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$\mathcal{M}$-SUBSPACES OF X_λ

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1. Introduction

Throughout this paper, n is a fixed positive integer, p, q, s, t nonnegative integers and α, λ are complex numbers related by $\lambda = -4n^2\alpha(1-\alpha)$.

1.1. Invariant Laplacian $\tilde{\Delta}$. B denotes the open unit ball of $\mathbf{C}^n$ with its boundary ∂B and $\operatorname{Aut}(B)$ the group of all bijective holomorphic maps of B onto itself. The invariant Laplacian $\tilde{\Delta}$ is defined by

$$(\tilde{\Delta} f)(z) = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z), \quad f \in C^2(B),$$

where δ_{jk} is the Kronecker's symbol. It is invariant under $\operatorname{Aut}(B)$ in the sense that

$$\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi, \quad \varphi \in \operatorname{Aut}(B).$$

1.2. $\mathcal{H}_s$ and $H(p,q)$. $\mathcal{H}_s$ denotes the space of all homogeneous polynomials on $\mathbf{C}^n$ of degree s that satisfy $\Delta f = 0$ where

$$\Delta = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

is the ordinary Laplacian. The term "homogeneous" refers here to real scalars: $f(tz) = t^s f(z)$, $t > 0$.

Being harmonic, each $f \in \mathcal{H}_s$ is uniquely determined by its restriction on ∂B. These restrictions are so-called spherical harmonics. We shall freely identify $\mathcal{H}_s$ with its restrictions on ∂B.

$H(p,q)$ denotes the vector space of all harmonic homogeneous polynomials on $\mathbf{C}^n$ that have total degree p in the variables $z_1, \ldots, z_n$ and total degree q in the variables $\bar{z}_1, \ldots, \bar{z}_n$. Some of the basic properties of $H(p,q)$

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which we need are:

(a) $H(p,q)$ has no proper nontrivial unitarily invariant subspace. That is $H(p,q)$ is $\mathscr{U}$-minimal [R1, 12.2.8].

(b) $\mathscr{H}_s$ is the sum of pairwise orthogonal spaces $H(p,q)$ with $p+q=s$ [R1, 12.2.2].

(c) The linear span of $\cup_{s=0}^{\infty}\mathscr{H}_s$ is dense in $C(\partial B)$ [R1, 12.1.3].

(d) $L^2(\partial B)$ is the direct sum of $H(p,q)$ with $0 \le p,\ q < \infty$ [R1, 12.2.3].

(e) For each (p,q), the projection $\pi_{p,q} : L^2(\partial B) \to H(p,q)$ is given by the kernel $K_{p,q}$ defined by

$$\pi_{p,q}f(\eta) = \int_{\partial B} K_{p,q}(\eta,\zeta)f(\zeta)\,d\sigma(\zeta), \quad f \in L^2(\partial B)\ [\text{R1, 12.2.5}].$$

Here σ denotes as usual the unique rotation-invariant probability measure on ∂B. For a fixed $\zeta \in \partial B$, $K_{p,q}(\cdot,\zeta)$ is a function in $H(p,q)$.

1.3. Differential operator L_{pq}. For a function $f(z) = y(|z|^2)h(z)$ with $y \in C^2([0,1))$ and $h \in H(p,q)$, $\tilde{\Delta} f$ has the form

$$(\tilde{\Delta} f)(z) = (L_{pq}y)(|z|^2)h(z)$$

where

$$(L_{pq}y)(t) = 4(1-t)\{t(1-t)y'' + [p+q+n-(p+q+1)t] \times y' - pqy\}\ (0<t<1);$$

see [R2, Prop. 2.4]. The differential equation $L_{pq}y = \lambda y$ has a singular point at $t=0$ and it is easy to check that it has a unique solution $y = R_{p,q,\lambda}(t)$ with $y(0)=1$. Thus

$$L_{pq}R_{p,q,\lambda}(t) = \lambda R_{p,q,\lambda}(t), \quad (0<t<1),$$
$$R_{pq\lambda}(0) = 1.$$

In particular, $R_{p,q,0}(|z|^2) = F(p,q;p+q+n;|z|^2)$ where

$$F(a,b;c;t) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{t^k}{k!}$$

is the Gauss hypergeometric series [F. p. 405]. $(a)_k = \Gamma(a+k)/\Gamma(a)$ as usual.

