)$$

where H_* denotes the reduced homology functor.

For a topological ring R, any continuous function ϕ from X to an R-module M can be extended uniquely to an R-module map $\tilde{\phi}: R[X] := SP_R(X_+) \to M$. That is,

$$\mathrm{Maps}(X, M) = \mathrm{Hom}_R(R[X], M)$$

As the extreme cases (1) and (2) satisfy similar universal properties, it is easy to see that $SP(X) = SP_{\mathbf{N}}(X)$ and that $\mathbf{C}[X] := SP_{\mathbf{C}}(X_+^\delta)$ has the inductive limit topology. Furthermore, for any cyclic space X.

$$(2.14) \qquad \begin{aligned} H^*_{\mathcal{T}}(X_.) &= H^*_{\mathcal{V}}(\mathbf{C}[X_.]) \\ HC^*_{\mathcal{T}}(X_.) &= HC^*_{\mathcal{V}}(\mathbf{C}[X_.]). \end{aligned}$$

Given a topological group G, $\mathbf{C}[G]$ is a topological algebra, and $\mathbf{C}[W_.G]$ and $N_.\mathbf{C}[G]$ are naturally isomorphic. Hence,

$$(2.15) \qquad \begin{aligned} H^*_{\mathcal{T}}(W_.G) &= HH^*_c\mathbf{C}[G] \\ HC^*_{\mathcal{T}}(W_.G) &= HC^*_c\mathbf{C}[G]. \end{aligned}$$

Dennis trace map. We are now in the position to define the Dennis trace map and its smooth dual. Let A be a Banach algebra, M_qA its $q \times q$ matrix algebra with units GL_qA. Consider the following sequence of maps of cyclic vector spaces

$$(2.16) \quad \mathbf{C}[B_.GL_qA] \xrightarrow{a} \mathbf{C}[W_.GL_qA] = N_.\mathbf{C}[GL_qA] \to N_.M_qA.$$

The last map is induced by the natural inclusion $GL_qA \hookrightarrow M_qA$. Applying the functor $H^*_{\mathcal{V}}$ respectively $HC^*_{\mathcal{V}}$ and taking the limit over q, yields maps

$$\begin{aligned} D_c &: HH^*_cA \to H^*_cGLA \\ ch_c &: HC^*_cA \to H^*_cGLA \otimes HC^*_c\mathbf{C}. \end{aligned}$$

Here we used the various identifications (Morita invariance 2.11, 2.10, 2.14, 2.15) and Theorem 2.6. A closer look at the map

$$\mathrm{Hom}(N_.M_qA, \mathbf{C}) \to \mathrm{Hom}(N_.\mathbf{C}[GL_qA], \mathbf{C}) = \mathrm{Maps}(W_.GL_qA, \mathbf{C})$$

shows that it factors through the set of smooth function $\mathrm{Maps}^\infty(W_.GL_qA, \mathbf{C})$, as every element in its image is "linear" on GL_qA. This leads to the definition of

$$D_{sm}: HH^*_cA \to H^*_{sm}GLA := H_*(\mathrm{Maps}^\infty(B_.GLA), b).$$

We have then the following sequence of maps

$$HH_c^*A \xrightarrow{D_{sm}} H_{sm}^*(GLA) \to H_c^*(GLA) \to H^*(GLA^\delta) \tag{2.17}$$

where the latter group is the cohomology of the discrete group GLA^δ, or equivalently, the cohomology groups of the classifying space $BGLA^\delta$. By definition, sequence (2.17) is then the dual of the Dennis trace map

$$D: K_*^{alg}A = \pi_*(BGLA^\delta)^+ \xrightarrow{h} H_*(BGLA^\delta)^+ = H_*(BGLA^\delta) \to HH_*A.$$

(Here + denotes Quillen's plus construction. Note that rationally the Hurewicz map h is an isomorphism onto the primitive elements in $H_*(BGLA^\delta)^+$.) In particular, the dual factors through the smooth and continuous cohomology of GLA.

3. Proof of Theorem A

Let A be a Banach algebra, M_qA the algebra of its $q \times q$ matrices, and GL_qA the group of units thereof. We will prove the following stronger statement.

