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HOMOLOGICAL PROPERTIES OF STRATIFIED SPACES

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In [4], Goresky and MacPherson introduced intersection homology in order to extend Poincaré Duality to some singular spaces. They also introduced the intersection cohomology from a differential point of view by means of intersection differential forms [3]. Using the sheaf axiomatic construction of [5], it is shown in [3] that the intersection homology is dual to the intersection cohomology. Moreover, a subcomplex of intersection differential forms is exhibited in [2] for which the usual integration $\int$ of differential forms on simplices realizes the above duality (deRham Theorem). The context of these works is the category of Thom-Mather stratified spaces.

Later, MacPherson introduced a more general notion of intersection homology, by enlarging the notion of perversity [8]. The aim of this work is to extend the previous deRham Theorem to this new context; we also give a weaker presentation of intersection differential forms. The description of the "allowability condition" for intersection differential forms uses the tubular neighborhoods of the strata, it is a *germ* condition. It seems more natural to give a presentation of intersection differential forms whose "allowability" is measured more directly on the strata, as for the intersection homology.

Since the differential forms cannot be defined on the singular part of A, the version we propose here uses a *blow up* $\pi\colon \tilde{A} \to A$ of the stratified space (essentially the resolution of singularities of Verona [14]). The allowability of the differential forms is measured on the desingularization $\pi^{-1}(S)$ of the strata S of A. This gives rise to weak intersection differential forms. We show that the complex of these differential forms calculates the intersection homology of A. The proof is direct; that is, we show that the usual integration $\int$ of differential forms on simplices realizes the isomorphism. We finish the work by giving a direct proof of the fact that the Poincaré Duality for intersection cohomology $(IH^{\bar{p}}_{*}(A) \cong IH^{\bar{q}}_{n-*}(A))$ can be realized by the integration $\int$ of the usual wedge product of differential forms (see also [3] for classical perversities).

In Section 1 we recall the notion of a stratified space A and we introduce the *blow up* of A, the unfolding (in fact, the resolution of singularities of Verona without faces). Remark that in some cases the unfolding of A

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appears more naturally than the tubular neighborhood system of A: compact Lie group actions, compact singular Riemannian foliations [10], etc... We recall in the second section the results of [7] and [2] about the intersection homology. Section 3 is devoted to the study of weak intersection differential forms. In the last Section we give the principal results of this note: the deRham Theorem (see §4.1.5) and Poincaré Duality (see §4.2.7).

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In this work all manifolds are considered smooth and without boundary, "differentiable" and smooth mean "of class C^∞" and the chains and cochains complexes are taken with coefficients in $\mathbf{R}$.

1. Stratified spaces and unfoldings

The stratified spaces used in [3] and [2] are Thom-Mather stratified spaces which are stratified pseudomanifolds. These spaces have a blow up in a manifold, which we called unfolding (see [14] and [2]).

1.1. Stratified spaces. We introduce the notion of singular space involved in this work.

Remember that a Thom Mather stratified space A is the union of smooth manifolds, called *strata*, each of which possesses a tubular neighborhood; these neighborhoods intersect each other in a *conical way*.[2] The *dimension* of A, written $\dim A$, is the greatest dimension of the strata.

A *stratified space* is a Thom-Mather stratified space A such that for each stratum S there exists a stratum R, with $\dim R = \dim A$, satisfying $S \subset \overline{R}$. These strata with maximal dimension are the *regular strata*, the others are the *singular strata*. We shall write $\mathcal{S}$ to represent the family of singular strata and $\Sigma \subset A$ the union of singular strata. The stratified space A is said to be *normal* if it possesses only one regular stratum. Notice that, if the codimension of singular strata is at least two, the stratified space A is a topological pseudomanifold (as defined in [5]). A useful concept in this work is the *depth* of A:

$$d(A) = \max\{i \in \{0, \ldots, \dim A\} / \text{there exists a family of strata } S_0, \ldots, S_i \text{ with } S_0 \subset \overline{S_1}, S_1 \subset \overline{S_2}, \ldots\}.$$

[2] For the relations related to Thom-Mather spaces we refer the reader to [12] and [9].

1.2. Unfoldings. We introduce the notion of unfolding, it is the resolution of singularities of [14] in the category of manifolds without boundary. It comes from A replacing each singular point by an unfolding of its link.

1.2.1. An unfolding of a 0-dimensional stratified space is a finite covering. An *unfolding* of a n-dimensional stratified space A is a continuous map π from a manifold $\tilde{A}$ onto A such that:

For each regular stratum R, the restriction $\pi\colon \pi^{-1}(R) \to R$ is a finite trivial smooth covering;

For each singular stratum S of dimension i, for $x \in S$ and for $\tilde{x} \in \pi^{-1}(x)$ there exists a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi"'] & \mathbf{R}^i \times \tilde{L} \times]-1,1[\arrow[d, "P"] \\ U \arrow[r, "\Phi"] & \mathbf{R}^i \times cL \end{tikzcd} \tag{1}$$

where

(i) U and V are neighborhoods of x and $\tilde{x}$ respectively,
(ii) $\pi_L\colon \tilde{L} \to L$ is an unfolding of L, compact stratified space,
(iii) Φ is a strata preserving homeomorphism whose restriction to each stratum is smooth and $\tilde{\Phi}$ is a diffeomorphism,
(iv) $P(x, \tilde{y}, r) = (x, [\pi_L(\tilde{y}), |r|])$.

Here cL denotes the *cone* $L \times [0,1[/L \times \{0\}$ and $[\ ,\ ]$ a point of cL.

It is shown in [2] that any stratified space possesses an unfolding. But in some cases the unfolding is a more natural structure than the Thom-Mather structure: the orbit space of an action of a compact Lie group, and the leaf space of a singular Riemannian manifold (see [10]).

1.2.2. The neighborhood U is called a *distinguished neighborhood* of x. The point x has a base for the family of neighborhoods formed by distinguished neighborhoods. To see this, it suffices to reparametrize $]-1,1[$ and the ratio of cL. As a consequence we get that each open set $W \subset A$ has the natural unfolding $\pi\colon \pi^{-1}(W) \to W$. The stratified spaces $M \times A$, where M is a manifold, and cA, for A compact, have also natural unfoldings:

$$\pi_1\colon M \times \tilde{A} \to M \times A \text{ defined by } \pi_1(x, \tilde{a}) = (x, \pi(\tilde{a})), \tag{2}$$

$$\pi_2\colon \tilde{A} \times]-1,1[\to cA \text{ defined by } \pi_2(\tilde{a}, t) = [\pi(\tilde{a}), |t|]. \tag{3}$$

1.2.3. An *isomorphism* between two stratified spaces A and A' with unfoldings $\tilde{A}$ and $\tilde{A}'$ is given by a stratum preserving homeomorphism f:

$A \to A'$ and by a diffeomorphism $\tilde{f}\colon \tilde{A} \to \tilde{A}'$ satisfying $\pi'\tilde{f} = f\pi$. For example $(\Phi, \tilde{\Phi})$ is an isomorphism.

Under the unfolding π each singular stratum S becomes a hypersurface of $\tilde{A}$ related to S by the following proposition.

PROPOSITION 1.2.4. *Let S be a singular stratum of A. Then, the restriction of π to a connected component of $\pi^{-1}(S)$ is a smooth locally trivial fibration with fiber $\tilde{L}$.*

Proof. It suffices to consider the diagram (1) for a point $x \in S$:

$$\begin{tikzcd} \pi^{-1}(S) \cap V \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi"'] & \mathbf{R}^i \times \tilde{L} \times \{0\} \arrow[d, "P \equiv \text{projection}"] \\ U \cap S \arrow[r, "\Phi"] & \mathbf{R}^i \times \{\text{vertex}\} \end{tikzcd}$$ ■

Throughout this work, we fix a n-dimensional stratified space A and an unfolding $\pi\colon \tilde{A} \to A$. In fact, all the results of this work still hold if A is a topological pseudomanifold, with smooth strata, admitting an unfolding.