1.4. $\mathscr{M}$-spaces. For $\lambda \in \mathbf{C}$, X_λ denotes the space of all $f \in C^2(B)$ that satisfy $\tilde{\Delta} f = \lambda f$. These eigenspaces X_λ are infinite dimensional, they are closed in the topology of uniform convergence on compact subsets of B and they are Moebius invariant: If $f \in X_\lambda$ and $\varphi \in \operatorname{Aut}(B)$ then $f \circ \varphi \in X_\lambda$. X_λ contains $H(p, q, \lambda)$ the space of all functions of the form

$$f(z) = R_{pq\lambda}(|z|^2)h(z), \quad h \in H(p, q).$$

If Ω is a set of lattice points (p, q) with $p \geq 0$ and $q \geq 0$, $Y(\Omega, \lambda)$ denotes the closed linear span of the spaces $H(p, q, \lambda)$ with $(p, q) \in \Omega$. W. Rudin [R2] characterized all $\mathcal{M}$-subspaces (closed Moebius invariant subspaces) of X_λ as follows:

(a) If $\lambda = 4m(m + n)$ for some integer $m \geq 0$, then the $\mathcal{M}$-subspaces of X_λ are $\{0\}$, X_λ and $Y_j = Y(\Omega_j, \lambda)$ where

$$\begin{aligned}
\Omega_1 &= \{(p, q) : 0 \leq p < \infty, 0 \leq q \leq m\}, \\
\Omega_2 &= \{(p, q) : 0 \leq p \leq m, 0 \leq q < \infty\}, \\
\Omega_3 &= \Omega_1 \cap \Omega_2, \\
\Omega_4 &= \Omega_1 \cup \Omega_2.
\end{aligned}$$

(b) For all other $\lambda \in \mathbf{C}$, $\{0\}$ and X_λ are the only $\mathcal{M}$-subspaces of X_λ.

For the case $\lambda = 0$, Y_1 is the space of all holomorphic functions on B, Y_2 the space of all conjugate-holomorphic functions, Y_3 the space of all constants and Y_4 the space of all pluriharmonic functions.

1.5. Integral $P^\alpha[\mu]$. For a complex Borel measure μ on ∂B we define

$$P^\alpha[\mu](z) = \int_{\partial B} P^\alpha(z, \zeta)\, d\mu(\zeta), \quad z \in B,$$

where

$$P(z, \zeta) = (1 - |z|^2)^n / |1 - \langle z, \zeta \rangle|^{2n}$$

is the Poisson-Szegö kernel for B and

$$P^\alpha(z, \zeta) = \exp\{\alpha \log P(z, \zeta)\}$$

is the principal branch. It is known that $P^\alpha[\mu] \in X_\lambda$ [R1, 4.2.2]. We denote by $\mathcal{M}_\alpha$ the vector space of all $P^\alpha[\mu]$'s where μ is a complex Borel measure on ∂B.

1.6. Results. We first determine the solution $R_{pq\lambda}(t)$ as a hypergeometric series and get the spherical harmonic expansion of $P^\alpha(z, \zeta)$ in Section 2. The case $\alpha = 1$ was obtained in [F]. As an application, we obtain an L^2-growth condition for a function in X_λ to be in Y_4 extending the corresponding result in [R3, AR] for X_0 in Section 4. In the process, we also prove a necessary and sufficient condition for a function $g \in X_\lambda$ to be represented by $P^\alpha[G]$ for some $G \in L^2(\partial B)$ when $\alpha > \frac{1}{2}$ in Section 3. Finally, we give a description of Y_3 in terms of $\mathscr{M}_\alpha$ when $\lambda = 4m(m+n)$, $m = 0, 1, 2, \cdots$, in Section 5.

2. Spherical harmonic expansion of P^α

2.1. LEMMA. *If $f \in H(p,q)$ then*

$$\int_{\partial B} \langle z, \zeta\rangle^s \langle \zeta, z\rangle^t f(\zeta)\, d\sigma(\zeta) = \begin{cases} \dfrac{s!t!(n-1)!}{(s-p)!(n+s+q-1)!}|z|^{2(s-p)} f(z), & s+q = t+p,\ p \le s,\ q \le t, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. If $f(\zeta) = \zeta_1^p \bar{\zeta}_2^q$, then the equality follows from the multinomial expansion of $\langle z, \zeta\rangle^s$ and $\langle \zeta, z\rangle^t$ and by using the orthogonality relations of [R1, 1.4.8, 1.4.9]. Since $H(p,q)$ is generated by functions obtained by unitary changes of variables of $\zeta_1^p \bar{\zeta}_2^q$, the lemma follows from the unitary invariance of $d\sigma$.

2.2. LEMMA. *If $f \in H(p,q)$ then*

$$P^\alpha[f](z) = R(|z|^2) f(z), \tag{1}$$

where

$$R(t) = (1-t^2)^{n\alpha} \sum_{j=0}^{\infty} \frac{(n\alpha)_{j+p}(n\alpha)_{j+q}\Gamma(n)}{\Gamma(p+q+n+j)} \frac{t^j}{j!} \tag{2}$$

Proof. Apply term-by-term integration on the binomial expansion of $P^\alpha(z,\zeta)$ and use Lemma 2.1.