THEOREM 3.1. *There is a (up to a constant) commutative diagram of cochain complexes for all $q > 0$*

$$\begin{array}{ccc} \tilde{C}^*(M_qA) & \xrightarrow{B} & C^{*-1}(M_qA)^T \\ {\scriptstyle D_{sm}}\downarrow & & \downarrow{\scriptstyle \mathscr{A}} \\ \tilde{F}(GL_qA)^{GL_qA} & \xrightarrow{\lambda} & \Lambda^*\mathfrak{gl}_qA \end{array}$$

Proof of Theorem A. After taking homology of the above cochain complexes and letting q tend to infinity, Theorem A follows now by Morita equivalence 2.11. □

Before embarking on the proof of Theorem 3.1, recall briefly the definitions of the various chain complexes and chain maps envolved. All vector spaces and algebras are considered to be over **R** while coefficients in the various cochain complexes may be taken to be **R** or **C**.

3.2. *Lie algebra cochains.* Let $\mathfrak{g}$ be a Lie algebra and $\Lambda^*\mathfrak{g}$ denote the continuous linear functionals on its exterior power algebra. If $f \in \Lambda^{n-1}\mathfrak{g}$ and

$x_1, \dots, x_n \in \mathfrak{g}$, its differential δf is defined as

$$\delta f(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j} f\big([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n\big).$$

The homology of this complex is the Lie algebra cohomology $H^*_{Lie}\mathfrak{g}$.

Next, recall the definition of the alternation operator $\mathscr{A}$ mapping cyclic cochains $C^{n-1}(M_qA)^T$ to $\Lambda^n \mathfrak{gl}_q A$. Let $f \in C^{n-1}(M_qA)^T$ and $x_1, \dots, x_n \in \mathfrak{gl}_q A$. Then

$$\mathscr{A}(f)(x_1, \dots, x_n) = \frac{(-1)^{n-1}}{n} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma f(x_{\sigma 1}, \dots, x_{\sigma n})$$

where Σ_n is the permutation group on n letters. It is straightforward to check that $\mathscr{A}$ is a chain map.

3.3. *Reduced Hochschild cochains.* In order to simplify computations, we prefer to work with the reduced Hochschild complex $\tilde{C}^*(M_qA)$. It is the quasi-isomorphic subcomplex of $C^*(M_qA)$ consisting of all functionals that vanish on the degenerate chains $(a_0, \dots, a_n)$, $a_i = 1$ for some i, $1 \leq i \leq n$. Then Connes' boundary operator

$$B : \tilde{C}^n(M_qA) \to C^{n-1}(M_qA)^T$$

simplifies to

$$B(f)(a_1, \dots, a_n) = (-1)^n \sum_{i=1}^{n} (-1)^{(n-1)(i-1)} f(1, a_i, \dots, a_n, a_1, \dots, a_{i-1})$$

where $f \in \tilde{C}^n(M_qA)$ and $a_i \in M_qA$. (See [LQ, Proposition 1.11]. We adjusted the signs so that B is a map of cochain complexes.)

3.4. *The map λ.* For an arbitrary Lie group G, define a map λ from the smooth functions $\mathrm{Maps}^\infty(G^{n+1})$ to the n-forms $\Omega^n(G)$ such that for $f \in \mathrm{Maps}^\infty(G^{n+1})$ and $x_1, \dots, x_n \in T_gG$,

$$\lambda(f)(x_1, \dots, x_n; g) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} f(g, \exp_g t_1 x_{\sigma 1}, \dots, \exp_g t_n x_{\sigma n})\Big|_{0 \cdots 0}$$

Here, $t \to \exp_g tx$ is the integral curve through g of the vector field obtained from x by left translation. In particular, we have

$$\exp_g tx = g \exp t\big(dL_{g^{-1}} x\big)$$

where exp is the exponential map at the identity $1 \in G$ and dL_g is the differential of left translation by g. If f is G-invariant then

$$\begin{aligned} f(g, \exp_g t_1 x_{\sigma 1}, \ldots, \exp_g t_n x_{\sigma n}) &= f\big(g, g \exp t_1 dL_{g^{-1}}(x_{\sigma 1}), \ldots, g \exp t_n dL_{g^{-1}}(x_{\sigma n})\big) \\ &= f\big(1, \exp t_1 dL_{g^{-1}}(x_{\sigma 1}), \ldots, \exp t_n dL_{g^{-1}}(x_{\sigma n})\big) \end{aligned}$$