2. Intersection homology

MacPherson has presented a weaker notion of perversity and generalized the simplicial intersection homology (see [8]). As we shall see, this is also the case for the singular intersection homology of [7]. In this section we show how the singular intersection homology of A can be computed by using the complex of singular intersection chains which have a lifting; this is an important tool for the deRham Theorem.

2.1. Singular intersection homology. We recall the definition of the notion of perversity of [8] and we present the corresponding adaptation of the singular intersection homology of [7].

2.1.1. A *perversity* is a function $\bar{p}\colon \mathcal{S} \to \mathbf{Z}$ from the set of singular strata to the integers. Two perversities $\bar{p}$ and $\bar{q}$ are *dual* if $\bar{p}(S) + \bar{q}(S) = \operatorname{codim} S - 2$, for each $S \in \mathcal{S}$. For example, the *zero perversity* $\bar{0}$, defined by $\bar{0}(S) = 0$, and the *top perversity* $\bar{0}$, defined by $\bar{t}(S) = \operatorname{codim} S - 2$, are dual.

A *classical perversity* of a topological pseudomanifold is a function p from the integers greater than one to the integers with the properties that $p(2) = 0$ and $p(i+1)$ is either $p(i) + 1$ or $p(i)$ for $i \geq 2$. The classical perversity p induces a perversity $\bar{p}$ by taking $\bar{p}(S) = p(\operatorname{codim} S)$ for each singular stratum.

From now on, we fix a perversity $\bar{p}$. The following definitions are adaptations of the notions of [7] to this new context.

2.1.2. A singular simplex $\sigma: \Delta \to A$ of dimension i is *$\bar{p}$-allowable* (or *allowable*) if

(a) σ sends the interior of Δ in a regular stratum of A, and

(b) $\sigma^{-1}(S) \subset (\dim \Delta - \operatorname{codim} S + \bar{p}(S))$-skeleton of Δ, for each singular stratum S of A.

When $\bar{p}$ is a classical perversity the condition (b) implies (a). Observe that each singular simplex obtained from σ by linear subdivision is still $\bar{p}$-allowable.

A singular chain $\xi = \sum_{j=1}^{m} r_j \sigma_j$ is *$\bar{p}$-allowable* (or *allowable*) if each singular simplex σ_j is $\bar{p}$-allowable. We shall say that ξ is a *$\bar{p}$-intersection* (or *intersection*) singular chain if ξ is $\bar{p}$-allowable and the boundary $\partial\xi$, where we have neglected all simplices not satisfying a), is also $\bar{p}$-allowable. When $\bar{p}$ is a classical perversity, any simplex of the boundary $\partial\xi$ verifies a).

Define $SC_*^{\bar{p}}(A)$ to be the complex of $\bar{p}$-intersection singular chains. Proceeding as in [7], we can prove that this differential complex computes the intersection homology of [8]. That is, we get $H_*(SC^{\bar{p}}(A)) \cong IH_*^{\bar{p}}(A)$. An isomorphism between two stratified spaces A and A' induces an isomorphism between $IH_*^{\bar{p}}(A)$ and $IH_*^{\bar{p}}(A')$.

The following local calculations will be used throughout this work. They are shown in [7] for a classical perversity, but the same proofs hold for a perversity.

Proposition 2.1.3. *If M is a contractible manifold, the map $a \mapsto (t_0, a)$, where t_0 is a fixed point of M, induces an isomorphism $IH_*^{\bar{p}}(A) \cong IH_*^{\bar{p}}(M \times A)$.*

Proposition 2.1.4. *If A is compact then the map $a \mapsto [t_0, a]$, where t_0 is a fixed point of the interval* $]0, 1[$, *induces an isomorphism*

$$IH_j^{\bar{p}}(cA) \cong \begin{cases} IH_j^{\bar{p}}(A) & \text{if } j < n - \bar{p}(\text{vertex of } cA) \\ 0 & \text{if } j \geq n - \bar{p}(\text{vertex of } cA) \end{cases}$$

By working with this new definition of perversity we loose some properties of [4], namely the stratification invariance of $IH_*^{\bar{p}}(A)$. However, the following property remains.

Proposition 2.1.5. *If A is manifold then $IH_*^{\bar{p}}(A) \cong H_*(A)$, for any stratification on A, provided that $\bar{0} \leq \bar{p} \leq \bar{t}$.*

Proof. Locally, the manifold A looks like $\mathbf{R}^i \times cL$ (see §1.2.1), where L is a homological sphere. The previous calculation shows that

$$IH_j^{\bar{p}}(\mathbf{R}^i \times cL) \cong \begin{cases} IH_j^{\bar{p}}(L) & \text{if } j \leq \operatorname{codim} S - 1 - \bar{p}(S) \\ 0 & \text{if } j \geq \operatorname{codim} S - \bar{p}(S) \end{cases}.$$

An argument by recurrence on the depth of A shows that $IH_*^{\bar{p}}(L) \cong H_*(L)$. Since $\bar{0} \leq \bar{p} \leq \bar{t}$, we get

$$IH_*^{\bar{p}}(\mathbf{R}^i \times cL) \cong H_*(\mathbf{R}^i \times cL).$$

Now, the passage from the local to the global can be done as in [7]. ∎

2.2. Liftable singular intersection chains. The notion of lifting of a singular chain arises from the notion of unfolding of a stratified space. This concept is useful because it allows us to integrate the intersection differential forms over the liftable singular intersection chains. We introduce in this section the notion of the lifting of the singular chains.

2.2.1. Let Δ be the standard simplex. An *unfolding* of Δ is given by a decomposition $\Delta = \Delta_0 * \cdots * \Delta_p$ and by the map μ from $\tilde{\Delta} = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p$ onto Δ defined by:

$$\begin{aligned} \mu([x_0, t_0], \ldots, [x_{p-1}, t_{p-1}], x_p) &= t_0 x_0 + (1 - t_0) t_1 x_1 + \cdots + (1 - t_0) \cdots (1 - t_{p-2}) t_{p-1} x_{p-1} \\ &\quad + (1 - t_0) \cdots (1 - t_{p-1}) x_p. \end{aligned}$$

Here $\bar{c}\Delta_i$ denotes the closed cone $\Delta_i \times [0, 1]/\Delta_i \times \{0\}$, and $[x_i, t_i]$ a point of it. The map μ is well defined and maps diffeomorphically the interior of $\tilde{\Delta}$ to the interior of Δ.

2.2.2. The boundary of $\tilde{\Delta}$ has the following decomposition $\partial\tilde{\Delta} = \widetilde{\partial\Delta} + \delta\tilde{\Delta}$ (see [2]), where $\widetilde{\partial\Delta}$ is the unfolding of the boundary $\partial\Delta$ with the induced decomposition, and $\delta\Delta$ is formed by the faces $\mu(F)$ of Δ with

$$F = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{i-1} \times (\Delta_i \times \{1\}) \times \bar{c}\Delta_{i+1} \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p.$$

Observe that the map μ, when restricted to the interior of $\tilde{\Delta}$, is a submersion.

2.2.3. Let $\sigma \colon \Delta \to A$ be a singular simplex. We shall say that σ is a *liftable singular simplex* if

(a) for each face C of Δ there exists a stratum S of A containing the image by σ of the interior of C, and

(b) there exists an unfolding $\mu\colon \tilde{\Delta} \to \Delta$ and a differentiable map $\tilde{\sigma}\colon \tilde{\Delta} \to \tilde{A}$ such that $\pi\tilde{\sigma} = \sigma\mu$.

The map $\tilde{\sigma}$ is a *lifting* of σ. It is shown in [2] that any singular simplex obtained from σ by linear subdivision of Δ has a lifting.

A singular chain $\xi = \sum_{j=1}^{m} r_j\sigma_j$ is *liftable* if each singular simplex σ_j is liftable. We define $RC_*^{\bar{p}}(A)$ to be

$$\{\xi \in SC_*^{\bar{p}}(A)/\xi \text{ is liftable}\}.$$

Notice that this complex is differential. An isomorphism between two stratified spaces A and A' induces an isomorphism between $H_*(RC^{\bar{p}}(A))$ and $H_*(RC^{\bar{p}}(A'))$.

The two following results are proved in [2] for a classical perversity, but the proofs still hold for any perversity.