2.3. THEOREM. *If $f \in H(p,q)$ then*

$$P^\alpha[f](z) = A_{p,q,\alpha} R_{p,q,\lambda}(|z|^2) f(z), \quad z \in B, \tag{3}$$

where

$$A_{p,q,\alpha} = \frac{(n\alpha)_p (n\alpha)_q \Gamma(n)}{\Gamma(n+p+q)}. \tag{4}$$

Proof. By Lemma 2.2, $R(|z|^2)f(z) = P^\alpha[f](z) \in X_\lambda$. Therefore

$$\tilde{\Delta}\big(R(|z|^2)f(z)\big) = \lambda R(|z|^2)f(z).$$

As noted in 1.3, $R(t)$ satisfies the differential equation $L_{pq}R(t) = \lambda R(t)$. Therefore

$$\begin{aligned} R_{p,q,\lambda}(t) = R(t)/R(0) \quad \text{and} \quad R(0) &= (n\alpha)_p(n\alpha)_q\Gamma(n)/\Gamma(n+p+q) \\ &= A_{p,q,\alpha}. \end{aligned}$$

2.4. COROLLARY.

(a)

$$\begin{aligned} R_{p,q,\lambda}(t) &= (1-t)^{n\alpha}F(n\alpha+p, n\alpha+q; n+p+q; t) \\ &= (1-t)^{n(1-\alpha)}F(n(1-\alpha)+p, n(1-\alpha)+q; n+p+q; t) \end{aligned}$$

(b) $P^\alpha[f]/(n\alpha)_p(n\alpha)_q = P^{1-\alpha}[f]/(n-n\alpha)_p(n-n\alpha)_q$, $f \in H(p,q)$, *unless one of the denominators is zero.*
In particular, $P^\alpha[1] = P^{1-\alpha}[1]$.

Proof. (a) The first equality follows from

$$\begin{aligned} R_{p,q,\lambda}(t) &= R(t)/A_{p,q,\alpha} \\ &= (1-t)^{n\alpha}\frac{\Gamma(n+p+q)}{(n\alpha)_p(n\alpha)_q\Gamma(n)}\sum_0^\infty \frac{(n\alpha)_{j+p}(n\alpha)_{j+q}\Gamma(n)}{\Gamma(n+p+q+j)}\frac{t^j}{j!} \\ &= (1-t)^{n\alpha}\sum_{j=0}^\infty \frac{(n\alpha+p)_j(n\alpha+q)_j}{(n+p+q)_j}\frac{t^j}{j!} \\ &= (1-t)^{n\alpha}F(n\alpha+p, n\alpha+q; n+p+q; t). \end{aligned}$$

The second equality follows from the identity (9.5.3) of [L].
(b) For $f \in H(p,q)$, we have, from Theorem 2.3,

$$\begin{aligned} A_{p,q,1-\alpha}P^\alpha[f](z) &= A_{p,q,1-\alpha}A_{p,q,\alpha}R_{p,q,\lambda}(|z|^2)f(z) \\ &= A_{p,q,\alpha}P^{1-\alpha}[f](z) \end{aligned}$$

Therefore (b) follows from (4). Finally if we take $p = q = 0$ and $f \equiv 1 \in H(0,0)$, we have $P^\alpha[1](z) = P^{1-\alpha}[1](z)$.

2.5. THEOREM. *For $\alpha \in \mathbf{C}$,*

$$P^{\alpha}(z,\zeta) = \sum_{p,q=0}^{\infty} G_{p,q,\alpha}(r) K_{p,q}(\eta,\zeta), \quad z = r\eta \in B, \zeta \in \partial B, \tag{5}$$

where $G_{p,q,\alpha}(r) = A_{p,q,\alpha} R_{p,q,\lambda}(r^2) r^{p+q}$. The series on the right of (5) converges absolutely and uniformly for η, $\zeta \in \partial B$ and $0 \le r \le \rho$ for each $\rho < 1$.