and thus, $\lambda(f)$ is G-invariant. Recall that the invariant forms can canonically be identified with the Lie algebra cochains. Hence, λ induces a map

$$\mathrm{Maps}^\infty(G^{n+1})^G \to \Omega^n_{inv} G = \Lambda^n \mathfrak{g},$$

which again we denote by λ. Direct computation shows that λ is a chain map, where $\mathrm{Maps}^\infty(G^{*+1})$ is given the boundary of the simplicial space $E_\cdot G$, that is that of $\tilde{F}(G)$.

The above computations apply also when $G = GL_q A$ since $GL_q A$ is an infinite dimensional smooth Lie group modelled on the Banach vector space $M_q A$. In particular, its exponential map is well-defined and smooth [M]. Hence, the above formula for $G = GL_q A$ gives the desired map λ.

3.5. *Dennis trace.* Let $g_0, \ldots, g_n \in GL_q A$ and let $f \in \tilde{C}^n(M_q A)$. From the definition of D_{sm} in 2.16, we deduce the explicit formula

$$\begin{aligned} D_{sm}(f)(g_0, \ldots, g_n) &= f(a \circ \pi(g_0, \ldots, g_n)) \\ &= f\big((g_n)^{-1} g_0, (g_0)^{-1} g_1, (g_1)^{-1} g_2, \ldots, (g_{n-1})^{-1} g_n\big) \end{aligned}$$

where $\pi : E_\cdot G/G \to B_\cdot G$ and $a : B_\cdot G \to W_\cdot G$ are the maps defined in examples 2.5 and 2.7.

Proof. Theorem 3.1 now follows from a straightforward diagram chase. Let $f \in \tilde{C}^n(M_q A)$ and $x_1, \ldots, x_n \in \mathfrak{gl}_q A$. Then, by definition of $\mathscr{A}$ and B,

$$\begin{aligned} &\langle \mathscr{A} \circ B(f), x_1 \wedge \cdots \wedge x_n \rangle \\ &= \langle B(f), \frac{(-1)^{n-1}}{n} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma (x_{\sigma 1}, \ldots, x_{\sigma n}) \rangle \\ &= \langle f, \frac{1}{n} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma \sum_{\tau \in \mathbf{Z}/n} \operatorname{sign} \tau (1, x_{\tau\sigma 1}, \ldots, x_{\tau\sigma n}) \rangle \\ &= \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma f(1, x_{\sigma 1}, \ldots, x_{\sigma n}). \end{aligned}$$

On the other hand, by the definition of λ and D_{sm}, we have

$$\begin{aligned}\langle \lambda \circ D_{sm}(f), x_1 \wedge \cdots \wedge x_n \rangle &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} D_{sm}(f)(1, \exp t_1 x_{\sigma 1}, \ldots, \exp t_n x_{\sigma n}) \Big|_{0 \cdots 0} \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \operatorname{sign} \sigma \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} f\big((\exp t_n x_{\sigma n})^{-1}, \exp t_1 x_{\sigma 1}, \\ &\qquad (\exp t_1 x_{\sigma 1})^{-1} \exp t_2 x_{\sigma 2}, \ldots, (\exp t_{n-1} x_{\sigma n-1})^{-1} \exp t_n x_{\sigma n}\big) \Big|_{0 \cdots 0}.\end{aligned}$$

LEMMA 3.6.

$$\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} f\big((\exp t_n x_n)^{-1}, \exp t_1 x_1, \ldots, (\exp t_{n-1} x_{n-1})^{-1} \exp t_n x_n\big) \Big|_{0 \cdots 0} = f(1, x_1, \ldots, x_n).$$

We can now finish our calculation:

$$\begin{aligned}\langle \lambda \circ D_{sm}(f), x_1 \wedge \cdots \wedge x_n \rangle &= \frac{1}{n!} \sum_{\sigma \Sigma_n} \operatorname{sign} \sigma f(1, x_{\sigma 1}, \ldots, x_{\sigma n}) \\ &= \frac{1}{n!} \langle \mathscr{A} \circ B(f), x_1 \wedge \cdots \wedge x_n \rangle. \qquad \square\end{aligned}$$

Proof of Lemma 3.6. We will need the following fact.