PROPOSITION 2.2.4. *Let I be an open interval of* $\mathbf{R}$. *Then the map $a \mapsto (t_0, a)$, where t_0 is a fixed point of I, induces an isomorphism $H_*(RC^{\bar{p}}(A)) \cong H_*(RC^{\bar{p}}(I \times A))$.*

PROPOSITION 2.2.5. *If A is compact then the map $a \mapsto [t_0, a]$, where t_0 is a fixed point of the interval* $]0, 1[$, *induces an isomorphism*

$$H_j(RC_*^{\bar{p}}(cA)) \cong \begin{cases} H_j(RC_*^{\bar{p}}(A)) & \text{if } j < n - \bar{p}(\text{vertex of } cA) \\ 0 & \text{if } j \geq n - \bar{p}(\text{vertex of } cA). \end{cases}$$

2.3. Relation between $IH_*^{\bar{p}}(A)$ and $H_*(RC^{\bar{p}}(A))$. This section is devoted to show that the inclusion $RC_*^{\bar{p}}(A) \hookrightarrow SC_*^{\bar{p}}(A)$ induces an isomorphism in homology (*quasi-isomorphism*). First, we introduce the Mayer-Vietoris argument, and we show how to localize the problem. Then we will use the local calculations made in the above section.

2.3.1. Let $\mathcal{U} = \{U_\alpha/\alpha \in J\}$ be an open cover of A. The complexes $S^{\mathcal{U}}C_*^{\bar{p}}(A)$ and $R^{\mathcal{U}}C_*^{\bar{p}}(A)$ of *$\mathcal{U}$-small chains* are defined as subcomplexes of $SC_*^{\bar{p}}(A)$ and $RC_*^{\bar{p}}(A)$ respectively, these are generated by the chains lying on some open of the cover $\mathcal{U}$. The exact sequences

$$(4)\quad 0 \leftarrow S^{\mathcal{U}}C_*^{\bar{p}}(A) \leftarrow \bigoplus_{\alpha_0} SC_*^{\bar{p}}(U_{\alpha_0}) \leftarrow \bigoplus_{\alpha_0<\alpha_1} SC_*^{\bar{p}}(U_{\alpha_0} \cap U_{\alpha_1}) \leftarrow \cdots$$

$$(5)\quad 0 \leftarrow R^{\mathcal{U}}C_*^{\bar{p}}(A) \leftarrow \bigoplus_{\alpha_0} RC_*^{\bar{p}}(U_{\alpha_0}) \leftarrow \bigoplus_{\alpha_0<\alpha_1} RC_*^{\bar{p}}(U_{\alpha_0} \cap U_{\alpha_1}) \leftarrow \cdots$$

are the *Mayer-Vietoris sequences* (see [1, page 186]).

The next step is to show that the subcomplex of $\mathcal{U}$-small chains is homologous to the original one (see Prop. 2.3.5). In order to do this we need some preliminary results.

A singular simplex $\sigma\colon \Delta \to A$ is *$\bar{p}$-good* (or *good*) if

(a) for each singular stratum S the minimal face C_S of Δ containing $\sigma^{-1}(S)$ satisfies

$$\dim C_S \leq \dim \Delta - \operatorname{codim} S + \bar{p}(S),$$

and

(b) the family $\{C_S/\dim C_S = \dim \Delta - \operatorname{codim} S + \bar{p}(S)\}$ is totally ordered by inclusion.

LEMMA 2.3.2. *Any maximal element of the barycentric subdivision of an allowable singular simplex is good.*

Proof. Let $\varphi\colon \nabla \to A$ be an allowable singular simplex and $\sigma\colon \Delta \to A$ an element of its barycentric subdivision with $\dim \nabla = \dim \Delta$. Recall that the trace on Δ of the j-skeleton of ∇ is a face of Δ with dimension lower or equal than j. So, for each singular stratum S, the minimal face C_S of Δ containing $\sigma^{-1}(S)$ satisfies (a).

If $\dim C_S = \dim \Delta - \operatorname{codim} S + \bar{p}(S)$ then the trace on Δ of the $(\dim \Delta - \operatorname{codim} S + \bar{p}(S))$-skeleton of ∇ is exactly C_S. The result follows now from the fact that the family $\{C \text{ face of } \Delta/C \subset (\dim C)\text{-skeleton of } \nabla\}$ is totally ordered by inclusion.

LEMMA 2.3.3. *Let $\sigma\colon \Delta \to A$ be a good allowable singular simplex. Suppose that $T(\sigma) = \min\{C_S/\dim C_S = \dim \Delta - \operatorname{codim} S + \bar{p}(S)\}$ exists. For each codimension one face $s\colon C \to \Delta$ of σ satisfying* §2.1.2(*a*), *we get*

$$s \text{ is not allowable if and only if } C \supset T(\sigma).$$

In this case, if $\sigma'\colon \Delta \to A$ is another good allowable singular simplex having s as a face, we have the relation $T(\sigma) = T(\sigma')$.

Proof. If s is not allowable then there exists a singular stratum S with $s^{-1}(S) \not\subset (\dim \Delta - 1 - \operatorname{codim} S + \bar{p}(S))$-skeleton of C. Since $s^{-1}(S) \subset C_S \cap C$ and $\dim C_S = \dim \Delta - \operatorname{codim} S + \bar{p}(S)$ we conclude that $C_S \subset C$ and therefore $T(\sigma) \subset C$.

On the other hand, if $C_S \subset C$ for some singular stratum then we have $\sigma^{-1}(S) = s^{-1}(S)$ and

$$\dim C_S = \dim \Delta - \operatorname{codim} S + \bar{p}(S).$$

Hence $s^{-1}(S) \not\subset (\dim \Delta - 1 - \operatorname{codim} S + \bar{p}(S))$-skeleton of C.

Finally, we prove $T(\sigma) = T(\sigma')$. Write S the singular stratum verifying $C_S = T(\sigma)$. The relation $\sigma^{-1}(S) = s^{-1}(S) \subset (\sigma')^{-1}(S)$ implies $T(\sigma) \subset C'_S$. Since $\dim C'_S \leq \dim \Delta - \operatorname{codim} S + \bar{p}(S) = \dim T(\sigma)$ we obtain $C'_S = T(\sigma)$. By the definition of $T(\sigma')$, we can write $T(\sigma') \subset C'_S$, that is, $T(\sigma') \subset T(\sigma)$. Similarly, we prove $T(\sigma) \subset T(\sigma')$ and therefore $T(\sigma) = T(\sigma')$. ∎

We have already noticed that the chain subdivision is an interior operator in $SC_*^{\bar{p}}(A)$ and $R^{\mathcal{U}}C_*^{\bar{p}}(A)$. We shall let $\mathscr{S}$: $SC_*^{\bar{p}}(A) \to SC_*^{\bar{p}}(A)$ and $\mathscr{S}$: $RC_*^{\bar{p}}(A) \to RC_*^{\bar{p}}(A)$ the barycentric subdivision (see [15, page 206]).

Lemma 2.3.4. *For each $\xi \in SC_*^{\bar{p}}(A)$ (resp. $RC_*^{\bar{p}}(A)$) there exists $l \geq 1$ such that $S^l(\xi) \in S^{\mathcal{U}}C_*^{\bar{p}}(A)$ (resp. $R^{\mathcal{U}}C_*^{\bar{p}}(A)$).*

Proof. The method used in [15, page 207] ensures the existence of $l \geq 1$ such that $\mathscr{S}^l(\xi) = \sum_{j=1}^m r_j \sigma_j$ is an element of $S_*^{\mathcal{U}}(A)$ satisfying: $\sigma_j(T(\sigma_j)) \subset U_\alpha \Rightarrow \operatorname{Im} \sigma_j \subset U_\alpha$, when $T(\sigma_j)$ exists. We need to prove that $S^l(\xi)$ belongs to $S^{\mathcal{U}}C_*^{\bar{p}}(A)$ (resp. $RC_*^{\bar{p}}(A)$).