Proof. For $p, q \ge n|\alpha|$, the following estimate of $F(n\alpha + p, n\alpha + q; n + p + q; r^2)$ follows easily from the formulas (9.5.3) and (9.3.4) in [L] for the hypergeometric functions:

$$\begin{aligned}
&|F(n\alpha + p, n\alpha + q; n + p + q; r^2)| \\
&\quad \le F(n|\alpha| + p, n|\alpha| + q; n + p + q; r^2) \\
&\quad \le F(n|\alpha| + p + n, n|\alpha| + q; n + p + q; r^2) \\
&\quad = (1 - r^2)^{-2n|\alpha|} F(q - n|\alpha|, n(1 - |\alpha|) + p; n + p + q; r^2) \\
&\quad \le (1 - r^2)^{-2n|\alpha|} F(q - n|\alpha|, n(1 - |\alpha|) + p; n + p + q; r^1) \\
&\quad = (1 - r^2)^{-2n|\alpha|} \frac{\Gamma(n + p + q)\Gamma(2n|\alpha|)}{\Gamma(n + n|\alpha| + p)\Gamma(n|\alpha| + q)}.
\end{aligned} \tag{6}$$

From (4), (6) and Corollary 2.4 (a), we have the following estimate for $G_{p,q,\alpha}$:

$$\begin{aligned}
|G_{p,q,\alpha}(r)| &\le \left| \frac{(n\alpha)_p (n\alpha)_q \Gamma(n)}{\Gamma(n + p + q)} R_{p,q,\lambda}(r^2) \right| \\
&\le \frac{(n|\alpha|)_p (n|\alpha|)_q \Gamma(n)}{\Gamma(n + p + q)} (1 - r^2)^{\operatorname{Re}\alpha} \\
&\qquad \times |F(n\alpha + p, n\alpha + q; n + p + q; r^2)| \\
&\le (1 - r^2)^{\operatorname{Re}\alpha - 2n|\alpha|} \frac{\Gamma(n|\alpha| + p)\Gamma(n|\alpha| + q)\Gamma(n)}{\Gamma(n|\alpha|)^2 \Gamma(n + p + q)} \\
&\qquad \times \frac{\Gamma(2n|\alpha|)}{\Gamma(n + n|\alpha| + p)\Gamma(n|\alpha| + q)} \\
&= (1 - r^2)^{\operatorname{Re}\alpha - 2n|\alpha|} \frac{\Gamma(n)\Gamma(2n|\alpha|)}{\Gamma(n|\alpha|)^2} \frac{\Gamma(n|\alpha| + p)}{\Gamma(n + p + n|\alpha|)} \\
&\le (1 - r^2)^{\operatorname{Re}\alpha - 2n|\alpha|} \frac{\Gamma(n)\Gamma(2n|\alpha|)}{\Gamma(n|\alpha|)^2}.
\end{aligned} \tag{7}$$

Now, since $K_{p,q}(\eta,\zeta)$ is uniformly dominated by $(p+q+1)^{2n}$ times a constant depending only on n, it follows from (7) that

$$\sum_{p,q>n|\alpha|} |G_{p,q,\alpha}(r)K_{p,q}(\eta,\zeta)| \leq C(n,\alpha)(1-r^2)^{\operatorname{Re}\alpha-2n|\alpha|} \sum_{k>2n|\alpha|} r^2(k+1)^{2n}$$

for some positive constant $C(n,\alpha)$ depending only on n and α. Therefore the series (5) converges absolutely and uniformly for ζ, $\eta \in \partial B$ and $r \leq \rho < 1$.

Now, fix $r < 1$. Let $f \in \mathscr{H}_s$. Then $f = \sum_{p+q=s} f_{p,q}$ where $f_{p,q} = \pi_{p,q} f \in H(p,q)$ [R1, 12.2.2]. Hence, by Theorem 2.3,

$$\begin{aligned} P^\alpha[f](z) &= \int_{\partial B} P^\alpha(r\eta,\zeta)f(\zeta)\,d\sigma(\zeta) \\ &= \sum_{p+q=s} \int_{\partial B} P^\alpha(r\eta,\zeta)f_{p,q}(\zeta)\,d\sigma(\zeta) \\ &= \sum_{p+q=s} A_{p,q,\alpha}R_{p,q,\lambda}(r^2)f_{p,q}(r\eta) \\ &= \sum_{p+q=s} G_{p,q,\alpha}(r)f_{p,q}(\eta). \end{aligned} \tag{8}$$

Since

$$f_{p,q}(\eta) = (\pi_{p,q}f)(\eta) = \int_{\partial B} K_{p,q}(\eta,\zeta)f(\zeta)\,d\sigma(\zeta),$$

(8) has the following form

$$\begin{aligned} P^\alpha[f](r\eta) &= \int_{\partial B} \sum_{p+q=s} G_{p,q,\alpha}(r)K_{p,q}(\eta,\zeta)f(\zeta)\,d\sigma(\zeta) \\ &= \int_{\partial B} \sum_{p,q=0}^{\infty} G_{p,q,\alpha}(r)K_{p,q}(\eta,\zeta)f(\zeta)\,d\sigma(\zeta) \end{aligned} \tag{9}$$

for $f \in \mathscr{H}_s$. Since the linear span of $\cup_{s=0}^{\infty}\mathscr{H}_s$ is dense in $C(\partial B)$, (9) is true for any $f \in C(\partial B)$. Therefore we have (5).