SUBLEMMA. *Let A be a topological algebra and $f : A^n \to \mathbf{C}$ be a multilinear continuous functional. Then its $w = (w_1, \ldots, w_n)$ directional derivative at the point $a = (a_1, \ldots, a_n)$ is given by the formula*

$$df.w|_a = \sum_{i=1}^{n} f(a_1, \ldots, a_{i-1}, w_i, a_{i+1}, \ldots, a_n).$$

Proof. By definition,

$$df.w|_a = \lim_{t \to 0} \frac{f(a + tw) - f(a)}{t}.$$

The denominator is a polynomial in t of degree n in which the constant term is zero and the coefficient of t is $\Sigma f(a_1, \ldots, a_{i-1}, w_i, a_{i+1}, \ldots, a_n)$. $\square$

Formula 3.6 can now be derived from the sublemma in conjunction with the chain rule and the fact that

$$\frac{\partial}{\partial t}(\exp tx)\Big|_0 = x.$$

We demonstrate the proof in the case $n = 2$. Recall that f is an element of the normalized Hochschild complex.

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} f\big((\exp t_2x_2)^{-1}, \exp t_1x_1, (\exp t_1x_1)^{-1}\exp t_2x_2\big)\Big|_{0,0} \\
&= \frac{\partial}{\partial t_1}\Big[df_{.}\big(-x_2, 0, (\exp t_1x_1)_{.}^{-1}x_2\big)|_{(1,\exp t_1x_1,(\exp t_1x_1)^{-1})}\Big]\Big|_0 \\
&= \frac{\partial}{\partial t_1}\Big[f\big(-x_2, \exp t_1x_1, (\exp t_1x_1)^{-1}\big) + f\big(1, \exp t_1x_1, (\exp t_1x_1)^{-1}x_2\big)\Big]\Big|_0 \\
&= df_{.}(0, x_1, -x_1)|_{(-x_2,1,1)} + df_{.}(0, x_1, -x_1x_2)|_{(1,1,x_2)} \\
&= f(1, x_1, x_2). \qquad \square
\end{aligned}$$

4. Commutativity of Diagram B

We will prove the commutativity of the remaining squares in Diagram B of the introduction. I trust that at least for finite dimensional Lie groups this is well-known. However, since we like to interpret Diagram B as the dual of Karoubi's diagram [K2, III], I hope that the reader will find it convenient to find the arguments reproduced here and generalized to the infinite dimensional setting.

Let G be a smooth, compactly generated Lie group modelled on a complete, locally convex vector space, possibly of infinite dimensions [M] [B]. Let $G_{.} = \mathrm{Maps}^{\infty}(\Delta^{\cdot}, G)$ denote the simplicial group of its smooth simplices, where Δ^n is the standard n-simplex with vertices labelled $0, 1, \ldots, n$. $G_{.}$ contains the discrete group G^{δ} as the constant simplicial group. We will not be careful to distinguish between the bisimplicial set $B_{.}G_{.}$, its diagonal, and the simplicial space $B_{.}G$ as their realizations are homotopic, which we will denote by BG. The inclusion $G^{\delta} \to G$ gives rise to a simplicial fibration

$$G_{.}/G^{\delta} \to B_{.}G^{\delta} \to B_{.}G_{.}$$

Identifying $B_{.}G^{\delta}$ with $E_{.}G^{\delta}/G^{\delta}$, the first map is given by evaluation of a simplex $\sigma : \Delta^n \to G$ at its vertices

$$\sigma \xrightarrow{\text{eval}} (\sigma(0), \ldots \sigma(n)).$$

Proposition 4.1. *The following diagram commutes*:

$$\begin{array}{ccc} H^*_{deR}B_{\bullet}G & \longrightarrow & H^*_{sm}G \\ {\scriptstyle deR}\big\downarrow & & \big\downarrow \\ H^*BG & \longrightarrow & H^*BG^{\delta} \end{array}$$