Notice that if $T(\sigma_j)$ does not exist, the singular simplex σ_j lies in $S^{\mathcal{U}}C_*^{\bar{p}}(A)$ (resp. $RC_*^{\bar{p}}(A)$). Thus, we can assume the existence of $T(\sigma_j)$, for $j = 1, \dots, n$. The lemma will be proved if we show that, for a fixed $\alpha \in J$, the chain $\xi_\alpha = \sum_{\sigma_j(T(\sigma_j)) \subset U_\alpha} r_j \sigma_j$ is an element of $SC_*^{\bar{p}}(U_\alpha)$ (resp. $RC_*^{\bar{p}}(U_\alpha)$). In fact we only need to show that the elements of $\partial \xi_\alpha$ satisfy §2.1.2(b). Let σ_j be an element of ξ_α. There exists a family $\{\sigma_{j_0}, \dots, \sigma_{j_p}\}$ of good allowable singular simplices of ξ which cancel the codimension one faces of σ_j not satisfying §2.1.2(b). From the previous lemma, we know that the simplices $\{\sigma_{j_0}, \dots, \sigma_{j_p}\}$ are in ξ_α. Thus the chain ξ_α is an intersection chain. ∎

The relationship between the $\mathcal{U}$-small chains and the original chains is given by:

Proposition 2.3.5. *The inclusions $S^{\mathcal{U}}C_*^{\bar{p}}(A) \hookrightarrow SC_*^{\bar{p}}(A)$ and $R^{\mathcal{U}}C_*^{\bar{p}}(A) \hookrightarrow RC_*{}^{\bar{p}}(A)$ are quasi-isomorphisms.*

Proof. For a proof of this fact we refer the reader to [15, appendix I, page 207]. The idea behind is quite intuitive: to get an inversion chain map, subdivide each chain in A until it becomes $\mathcal{U}$-small, and this is possible by the lemma above. Now, we only need to show that the homotopy operator is an interior operator in the complexes $SC_*^{\bar{p}}(A)$ and $RC_*^{\bar{p}}(A)$.

Let $\sigma\colon \Delta \to A$ be a liftable $\bar{p}$-allowable singular simplex (the same proof holds for a $\bar{p}$-allowable singular simplex). Consider the *cone singular simplex* $c\sigma\colon \bar{c}\Delta \to A$ defined by $c\sigma([x,t]) = \sigma(tx + (1-t)B)$, where B is the barycenter of Δ. We must proof that $c\sigma$ is also a liftable $\bar{p}$-allowable singular simplex. This arises from the following remarks:

For each face C of Δ we have

$$c\sigma(\text{interior of } C \times]0,1[) = \sigma(\text{interior of } C) \subset A - \Sigma,$$

and

$$c\sigma(\text{interior of } C \times \{1\}) = \sigma(\text{interior of } C) \subset S,$$

for some stratum S.

$c\sigma(\text{vertex of } \bar{c}\Delta) = \sigma(B) \subset A - \Sigma$.

For any singular stratum S,

1. $(c\sigma)^{-1}(S) = \sigma^{-1}(S) \times \{1\} \subset (\dim \Delta - \operatorname{codim} S + \bar{p}(S))$-skeleton of $\Delta \times \{1\}$, if $\bar{p}(S) < \operatorname{codim} S$,
2. $(c\sigma)^{-1}(S) \subset \bar{c}\Delta \subset (\dim \bar{c}\Delta - \operatorname{codim} S + \bar{p}(S))$-skeleton of $\bar{c}\Delta$, if $\bar{p}(S) \geq \operatorname{codim} S$.

Consider $\tilde{\sigma}\colon \tilde{\Delta} = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p \to \tilde{A}$ a lifting of σ. In $\bar{c}\Delta$ we have the decomposition

$$\Delta_0 * \cdots * \Delta_{p-1} * (\{S\} * \Delta_p),$$

where S is the vertex of the cone $\bar{c}\Delta$. The unfolding $\mu'\colon \widetilde{\bar{c}\Delta} \to \bar{c}\Delta$ is defined by

$$\begin{aligned}\mu'\big(x = [x_0, t_0], \ldots, [x_{p-1}, t_{p-1}], tx_p + (1-t)S\big)\\ = t_0 x_0 + (1-t_0)t_1 x_1 + \cdots + (1-t_0)\cdots(1-t_{p-2})t_{p-1}x_{p-1}\\ + (1-t_0)\cdots(1-t_{p-1})\big(tx_p + (1-t)S\big).\end{aligned}$$

We define the lifting $\widetilde{c\sigma}\colon \widetilde{\bar{c}\Delta} \to \tilde{A}$ by $\widetilde{c\sigma}(x) = \tilde{\sigma}(P_0, \ldots, P_p)$ with

$$\begin{aligned}-P_i = \{1 - (1-t_i)\cdots(1-t_{p-1})(1-t)(1-(\alpha_i + \cdots + \alpha_p))\}^{-1}\\ \times\big(t_i x_i + (1-t_i)\cdots(1-t_{p-1})(1-t)\alpha_i B_i\\ + \{(1-t_i) - (1-t_i)\cdots(1-t_{p-1})(1-t)(\alpha_0 + \cdots + \alpha_i)\}S_i\big)\\ \text{for } i \in \{0, \ldots, p-1\}\end{aligned}$$

and

$$-P_p = \{1 - (1-t)(1-\alpha_p)\}^{-1}(tx_p + (1-t)\alpha_p B_p).$$

Here $B = \sum_{i=0}^{p} \alpha_i B_i$, where B_i is the barycenter Δ_i, and S_i is the vertex of $\bar{c}\Delta_i$. This map is well defined because

(a) P_i depends on $\{t_i x_i, t_i, \dots, t_{p-1}, t\}$, and
(b) $1 \neq (1-t)(1-\alpha_p)$ and $1 \neq (1-t_i)\cdots(1-t_{p-1})(1-t)(1-(\alpha_i + \cdots + \alpha_p))$ for $i \in \{0, \dots, p-1\}$;

and it is a differentiable map. Since $\pi\tilde{\sigma} = \sigma\mu$, a straightforward computation shows $\pi\widetilde{c\sigma} = c\sigma\mu'$. Therefore, the simplex $c\sigma$ has a lifting.

To get the main result of this section we also need the following lemma:

LEMMA 2.3.6. *Suppose A is compact. Then the first statement implies the second one*:

(a) *the inclusion $RC_*^{\bar{p}}(W) \hookrightarrow SC_*^{\bar{p}}(W)$ is a quasi-isomorphism for each open $W \subset A$,*
(b) *the inclusion $RC_*^{\bar{p}}(V) \hookrightarrow SC_*^{\bar{p}}(V)$ is a quasi-isomorphism for each open $V \subset \mathbf{R}^m \times cA$.*

Proof. We proceed in four steps.

(1) $V = \mathbf{R}^m \times cA$. We apply §2.1.2, §2.1.3, §2.2.4, §2.2.5 and the hypothesis (a) for $W = A$.

(2) $V =]a_1, b_1[, \dots,]a_m, b_m[\times c_\varepsilon A$, where $a_i, b_i \in \mathbf{R}$, $\varepsilon \in]0,1[$ and $c_\varepsilon A = A \times [0, \varepsilon[/A \times \{0\}$. Since V is isomorphic to $\mathbf{R}^m \times cA$ it suffices to apply (1).

(3) $V =]a_1, b_1[a_m, b_m[\times]\varepsilon, \varepsilon'[\times W$, where $a_i, b_i \in \mathbf{R}$, $\varepsilon, \varepsilon' \in]0,1[$ and $W \subset A$. In this case it follows from §2.1.2, §2.2.4 and the hypothesis (a).