3. Integral representations of functions in X_λ

For a function f continuous on B and $0 \leq r < 1$, we let f_r denote the function defined on ∂B by

$$f_r(\zeta) = f(r\zeta) \quad (\zeta \in S)$$

and we define $\tilde{\pi}_{pq}f$ by

$$(\tilde{\pi}_{pq}f)(z) = (\pi_{pq}f_r)(\zeta) \quad (z = r\zeta).$$

For $f \in L^2(\partial B)$, we denote, as usual, $\int_{\partial B}|f(\zeta)|^2\, d\sigma(\zeta)$ by $\|f\|_2^2$. For $\alpha > \frac{1}{2}$, we have the following characterization of functions of the form $g = P^\alpha[G]$ for $G \in L^2(\partial B)$.

3.1. PROPOSITION. *Let $\alpha > \frac{1}{2}$. Then $g = P^\alpha[G]$ for some $G \in L^2(\partial B)$ if and only if $g \in X_\lambda$ and*

$$\sup_{0 \le r < 1} \int_{\partial B} |(1 - r^2)^{n(\alpha - 1)} g(r\zeta)|^2\, d\sigma(\zeta) < \infty. \tag{1}$$

Proof. Suppose $g = P^\alpha[G]$ and $G \in L^2(\partial B)$. It is known that $g \in X_\lambda$. We recall that if $\alpha > \frac{1}{2}$ then

$$\int_{\partial B} P^\alpha(r\zeta, \eta)\, d\sigma(\eta) = \int_{\partial B} \frac{(1 - r^2)^{n\alpha}}{|1 - r\eta_1|^{2n\alpha}}\, d\sigma(\eta) \approx (1 - r^2)^{n(1-\alpha)}.$$

We denote the integral on the left by $\Delta(n, \alpha, r)$ for convenience. We use Jensen's inequality to get

$$\begin{aligned}
&\int_{\partial B} |g(r\zeta)|^2\, d\sigma(\zeta) \\
&= \int_{\partial B} \Delta(n, \alpha, r)^2 \left| \frac{1}{\Delta(n, \alpha, r)} \int_{\partial B} P^\alpha(r\zeta, \eta) G(\eta)\, d\sigma(\eta) \right|^2 d\sigma(\zeta) \\
&\le \Delta(n, \alpha, r)^2 \frac{1}{\Delta(n, \alpha, r)} \int_{\partial B} |G(\eta)|^2\, d\sigma(\eta) \int_{\partial B} P^\alpha(r\zeta, \eta)\, d\sigma(\zeta) \\
&= \Delta(n, \alpha, r)^2 \|G\|_2^2 \approx (1 - r^2)^{2n(1-\alpha)} \|G\|_2^2.
\end{aligned}$$

Therefore (1) follows.

Suppose $g \in X_\lambda$ and (1) holds. It follows from [R2, Theorem 2.6] that

$$(\tilde{\pi}_{pq}g)(z) = R_{p,q,\lambda}(|z|^2) g_{pq}(z) \quad (z \in B)$$

for some $g_{pq} \in H(p, q)$. Since g is real-analytic in B, g lies in the closed linear span of $\tilde{\pi}_{pq}g$ [R2, Theorem 2.3]. Hence

$$g(z) = \lim_{N \to \infty} \sum_{p+q \le N} R_{p,q,\lambda}(|z|^2) g_{pq}(z) \tag{2}$$

in the topology of uniform convergence on compact subsets of B. In particular, (2) holds pointwise.

We will show that the following defines a function G in $L^2(\partial B)$:

$$G(\zeta) = \sum_{p,q} A_{p,q,\alpha}^{-1} g_{pq}(\zeta) \quad (\zeta \in \partial B) \tag{3}$$

From (1) and (2), we have

$$\begin{aligned} \infty > C &\geq \int_{\partial B} (1-r^2)^{2n(\alpha-1)} |g(r\zeta)|^2 \, d\sigma(\zeta) \\ &= \sum_{p,q=0}^{\infty} (1-r^2)^{2n(\alpha-1)} R_{p,q,\lambda}(r^2)^2 r^{2(p+q)} \|g_{p,q}\|_2^2. \end{aligned} \tag{4}$$