Proof. Consider the double complex of smooth forms $\Omega^*(B_{\bullet}G)$ and the double complex of singular cochains $S^*(B_{\bullet}G)$ on the simplicial space $B_{\bullet}G$. Integration of forms defines the de Rham map

$$de R : \Omega^*(G^n) \to S^*(G^n).$$

The horizontal maps are given by projection onto the first columns $\Omega^0(B_{\bullet}G)$ and $S^0(B_{\bullet}G)$. The diagram then clearly commutes. □

Proposition 4.2. *The following diagram commutes*:

$$\begin{array}{ccc} H^*_{Lie}\mathfrak{g} & \longrightarrow & H^*_{de R}G \\ {\scriptstyle de R}\big\downarrow & & {\scriptstyle de R}\big\downarrow \\ H^*G/G^{\delta} & \longrightarrow & H^*G \end{array}$$

Proof. More or less by definition H^*G/G^{δ} is the homology of $(S^*G)^{G^{\delta}}$, the G^{δ}-invariant singular cochains. Their inclusion into S^*G defines the map to H^*G on the bottom. Recall that the Lie algebra cochains $\Lambda^*\mathfrak{g}$ can canonically be identified with the left invariant forms $(\Omega^*G)^G$. Clearly, if $\omega \in \Omega^n G$ is G-invariant, then so is its image under the de Rham map, which proves the proposition. □

Theorem 4.3. *The following diagram commutes*:

$$\begin{array}{ccc} H^*_{sm}G & \xrightarrow{\lambda} & H^*_{Lie}\mathfrak{g} \\ \big\downarrow & & {\scriptstyle de R}\big\downarrow \\ H^*BG^{\delta} & \underset{\text{eval}}{\longrightarrow} & H^*G/G^{\delta} \end{array}$$

λ is defined by formula 3.4. Unlike the previous two propositions, here the diagram of the underlying chain complex does not commute directly. Indeed, if $\sigma : \Delta^n \to G$ is an n-simplex and $f \in \mathrm{Maps}^{\infty}(G^{n+1})$ is a homogeneous n-cochain, then we would need

$$\int_{\sigma} \lambda(f) = f(\sigma(0), \ldots, \sigma(n)).$$

But this cannot hold in general as integration is additive in the sense that $\int_\sigma = \int_{\sigma_1} + \int_{\sigma_2}$ when σ is the union of two n-simplices σ_1 and σ_2, while there is no such linearity property on the right hand side for generic f.

Proof. As before, let $S^*(G)$ and $\Omega^*(G)$ denote the complex of singular and de Rham cochains on G with coefficients in $\mathbf{C}$. Adopting the notation of Theorem 2.6, for a smooth G-module A, let $\tilde{F}^*(G, A) = (\mathrm{Maps}^\infty(E_\cdot G, A), b)$ and $F^*(G, A) = (\mathrm{Maps}^\infty(B_\cdot G, A), b)$. Recall that there is a canonical isomorphism between the homogeneous and inhomogeneous group cochains $(\tilde{F}^*(G, A))^G \simeq F^*(G, A)$. Now consider the following diagram:

$$
\begin{array}{ccccc}
F^*(G,\mathbf{C}) & \xrightarrow{e} & F^*(G,\Omega^*G) & \xleftarrow{\simeq} & (\Omega^*G)^G \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle de R} \\
F^*(G^\delta,\mathbf{C}) & \xrightarrow{e^\delta} & F^*(G^\delta,S^*G) & \xleftarrow{\simeq} & (S^*G)^{G^\delta}
\end{array}
\tag{4.4}
$$

where e and e^δ are the edge homomorphisms and $\mathbf{C}$ is identified with the constant functions in $\Omega^0 G$ and $S^0 G$. The vertical maps are induced by the inclusion $G^\delta \to G$ and the de Rham map. Clearly the diagram commutes. Furthermore, both horizontal arrows on the right hand side are homotopy equivalences as $\Omega^n G$ is an injective G-module[4] and $S^n G$ is an injective G^δ-module for $S^n G = \prod \mathrm{Hom}(\mathbf{C}[G^\delta], \mathbf{C})$, where the product is taken over all G^δ-orbits of $\mathrm{Maps}(\Delta^n, G)$.