(4) General case. Let $\mathscr{U} = \{U_\alpha/\alpha \in J\}$ an open cover of V with each U_α satisfying (1). Observe that the intersections $U_{\alpha_0} \cap U_{\alpha_1}$ satisfy (2) or (3). Then, from (4) and (5) we get the following commutative diagram:

$$\begin{tikzcd}
0 & R^{\mathscr{U}}C_*^{\bar{p}}(A) \arrow[l] & \bigoplus_{\alpha_0} RC_*^{\bar{p}}(U_{\alpha_0}) \arrow[l] & \bigoplus_{\alpha_0<\alpha_1} RC_*^{\bar{p}}(U_{\alpha_0} \cap U_{\alpha_1}) \arrow[l] & \cdots \arrow[l] \\
0 & S^{\mathscr{U}}C_*^{\bar{p}}(A) \arrow[l] \arrow[u, "\iota_1"] & \bigoplus_{\alpha_0} SC_*^{\bar{p}}(U_{\alpha_0}) \arrow[l] \arrow[u, "\iota_2"'] & \bigoplus_{\alpha_0<\alpha_1} SC_*^{\bar{p}}(U_{\alpha_0} \cap U_{\alpha_1}) \arrow[l] \arrow[u, "\iota_3"'] & \cdots, \arrow[l]
\end{tikzcd}$$

where $\iota_.$ is the inclusion. According to (2) and (3) the maps ι_2 and ι_3 are quasi-isomorphisms. By the Five Lemma the inclusion ι_1 is also a quasi-isomorphism. The proof finishes after applying the Lemma 2.3.4. ∎

We arrive at the principal result of this section.

PROPOSITION 2.3.7. *The inclusion $RC_*^{\bar{p}}(A) \hookrightarrow SC_*^{\bar{p}}(A)$ is a quasi-isomorphism.*

Proof. We proceed by induction over the depth of A. If $d(A) = 1$ then the unfolding of A is a trivial covering over each connected component of A. So, the complex $SC_*^{\bar{p}}(A)$ (resp. $RC_*^{\bar{p}}(A)$) is the complex of singular chains of A (resp. differentiable singular chains), and the result holds (see [6]).

Suppose the proposition were proved for every B with $d(B) < d(A)$. Consider an open cover $\mathcal{U} = \{U_\alpha/\alpha \in J\}$ of A by distinguished neighborhoods (see §1.2.2). Following (4) and (5) we have the commutative diagram of the above lemma. Here, each open U_α is isomorphic to some $\mathbf{R}^m \times cL$. Since for each open $W \subset L$ we have: $d(W) \leq d(L) < d(cL) \leq d(U_\alpha) \leq d(A)$ we can apply §2.3.5 and get that ι_2 is a quasi-isomorphism. The same argument shows that the operator ι_3 is a quasi-isomorphism. The proof follows from the Five Lemma and Lemma 2.3.4. ∎

3. Intersection cohomology

Goresky and MacPherson introduced the intersection cohomology from the point of view of differential forms (deRham intersection cohomology) for a Thom-Mather stratified space (see [3]). In [2] we showed how to calculate this cohomology with the subcomplex of liftable forms. The allowability of an intersection differential form ω is reflected on the behavior of the germ of ω near Σ. We introduce the notion of weak intersection differential form, whose allowability is measured directly on the singular part by means of the unfolding.

3.1. Weak intersection differential forms. From now on $\bar{q}$ will denote the dual perversity of $\bar{p}$ (see [4]), that is, $\bar{q}(S) = \operatorname{codim} S - 2 - \bar{p}(S)$ for each singular stratum S.

3.1.1. A differential form ω in $A - \Sigma$ is *liftable* if there exists a differential form $\tilde{\omega}$ on $\tilde{A}$, called the *lifting* of ω, coinciding with $\pi^*\omega$ on $\pi^{-1}(A - \Sigma)$. By density this form is unique.

If the forms ω and η are liftable then the forms $\omega + \eta$, $\omega \wedge \eta$ and $d\omega$ are also liftable, and we have the following relations:

$$\widetilde{\omega + \eta} = \tilde{\omega} + \tilde{\eta}, \widetilde{\omega \wedge \eta} = \tilde{\omega} \wedge \tilde{\eta} \quad \text{and} \quad \widetilde{d\omega} = d\tilde{\omega}.$$

Hence, the family of liftable differential forms is a differential subcomplex of the deRham complex of $\tilde{A}$.

3.1.2. Cartan's filtration. Let $\tau: M \to B$ be a differential submersion with M and B manifolds. For each $k \geq 0$ we denote $F_k\Omega^*_M$ the subcomplex of differential forms on M satisfying:

(6) If $\xi_0, \ldots, \xi_k$ are vector fields on M, tangents to the fibers of τ, then $i_{\xi_0} \cdots i_{\xi_k}\omega \equiv i_{\xi_0} \cdots i_{\xi_k} d\omega \equiv 0$. Here i_ξ denotes the interior product by ξ.. This is *Cartan's filtration* of τ (see [3]). We shall write $\|\omega\|_B$ the smallest integer j satisfying $i_{\xi_0} \cdots i_{\xi_j}\omega \equiv 0$, where $\xi_0, \ldots, \xi_j$ are as in (6). Then,

$$F_k\Omega^*_M = \{\omega \in \Omega^*_M / \|\omega\|_B \leq k \quad \text{and} \quad \|d\omega\|_B \leq k\}.$$

Notice that if $\alpha \in F_k\Omega^*_M$ and $\beta \in F_{k'}\Omega^*_M$ then

$$\alpha + \beta \in F_{\max(k,k')}\Omega^*_M \quad \text{and} \quad \alpha \wedge \beta \in F_{k+k'}\Omega^*_M. \tag{7}$$

The allowability condition is written in terms of the Cartan's filtration of the fibration $\pi: \pi^{-1}(S) \to S$ (see §1.2.4).

3.1.3. A liftable differential form ω is a *$\bar{p}$-weak intersection differential form* (or *weak intersection differential form*) if for each singular stratum S, the restriction of $\tilde{\omega}$ to $\pi^{-1}(S)$ belongs to $F_{\bar{q}(S)}\Omega^*_{\pi^{-1}(S)}$. We shall write $\mathscr{K}^*_{\bar{q}}(A)$ the complex of $\bar{q}$-weak intersection differential forms. It is a differential subcomplex of the deRham complex of $\tilde{A}$, but it is not always an algebra. It coincides with the complex $\Omega^*(A)$ of differential forms of A if $\Sigma = \emptyset$.

3.1.4. Remarks. (1) In spite of the fact that $\mathscr{K}^*_{\bar{q}}(A)$ depends of the unfolding chosen $\tilde{A}$, the cohomology of the complex does not (see Theorem 4.1.5).

(2) Since the allowability condition is a local condition then, for each open set $U \subset A$, the restriction

$$\rho: \mathscr{K}^*_{\bar{q}}(A) \to \mathscr{K}^*_{\bar{q}}(U)$$

is a well defined differential operator.

(3)With the notations of [2], $\Omega^*_{\bar{q}}(A) \cap \mathscr{K}^*_{\bar{q}}(A) = K^*_{\bar{q}}(A)$.

(4) For $\bar{q} = \bar{0}$ the complex $\mathscr{K}^*_{\bar{q}}(A)$ contains the Verona's complex (see [13]) and it can be seen as the limit of the Verona's complex when ρ_X goes to 0.

(5) An isomorphism between two stratified spaces A and A' induces an isomorphism between $H^*(\mathscr{K}_{\bar{q}}(A))$ and $H^*(\mathscr{K}_{\bar{q}}(A'))$.

3.2. Local calculations. We compute the cohomology of $\mathscr{K}^*_{\bar{q}}(I \times A)$ and $\mathscr{K}^*_{\bar{q}}(cA)$ in terms of that of $\mathscr{K}^*_{\bar{q}}(A)$. Since the proofs are similar to those of [2] we only give a sketch.