By Corollary 2.4,

$$(1-r^2)^{n(\alpha-1)} R_{p,q,\lambda}(r^2) = F(n(1-\alpha)+p, n(1-\alpha)+q; n+p+q; r^2),$$

which increases to

$$B_{p,q,\alpha} \equiv \frac{\Gamma(n+p+q)\Gamma(2n\alpha-n)}{\Gamma(n\alpha+p)\Gamma(n\alpha+q)}$$

as $r \nearrow 1$ since $\alpha > \frac{1}{2}$. Therefore if we take limit as $r \nearrow 1$ in (4) we get

$$\sum B_{p,q,\alpha} \|g_{p,q}\|_2^2 < \infty.$$

Since

$$A_{p,q,\alpha} \cdot B_{p,q,\alpha} = \frac{\Gamma(n)\Gamma(2n\alpha-n)}{\Gamma(n\alpha)^2}$$

is a constant depending only on n and α, (3) and (5) imply $G \in L^2(\partial B)$. If we let

$$G_N(\zeta) = \sum_{p+q \leq N} A_{p,q,\alpha}^{-1} g_{pq}(\zeta) \quad (\zeta \in \partial B) \tag{6}$$

and fix $z \in B$, it is easy to see, via Schwarz inequality and the fact that $G_N \to G$ in $L^2(\partial B)$, that

$$\lim_{N\to\infty} P^\alpha[G_N](z) = P^\alpha[G](z). \tag{7}$$

Therefore by (7), (6) and Theorem 2.3, we have

$$P^\alpha[G](z) = \lim_{N\to\infty} \sum_{p+q\le N} R_{p,q,\lambda}(|z|^2) g_{p,q}(z) = g(z).$$

This completes the proof.

4. The $\mathscr{M}$-subspace Y_4

If f is real-analytic in B then f has a homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} P_k(z, \bar{z})$$

where P_k is a homogeneous polynomial in $z_1, \cdots, z_n$ and $\bar{z}_1, \cdots, \bar{z}_n$ of total degree k. Let $\beta > 0$ be real. We define the radial derivative $\mathscr{D}^\beta f$ of f of order β by

$$\mathscr{D}^\beta f(z) = \sum_k (k+1)^\beta P_k(z, \bar{z}).$$

We give a sufficient condition for a function f in X_λ to be in Y_4. When $\alpha = 1$, this reduces to a result in [AR, R3], which gives a sufficient condition for an $\mathscr{M}$-harmonic function to be pluriharmonic.

4.1. THEOREM. *Let $\alpha > \frac{1}{2}$ and let $f \in X_\lambda$. If*

$$\int_{\partial B} |\mathscr{D}^{n(2\alpha-1)}(1-r^2)^{n(\alpha-1)} f(r\zeta)|^2 \, d\sigma(\zeta) = o\left(\log^2 \frac{1}{1-r}\right) \tag{1}$$

as $r \to 1$, then $f \in Y_4$. In fact, if $\alpha = 1 + m/n$, or if $\lambda = 4m(n+m)$, $m = 0, 1, 2, \cdots$, then $f = P^\alpha[F]$ for some $F = \sum_{\Omega_1 \cup \Omega_2} F_{pq} \in L^2(\partial B)$ where $F_{p,q} \in H(p,q)$ and Ω_1, Ω_2 are as in 1.4; if $\alpha \ne 1 + m/n$, or if $\lambda \ne 4m(n+m)$, $m = 0, 1, 2, \cdots$, then $f \equiv 0$.

Proof. We first note that (1) implies that f satisfies the hypothesis of Proposition 3.1. In fact, if we let $u(z) = (1-|z|^2)^{n(\alpha-1)} f(z)$ and $h(z) = \mathscr{D}^{n(2\alpha-1)} u(z)$, then

$$u(z) = \frac{1}{\Gamma(n(2\alpha-1))} \int_0^1 \left(\log \frac{1}{t}\right)^{n(2\alpha-1)-1} h(tz) \, dt.$$

Therefore $\int_{\partial B}|u(r\zeta)|^2\,d\sigma(\zeta)$ is bounded if

$$\int_{\partial B} d\sigma(\zeta)\left\{\int_0^1 (1-t)^{n(2\alpha-1)-1}|h(tr\zeta)|\,dt\right\}^2 \tag{2}$$

is bounded. (2) is, by Minkowski inequality, at most

$$\left\{\int_0^1 (1-t)^{n(2\alpha-1)-1}\,dt\left(\int_{\partial B}|h(tr\zeta)|^2\,d\sigma(\zeta)\right)^{1/2}\right\}^2,$$

which is bounded by

$$\left(\int_0^1 (1-t)^{n(2\alpha-1)-1}\log\frac{1}{1-t}\,dt\right)^2 < \infty$$

uniformly on r by (1).