This proves the proposition as soon as we show that in cohomology e and e^δ are the same as the given maps λ and eval. For this consider

$$
\begin{array}{ccccc}
F^*(G,\mathbf{C}) & \xrightarrow{\simeq} & F^*(G,\tilde{F}^*(G,\mathbf{C})) & \xleftarrow{\simeq} & F^*(G,\mathbf{C}) \\
\| & & \downarrow & & \downarrow{\scriptstyle \lambda} \\
F^*(G,\mathbf{C}) & \xrightarrow{e} & F^*(G,\Omega^*G) & \xleftarrow{\simeq} & (\Omega^*G)^G
\end{array}
$$

Note the two different maps of $F^*(G,\mathbf{C})$ into $F^*(G,\tilde{F}^*(G,\mathbf{C}))$ in the top row. While the left one is again an edge homomorphism identifying $\mathbf{C}$ with the constant functions in $\tilde{F}^0(G,\mathbf{C})$, the map from the right is given by the inclusion of the G-invariant submodule $\tilde{F}^*(G,\mathbf{C})^G$ into $F^0(G,\tilde{F}^*(G,\mathbf{C}))$. Both are natural equivalences: For the map on the left, consider the (second) spectral sequence associated to filtering the double complex with respect to columns. Since $\tilde{F}^*(G,\mathbf{C})$ is a split G-resolution of $\mathbf{C}$ (see the proof of Theorem 2.6), $E_1^{**} = F^*(G, H_* \tilde{F}^*(G,\mathbf{C})) = F^*(G,\mathbf{C})$. The right map is a homotopy equivalence as $\tilde{F}^n(G,\mathbf{C})$ is continuously G-injective for all n.[5]

---

[4] See [B, Lemma 8.3] or footnote 2 in the proof of Theorem 2.6.

[5] See footnote 2 in the proof of Theorem 2.6.

This proves that e is equal to λ in cohomology. A similar argument shows that e^{δ} is equal to eval in cohomology. □

Remark 4.5. Van Est spectral sequence [B]. If the de Rham complex $\Omega^* G$ splits, then there is a van Est spectral sequence converging to $H^*_{Lie}\mathfrak{g}$ with E_2-term

$$E_2^{pq} = H^p_{sm}(G, H^q_{de\,R}(G)).$$

For the proof one considers the double complex $F^*(G,\Omega^* G)$ of (4.4) with its two natural filtrations. Then, by definition, e, and hence λ, is the first and $H^*_{Lie}\mathfrak{g} \to H^* G$ is the second edge homomorphism in the van Est spectral sequence.

By [B, Theorem 6.7], the de Rham complex $\Omega^* GL_q A$ splits whenever A has a countable basis. In this case, we also have $H^*_{de\,R} GL_q A = H^* GL_q A$ and hence $H^*_{de\,R} B_{.} GL_q A = H^* BGL_q A$ in Proposition 4.2 and Proposition 4.1.

5. Final remarks

Looking back at Diagram B, one is led to ask whether

$$\begin{array}{ccc} HC^*_c A & \xrightarrow{I} & HH^*_c A \\ {\scriptstyle ch'}\downarrow & & \downarrow{\scriptstyle D_{sm}} \\ H^* BGLA & \longrightarrow & H^*_{sm} GLA \end{array}$$

commutes, where ch' is the dual of the Connes-Karoubi Chern character

$$ch: K_* A \to HC_* A.$$

An argument based on Karoubi's proof that the Dennis trace map is compatible with the Chern character can probably be pushed [K3, Theorem 5.20]. However, this does not seem to be very satisfactory in our topological setting as ch is not a map of chain complexes.

Recall, ch is defined as the trace of the curvature tensor taking values in the non-commutative de Rham homology $H^{de\,R}{}_* A$ which in turn is identified as a subgroup of $HC_* A$. While $H^{de\,R}{}_*$ is the homology of a chain complex with differential of degree 1, $HC_* A$ is the homology of a chain complex with differential of degree -1. Hence, ch cannot be a map of chain complexes if defined in this geometric manner.

We are thus left with the search for a natural map $HC^*_c A \to H^* BGLA$ induced by a map of spaces or chain complexes. Similarly, or alternatively, one might hope to find a natural map $HC^{*+1}_c A \to H^* GLA$.

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