PROPOSITION 3.2.1. *Fix $I =]-\varepsilon, \varepsilon[$ an interval of* $\mathbf{R}$. *The maps*

$$pr\colon I \times (A - \Sigma) \to A - \Sigma \quad \text{and} \quad J\colon A - \Sigma \to I \times (A - \Sigma),$$

defined respectively by $pr(t, a) = a$ and $J(a) = (t_0, a)$, for a fixed $t_0 \in I$, induce the quasi-isomorphisms

$$pr^*\colon \mathscr{K}_{\bar{q}}^*(A) \to \mathscr{K}_{\bar{q}}^*(I \times A) \quad \text{and} \quad J^*\colon \mathscr{K}_{\bar{q}}^*(I \times A) \to \mathscr{K}_{\bar{q}}^*(A).$$

Proof (*sketch*). Consider $\widetilde{pr}\colon I \times \tilde{A} \to \tilde{A}$ and $\tilde{J}\colon \tilde{A} \to I \times \tilde{A}$ defined by $\widetilde{pr}(t, \tilde{a}) = \tilde{a}$ and $\tilde{J}(\tilde{a}) = (t_0, \tilde{a})$. The two operators pr^* and J^* are well defined because, for each stratum S of A, we have

$$\|\widetilde{pr^*\omega} = \widetilde{pr}^*\tilde{\omega}\|_{I\times S} \leq \|\tilde{\omega}\|_S \quad \text{and} \quad \|\widetilde{J^*\eta} = \tilde{J}^*\tilde{\eta}\|_S \leq \|\tilde{\eta}\|_{I\times S},$$

for any liftable form $\omega \in \Omega^*(A - \Sigma)$ and $\eta \in \Omega^*(I \times (A - \Sigma))$. In fact these two operators are homotopic; a homotopy operator is given by $H\eta = \int_{t_0}^{-} \eta$. This comes from the following facts:

$$\widetilde{H\eta} = \int_{t_0}^{-} \tilde{\eta} \ (\text{on } I \times \tilde{A}),$$

$$\|\widetilde{H\eta}\|_{I\times S} \leq \|\tilde{\eta}\|_{I\times S},$$

$$dH\eta = Hd\eta + (-1)^{i-1}(\eta - pr^*J^*\eta)$$

where $\eta \in \Omega^i(I \times (A - \Sigma))$ is a liftable form. ■

PROPOSITION 3.2.2. *Suppose A is compact. Then*

$$H^i\big(\mathscr{K}_{\bar{q}}^*(cA)\big) \cong \begin{cases} H^i\big(\mathscr{K}_{\bar{q}}^*(A)\big) & \text{if } i \leq \bar{q}(\text{vertex of } cA) \\ 0 & \text{if } i > \bar{q}(\text{vertex of } cA). \end{cases}$$

where the isomorphism is induced by the canonical projection $pr\colon (A - \Sigma) \times]0, 1[\to (A - \Sigma)$.

Proof (*sketch*). The complex $\mathscr{K}_{\bar{q}}^*(cA)$ is naturally isomorphic to the subcomplex $\mathscr{C}^*$ of $\mathscr{K}_{\bar{q}}^*(A \times]-1, 1[)$ made up of the forms η satisfying

(1) $\eta = 0$ on $(A - \Sigma) \times \{0\}$ if (degree of η) $> \bar{q}$(vertex of cA),
(2) $d\eta = 0$ on $(A - \Sigma) \times \{0\}$ if (degree of η) $= \bar{q}$(vertex of cA), and
(3) $\sigma^*\eta = \eta$ on $(A - \Sigma) \times (]-1, 1[- \{0\})$ where $\sigma\colon A \times]-1, 1[\to A \times]-1, 1[$ is given by $\sigma(a, t) = (a, -t)$.

With the notations of the above proposition (for $\varepsilon = 1$ and $t_0 = 0$) we get

$$pr^*\left(\mathscr{K}^i_{\bar{q}}(A)\right) \subset \mathscr{C}^i \quad \text{for } i < \bar{q}(\text{vertex of } cA);$$
$$pr^*\left(\mathscr{K}^i_{\bar{q}}(A) \cap d^{-1}\{0\}\right) \subset \mathscr{C}^i \quad \text{for } i = \bar{q}(\text{vertex of } cA);$$
$$J^*\mathscr{C}^i = \{0\} \quad \text{for } i > \bar{q}(\text{vertex of } cA)$$

and $H^*(\mathscr{C}^*) = \mathscr{C}^*$. The same procedure used in §3.2.1 finishes the proof. ∎

4. Intersection cohomology of stratified spaces

We prove in this section the two principal results of this work: the deRham Theorem and Poincaré Duality.

4.1. The deRham Theorem. In this section we show that we can use the complex of weak intersection differential forms to compute the intersection cohomology of A. The isomorphism is given by the integration of differential forms over simplices. This integration is well defined because it is calculated on $\tilde{A}$.

4.1.1. Integration over simplices. Let ω be an element of $\mathscr{K}^*_{\bar{q}}(A)$ and let $\sigma\colon \Delta \to A$ be a liftable singular simplex with $\sigma(i(\Delta)) \cap \Sigma = \emptyset$, where $i(\Delta)$ denotes the interior of Δ. We define the *integral of ω over σ* by

$$\int_\sigma \omega = \int_{i(\Delta)} \sigma^* \omega.$$

Does this integral makes sense? Let

$$\Delta \xleftarrow{\mu} \tilde{\Delta} \xrightarrow{\tilde{\sigma}} \tilde{A}$$

be a lifting of σ (see §2.2.3). We recall that the restriction of μ to $i(\tilde{\Delta})$, the interior of $\tilde{\Delta}$, is a diffeomorphism. Then, the map $\sigma\colon i(\Delta) \to (A - \Sigma)$ is differentiable and we can write

$$\int_{i(\Delta)} \sigma^*\omega = \int_{i(\tilde{\Delta})} \mu^*\sigma^*\omega = \int_{i(\tilde{\Delta})} \tilde{\sigma}^*\pi^*\omega.$$

Since $\tilde{\omega}$ is a global differential form on $\tilde{A}$, we get

$$\int_\sigma \omega = \int_{\tilde{\Delta}} \tilde{\sigma}^*\tilde{\omega}, \tag{8}$$

which is finite. We shall write $\int_\sigma \omega = 0$ if $\sigma(\Delta) \subset \Sigma$.

4.1.2. Integration over chains. For each differential form $\omega \in \mathscr{K}_{\bar{q}}^{*}(A)$ and each chain $c = \sum_{j=1}^{m} r_j \sigma_j \in RC_{*}^{\bar{p}}(A)$ we define

$$\int_c \omega = \sum_{j=1}^{m} r_j \int_{\sigma_j} \omega .$$

This makes sense because $\sigma_j(i(\Delta))$ is included in $A - \Sigma$ (see §2.1.1), and we can apply the above definition. So we get the integration operator

$$\int : \mathscr{K}_{\bar{q}}^{*}(A) \to \operatorname{Hom}(RC_{*}^{\bar{p}}(A); \mathbf{R})$$

defined by $(\omega \mapsto (c \mapsto \int_c \omega))$.

The following Stokes formula shows that this operator is a differential operator.

Proposition 4.1.3. *For each differential form $\omega \in \mathscr{K}_{\bar{q}}^{*}(A)$ and each singular chain $c \in RC_{*}^{\bar{p}}(A)$ we have $\int_{\partial c} \omega = \int_c d\omega$.*

Proof. By linearity it suffices to show $\int_{\partial\sigma} \omega = \int_\sigma d\omega$ for a liftable $\bar{p}$-allowable singular simplex $\sigma\colon \Delta \to A$ of dimension i and a differential form $\omega \in \mathscr{K}_{\bar{q}}^{i-1}(A)$. Observe that $\int_{\partial\sigma} \omega$ is well defined because each $(i-1)$-face $\sigma\colon C \to A$ of σ satisfies $\sigma(i(C)) \subset A - \Sigma$ or $\sigma(C) \subset \Sigma$ (see §2.2.3).

We first prove (8) for a codimension one face $\sigma\colon C \to A$ of σ with $\sigma(i(C)) \subset \Sigma$, for some singular stratum S. The relation $i(C) \subset \sigma^{-1}(S) \subset$ (dim Δ − codim $S + \bar{p}(S)$)-skeleton of Δ implies $\bar{q}(S) < 0$ and therefore $\tilde{\omega}|_{\pi^{-1}(S)} = 0$. We get $\int_{\tilde{C}} \tilde{\sigma}^* \tilde{\omega} = 0 = \int_C \omega$, because $\pi\tilde{\sigma}(i(\tilde{C})) = \sigma(i(C)) \subset S$.