Now, by Proposition 3.1, there is an $F \in L^2(\partial B)$ such that $f = P^\alpha[F]$. From 1.2.(d),

$$F(\zeta) = \sum_{p,q} F_{p,q}(\zeta)$$

in $L^2(\partial B)$, where $F_{p,q} \in H(p,q)$. Let

$$F_N = \sum_{p+q\le N} F_{p,q}$$

and let $f_N(z) = P^\alpha[F_N]$. Then

$$f_N(z) = \sum_{p+q\le N} A_{p,q,\alpha} R_{p,q,\lambda}(|z|^2) F_{pq}(z) \quad (z \in B). \tag{3}$$

On the other hand, since $F_N \to F$ in $L^2(\partial B)$, the difference

$$\begin{aligned}&\mathscr{D}^{n(2\alpha-1)}u(r\eta) - \mathscr{D}^{n(2\alpha-1)}(1-r^2)^{n(\alpha-1)} f_N(r\eta)\\ &\quad= \int_{\partial B}\mathscr{D}^{n(2\alpha-1)}\left[(1-r^2)^{n(\alpha-1)}P^\alpha(r\eta,\zeta)\right](F-F_N)(\zeta)\,d\sigma(\zeta)\end{aligned}$$

tends to 0 in $L^2(\partial B)$ once r is fixed. Hence, by the orthogonality of $\{F_{p,q}\}$ and by (3), we have

$$\begin{aligned}&\int_{\partial B}|\mathscr{D}^{n(2\alpha-1)}u(r\zeta)|^2\,d\sigma(\zeta)\\ &\quad= \sum_{p,q}|A_{p,q,\alpha}|^2\|F_{p,q}\|_2^2\left|\mathscr{D}^{n(2\alpha-1)}\left[(1-r^2)^{n(\alpha-1)}R_{p,q,\lambda}(r^2)r^{p+q}\right]\right|^2. \end{aligned}\tag{4}$$

Now, by Corollary 2.4,

$$\begin{aligned}
&\mathscr{D}^{n(2\alpha-1)}\left[\left(1-r^2\right)^{n(\alpha-1)} R_{p,q,\lambda}(r^2) r^{p+q}\right] \\
&\quad = \mathscr{D}^{n(2\alpha-1)}\left[F(n(1-\alpha)+p, n(1-\alpha)+q; n+p+q; r^2) r^{p+q}\right] \\
&\quad = \sum_k \frac{(n-n\alpha+p)_k (n-n\alpha+q)_k}{(n+p+q)_k \cdot k!} (2k+p+q+1)^{n(2\alpha-1)} r^{2k+p+q}.
\end{aligned} \tag{5}$$

We note that if neither $n - n\alpha + p$ nor $n - n\alpha + q$ is a nonpositive integer then

$$\frac{(n-n\alpha+p)_k (n-n\alpha+q)_k}{(n+p+q)_k k!} (2k+p+q+1)^{n(2\alpha-1)} \approx \frac{1}{k},$$

as $k \to \infty$; so that (5) $> C \log(1/1-r)$ for some positive constant $C = C(n, \alpha, p, q)$. The hypothesis (1) now implies by (4) and (5) that $F_{pq} = 0$ unless either $n - n\alpha + p$ or $n - n\alpha + q$ is nonpositive integer. Therefore if $\alpha \neq 1 + m/n$, $m = 0, 1, 2 \ldots$ then $f \equiv 0$ and if $\alpha = 1 + m/n$, $m = 0, 1, 2, \ldots$ then $F_{pq} = 0$ unless either $0 \leq p \leq m$ or $0 \leq q \leq m$; so $f \in Y_4$. This completes the proof.

4.2. *Remark*. The function $f(z) = R_{p,q,\lambda}(|z|^2) z_1^p \bar{z}_2^q$ belongs to X_λ but

$$\int_{\partial B} |\mathscr{D}^{n(2\alpha-1)}(1-r^2)^{n(\alpha-1)} f(r\zeta)|^2 \, d\sigma \approx \left(\log^2 \frac{1}{1-r}\right)$$

as $r \to 1$ for large p and q. Since such f is not in Y_4, we can say that the growth condition (1) is best possible.

5. $\mathscr{M}$-subspace Y_3

Finally, we have the following characterization of Y_3 for the case $\lambda = 4m(m+n)$ or $\alpha = -m/n$, $m = 0, 1, 2 \ldots$