In view of (8) we may write $\int_{\partial\sigma} \omega = \int_{\widetilde{\partial\Delta}} \tilde{\sigma}^* \tilde{\omega}$ and $\int_\sigma d\omega = \int_{\tilde{\Delta}} d\tilde{\sigma}^* \tilde{\omega}$. According to the usual Stokes formula we get $\int_{\partial\tilde{\Delta}} \tilde{\sigma}\tilde{\omega} = \int_{\tilde{\Delta}} d\tilde{\sigma}^* \tilde{\omega}$, the proposition will be proved if we show $\int_{\delta\tilde{\Delta}} \tilde{\sigma}^* \tilde{\omega} = 0$ (see §2.2.2).

To see this, we consider a face F of $\delta\tilde{\Delta}$ of dimension $i-1$ and we verify that $\tilde{\sigma}^* \tilde{\omega}$ is 0 on F. We shall let $C = \mu(F)$ and S a stratum of A with $\sigma(i(C)) \subset S$ (see §2.2.3(a))). We have the following commutative diagram:

$$\begin{array}{ccc} i(F) & \xrightarrow{\tilde{\sigma}} & \pi^{-1}(S) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\pi} \\ i(C) & \xrightarrow{\sigma} & S. \end{array}$$

Now we distinguish two cases:

Case 1. $S \subset A - \Sigma$. The differential form ω is defined on S and we may write $\tilde{\sigma}^* \tilde{\omega} = \tilde{\sigma}^* \pi^* \omega = \mu^* \sigma^* \omega$ where $\sigma^* \omega$ is defined on $i(C)$. But $\dim C < \dim F = i - 1 =$ degree of ω = degree of $\sigma^* \omega$. Then $\sigma^* \omega = 0$ on $i(C)$ and therefore $\tilde{\sigma}^* \tilde{\omega} = 0$ on $i(F)$.

Case 2. $S \subset \Sigma$. The allowability condition of σ implies $\dim C \leq \dim F + 1 - \operatorname{codim} S + \bar{p}(S)$. Hence,

$$\dim F - \dim C > \bar{q}(S).$$

On the other hand, since $\tilde{\omega} \in F_{\bar{q}(S)}\Omega^*_{\pi^{-1}(S)}$, we have $\tilde{\sigma}^*\tilde{\omega} \in F_{\bar{q}(S)}\Omega^*_{i(F)}$ for the submersion μ: $i(F) \to i(C)$. These conditions imply that $\tilde{\sigma}^*\tilde{\omega}$ vanishes on $i(F)$. ■

4.1.4. Remark. In the same way we can show a converse for this proposition:

$$\mathscr{K}_{\bar{q}}^*(A) = \left\{\omega \in \Omega^*(A - \Sigma)/\omega \text{ liftable and } \int_{\partial\sigma} \omega = \int_\sigma d\omega \right.$$
$$\left. \text{for each } \bar{p}\text{-allowable liftable singular simplex of } A\right\}.$$

We arrive at the first result of this paper.

THEOREM (DERHAM THEOREM) 4.1.5. *Consider a stratified space A, an unfolding π: $\tilde{A} \to A$ and two dual perversities $(\bar{p}, \bar{q})$. The homology of the complex of intersection chains $SC^{\bar{p}}_*(A)$ and the cohomology of weak intersection differential forms $\mathscr{K}^*_{\bar{p}}(A)$ are isomorphic to the intersection homology $IH^{\bar{p}}_*(A)$.*

The integration of the differential forms of $\mathscr{K}_{\bar{q}}^*(A)$ over the liftable chains of $SC^{\bar{p}}_*(A)$ is well defined, and the maps

$$\mathscr{K}_{\bar{q}}^*(A) \xrightarrow{\int} \operatorname{Hom}(RC^{\bar{p}}_*(A); \mathbf{R}) \xleftarrow{\rho} \operatorname{Hom}(SC^{\bar{p}}_*(A); \mathbf{R}),$$

where ρ is the restriction, are quasi-isomorphisms.

Proof. Following [7] and §2.3.7 it suffices to prove that the operator $\int$: $\mathscr{K}_{\bar{q}}^*(A) \to \operatorname{Hom}(RC^{\bar{p}}_*(A); \mathbf{R})$ induces a quasi-isomorphism in cohomology.

Suppose that for each open cover $\mathscr{U} = \{U_\alpha/\alpha \in J\}$ of A we have the following commutative diagram made up of exact sequences:

$$\text{(9)}\qquad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathscr{K}_{\bar{q}}^*(A) & \xrightarrow{\int_1} & \operatorname{Hom}(R^{\mathscr{U}}C^{\bar{p}}_*(A); \mathbf{R}) \\ \downarrow & & \downarrow \\ \prod_{\alpha_0} \mathscr{K}_{\bar{q}}^*(U_{\alpha_0}) & \xrightarrow{\int_2} & \prod_{\alpha_0} \operatorname{Hom}(RC^{\bar{p}}_*(U_{\alpha_0}); \mathbf{R}) \\ \downarrow & & \downarrow \\ \prod_{\alpha_0<\alpha_1} \mathscr{K}_{\bar{q}}^*(U_{\alpha_0} \cap U_{\alpha_1}) & \xrightarrow{\int_3} & \prod_{\alpha_0<\alpha_1} \operatorname{Hom}(RC^{\bar{p}}_*(U_{\alpha_0} \cap U_{\alpha_1}); \mathbf{R}) \\ \downarrow & & \downarrow \\ \cdots & & \cdots \end{array}$$

where the vertical arrows are the restrictions and the horizontal arrows are the integrations. Using the procedure followed in Proposition 2.3.7 and using Propositions 2.2.4, 2.2.5, 3.2.1 and 3.2.2 it is easy to prove the Theorem (for $d(A) = 1$ we get the usual deRham Theorem for manifolds).

So, we must prove that the rows of (9) are exact. Applying the functor Hom to (5) we get that the bottom row of (9) is exact. In order to show that the top row is exact we need (following [1, page 94]) to find a partition of unity $\{f_\alpha/\alpha \in J\}$ subordinated to the cover $\mathscr{U}$ satisfying $f_\alpha\omega \in \mathscr{K}_{\bar{q}}^*(U_{\alpha_0})$ for each $\omega \in \mathscr{K}_{\bar{q}}^*(U_{\alpha_0})$. Since the covers by distinguished neighborhoods are cofinal in the set of all open covers of A (see §1.2.2), it suffices to show:

There exists a continuous map $f\colon \mathbf{R}^m \times cA \to [0,1]$ *and two numbers* $r, s \in]0,1[$ *with*
(a) $f \equiv 1$ *on* $]-r, r[^m \times c_rA$,
(b) $f \equiv 0$ *on the complement of* $]-s, s[^m \times c_sA$, *and*
(c) $f\omega \in \mathscr{K}_{\bar{q}}^*(\mathbf{R}^m \times cA)$ *for each* $\omega \in \mathscr{K}_{\bar{q}}^*(\mathbf{R}^m \times cA)$.

To see this, fix two numbers $r, s \in]0,1[$ and two smooth maps $f_1\colon \mathbf{R} \to [0,1]$ and $f_2\colon]-1,1[\to [0,1]$ with $f_i \equiv 1$ on $[-r,r]$ and $f_i \equiv 0$ off $]-s,s[$, $i = 1,2$. The map $f\colon \mathbf{R}^m \times cA \to [0,1]$ defined by

$$f(x_1,\dots,x_m,[a,t]) = f_1(x_1)\cdots f_1(x_m)f_2(t)$$

is continuous and has a smooth lifting $\tilde{f}\colon \mathbf{R}^m \times \tilde{A} \times]-1,1[\to [0,1]$, given by

$$\tilde{f}(x_1,\dots,x_m,\tilde{a},t) = f_1(x_1)\cdots f_1(x_m)f_2(|t|).$$

By construction we have a) and b). Let ω be an element of $\mathscr{K}_{\bar{q}}^*(\mathbf{R}^m \times cA)$, it remains to show that $f\omega$ belongs to $\mathscr{K}_{\bar{q}}^*(\mathbf{R}^m \times cA)$. Let S be a singular stratum of $\mathbf{R}^m \times cA$ and P the unfolding of $\mathbf{R}^m \times cA$ given in §1.2.2; the fiber of $P\colon P^{-1}(S) \to S$ over $(x,[y,t]) \in S$ is

$$\begin{array}{ll} \{x\} \times \tilde{A} \times \{0\} & \text{if } S = \mathbf{R}^m \times \{\text{vertex}\} \\ \{x\} \times \pi^{-1}(y) \times \{-t,t\} & \text{if } S = \mathbf{R}^m \times S' \times]0,1[,\ S' \text{ stratum of } A. \end{array}$$

In any case the function $\tilde{f}$ is constant on the fibers of $P\colon P^{-1}(S) \to S$. The map $\tilde{f}$ belongs to $F_0\Omega^*_{P^{-1}(S)}$ and therefore $\tilde{f}\tilde{\omega}$ lies on $F_{\bar{q}(S)}\Omega^*_{P^{-1}(S)}$ (see (7)). This shows (c). ∎

4.2. Poincaré duality. The intersection homology was introduced with the purpose of extending the Poincaré Duality to singular manifolds (see [4]). The pairing is given there by the intersection of cycles. In the deRham theory of manifolds the Duality derives from the integration of the wedge product of differential forms of arbitrary and compact support. In fact, this point of view is still available in the intersection homology context. This is shown in this section (see also [3]).