5.1. THEOREM. *If* $\lambda = 4m(m+n)$ *or* $\alpha = -m/n$, $m = 0, 1, 2 \ldots$ *then* $Y_3 = \mathscr{M}_\alpha$.

5.2. LEMMA. $\mathscr{M}_\alpha$ *is a subspace of* X_λ *which is invariant under* Aut(B).

Proof. We have seen that $\mathscr{M}_\alpha$ is a subspace of X_λ in 1.5. For $\psi \in$ Aut(B) and

$$f(z) = \int_{\partial B} P^\alpha(z, \zeta) \, d\mu(\zeta), \quad z \in B,$$

where μ is a complex Borel measure on ∂B, we shall show that $f \circ \varphi \in \mathscr{M}_\alpha$. Let $\psi(a) = 0$ with $|a| < 1$. Then $\psi = U\varphi_a$ where U is a unitary transformation of $\mathbf{C}^n$ and

$$\varphi_a(z) = \frac{a - |a|^{-2}\langle z, a\rangle a - \sqrt{1 - |a|^2}\,(z - |a|^{-2}\langle z, a\rangle a)}{1 - \langle z, a\rangle} \qquad (a \neq 0)$$

and $\varphi_a(z) = -z (a = 0)$. By a familiar calculation as in [R1], we have, for $\eta = \varphi_a(U^{-1}\zeta)$,

$$\begin{aligned}
(f \circ \psi)(z) &= (f \circ U\varphi_a)(z) = \int_{\partial B} P^\alpha(U\varphi_a(z), \zeta)\, d\mu(\zeta) \\
&= \int_{\partial B} P^\alpha(\varphi_a(z), U^{-1}\zeta)\, d\mu(\zeta) \\
&= \int_{\partial B} \left(\frac{1 - |\varphi_a(z)|^2}{|1 - \langle \varphi_a(z), U^{-1}\zeta\rangle|^2}\right)^{n\alpha} d\mu(\zeta) \\
&= \int_{\partial B} \left(\frac{1 - |\varphi_a(z)|^2}{|1 - \langle \varphi_a(z), \varphi_a(\eta)\rangle|^2}\right)^{n\alpha} d\mu(U\varphi_a(\eta)) \\
&= \int_{\partial B} \left(\frac{1 - |z|^2}{|1 - \langle z, \eta\rangle|^2}\right)^{n\alpha} \left(\frac{|1 - \langle a, \eta\rangle|^2}{1 - |a|^2}\right)^{n\alpha} d\mu(U\varphi_a(\eta)) \\
&= \int_{\partial B} P^\alpha(z, \eta) \left(\frac{|1 - \langle a, \eta\rangle|^2}{1 - |a|^2}\right)^{n\alpha} d\mu(U\varphi_a(\eta)).
\end{aligned}$$

We used the identities in Theorem 2.2.2 of [R1]. We note for $\eta \in S$,

$$1 - |a| \le |1 - \langle a, \eta\rangle| \le 1 + |a|.$$

Therefore if $a \in B$ is fixed then

$$\left|\left(\frac{|1 - \langle a, \eta\rangle|^2}{1 - |a|^2}\right)^{n\alpha}\right|$$

is uniformly bounded on ∂B. Now we define

$$(\mu \circ \psi)(E) = \int_E \left(\frac{|1 - \langle a, \eta\rangle|^2}{1 - |a|^2}\right)^{n\alpha} d\mu(U\varphi_a(\eta)), \quad E \subset S,$$

then $\mu \circ \psi$ is a complex Borel measure on ∂B. Thus $f \circ \varphi \in \mathscr{M}_\alpha$.

Proof of Theorem 5.1. Since $\alpha = -m/n$, we have

$$\begin{aligned}\int_{\partial B} P^{\alpha}(z,\zeta)\,d\mu(\zeta) &= \int_{\partial B}\left(\frac{1-|z|^2}{|1-\langle z,\eta\rangle|^2}\right)^{-m} d\mu(\zeta)\\ &= \frac{1}{(1-|z|^2)^m}\int_{\partial B}(1-\langle z,\zeta\rangle)^m(1-\langle \zeta,z\rangle)^m\,d\mu(\zeta)\\ &= \frac{1}{(1-|z|^2)^m}\int_{\partial B}\sum_{j,k=0}^{m}\binom{m}{j}\binom{m}{k}\\ &\quad\times(-1)^{j+k}\langle z,\zeta\rangle^j\langle \zeta,z\rangle^k\,d\mu(\zeta)\\ &= \frac{1}{(1-|z|^2)^m}\sum_{|\alpha|,|\beta|=0}^{m} C(\alpha,\beta)z^{\alpha}\bar{z}^{\beta}\int_{\partial B}\bar{\zeta}^{\alpha}\zeta^{\beta}\,d\mu(\zeta)\\ &= \frac{1}{(1-|z|^2)^m}\sum_{|\alpha|,|\beta|=0}^{m} C'(\alpha,\beta)z^{\alpha}\bar{z}^{\beta}\end{aligned}$$

where $C(\alpha,\beta)$ and $C'(\alpha,\beta)$ are constants depending on the multiindices α and β. This shows that $\mathscr{M}_\alpha$ is a finite dimensional subspace of X_λ which is invariant under Aut(B). Therefore it is also closed. Hence $\mathscr{M}_\alpha = Y_3$ from 1.4.

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