We consider in the following a stratified space A, an unfolding $\pi: \tilde{A} \to A$ and two dual perversities $\bar{p}$ and $\bar{q}$. We shall suppose also that A is *orientable*, that is, the manifold $A - \Sigma$ is an orientable manifold. This is used to integrate differential forms on $A - \Sigma$. In the following we will use the facts: (1) $M \times A$ is orientable, if M is an orientable manifold M, and (2) cA is orientable, where A is compact. Each link L (see §1.2.1) is orientable.

4.2.1. For each differential form ω on $A - \Sigma$ we define the *support of* ω, written $\operatorname{supp}(\omega)$, as the closure on A of the set $\{x \in A - \Sigma / \omega(x) \not\equiv 0\}$. This notion coincides with the usual one if $\Sigma = \emptyset$.

We define $\mathscr{K}^*_{c,\bar{p}}(A)$ as the subcomplex of $\mathscr{K}^*_{\bar{p}}(A)$ made up of the differential forms with compact support. The relation $\operatorname{supp}(d\omega) \subset \operatorname{supp}(\omega)$ shows that this subcomplex is a differential complex. If A is compact then $\mathscr{K}^*_{c,\bar{p}}(A)$ coincides with $\mathscr{K}^*_{\bar{p}}(A)$. For $\Sigma = \emptyset$ we have that $\mathscr{K}^*_{c,\bar{p}}(A)$ is just the complex of differential forms of A with compact support.

For each open set $U \subset A$ there is a natural inclusion $\mathscr{K}^*_{c,\bar{p}}(U) \hookrightarrow \mathscr{K}^*_{c,\bar{p}}(A)$, extending a form on $U - \Sigma$ by zero to a form on $A - \Sigma$. The same method used in [1, page 139] applies here to show that, for an open cover $\mathscr{U} = \{U_\alpha / \alpha \in J\}$, the Mayer-Vietoris sequence:

$$0 \leftarrow \mathscr{K}^*_{c,\bar{p}}(A) \leftarrow \bigoplus_{\alpha_0} \mathscr{K}^*_{c,\bar{p}}(U_{\alpha_0}) \leftarrow \bigoplus_{\alpha_0 < \alpha_1} \mathscr{K}^*_{c,\bar{p}}(U_{\alpha_0} \cap U_{\alpha_1}) \leftarrow \cdots$$

is exact (see proof of §4.1.5).

The following lemma will be needed in the definition of the Poincaré pairing.

LEMMA 4.2.2. *Let ω be a liftable differential form on $A - \Sigma$ with compact support. Then*

$$(a) \int_{A-\Sigma} \omega < +\infty \quad \textit{and} \quad (b) \int_{A-\Sigma} d\omega = 0.$$

Proof. Let l the number of connected components of $\tilde{A} - \pi^{-1}(\Sigma)$. By definition of $\tilde{\omega}$ we get

$$\int_{A-\Sigma} \omega = l^{-1} \int_{\tilde{A}-\pi^{-1}(\Sigma)} \pi^*\omega = l^{-1} \int_{\tilde{A}-\pi^{-1}(\Sigma)} \tilde{\omega} = l^{-1} \int_{\tilde{A}} \tilde{\omega}.$$

(a) It suffices to prove $\int_{\tilde{A}} \tilde{\omega} < \infty$. Since the map π is a proper map (this is shown using the local description (1) of π and the fact that $\tilde{L}$ is compact) then the support of $\tilde{\omega}$ is compact ($\operatorname{supp}(\tilde{\omega}) = \pi^{-1}(\operatorname{supp}(\omega))$).

(b) Since $\tilde{A}$ has not a boundary, we obtain $\int_{\tilde{A}} d\tilde{\omega} = 0$. ■

4.2.3. The above lemma shows that the pairing $\int: \mathscr{K}_{\bar{q}}^{*}(A) \otimes \mathscr{K}_{c,\bar{p}}^{n-*}(A) \to \mathbf{R}$ given by

$$(\omega, \eta) \mapsto \int_{A-\Sigma} \omega \wedge \eta$$

is well defined and induces a pairing in cohomology

$$\int : H^*\big(\mathscr{K}_{\bar{q}}(A)\big) \otimes H^{n-*}\big(\mathscr{K}_{c,\bar{p}}(A)\big) \to \mathbf{R},$$

called *Poincaré pairing*. We are going to show that it is nondegenerate; or equivalently, the map

$$\int : \mathscr{K}_{\bar{q}}^{*}(A) \to \mathrm{Hom}\big(\mathscr{K}_{c,\bar{p}}^{n-*}(A); \mathbf{R}\big)$$

given by

$$\left(\omega \mapsto \left(\eta \mapsto \int_{A-\Sigma} \omega \wedge \eta\right)\right)$$

is a quasi-isomorphism. First, we do the local calculations characteristic to the intersection homology, those of $I \times A$ and cA.

LEMMA 4.2.4. *Let $I =]u, v[$ be an open interval of $\mathbf{R}$ and let $e = e(t)\,dt$ be compactly supported 1-form on I with total integral 1. We let $pr_1\colon I \times (A - \Sigma) \to I$ and $pr_2\colon I \times (A - \Sigma) \to (A - \Sigma)$ be the canonical projections. Then the following operators (see [1, page 38]) are well defined:*

(a) $e_*\colon \mathscr{K}_{c,\bar{p}}^{*}(A) \to \mathscr{K}_{c,\bar{p}}^{*+1}(I \times A)$ *given by* $e_*(\omega) = pr_1^* e \wedge pr_2^* \omega$,

(b) $\phi\colon \mathscr{K}_{c,\bar{p}}^{*}(I \times A) \to \mathscr{K}_{c,\bar{p}}^{*-1}(A)$ *given by* $\phi\omega = \int_u^v \omega$, *and*

(c) $K\colon \mathscr{K}_{c,\bar{p}}^{*}(I \times A) \to \mathscr{K}_{c,\bar{p}}^{*-1}(I \times A)$ *given by* $K\omega = H\omega - H(pr_1^* e) \cdot pr_2^*(\phi\omega)$, *where* $H\omega = \int_u^{-} \omega$ *(see §3.2.2).*

And they satisfy the relation

(d) $1 - e_*\phi = (-1)^{i-1}(dK - Kd)$ *on* $\mathscr{K}_{c,\bar{p}}^{i}(I \times A)$.

Proof. (a) Since $pr_1^* e \in \mathscr{K}_{\bar{0}}^{1}(I \times A)$, we have $e_*(\omega) \in \mathscr{K}_{\bar{p}}^{*+1}(I \times A)$ (see §3.2.1). Its support is compact because

$$\mathrm{supp}(e_*(\omega)) \subset \mathrm{supp}(e) \times \mathrm{supp}(\omega).$$

(b) The same technique used in §3.2.2 shows that $\phi\omega \in \mathscr{K}_{\bar{p}}^{*-1}(A)$. For the support we get

$$\mathrm{supp}(\phi\omega) \subset pr_2(\mathrm{supp}(\omega)).$$

(c) The above remarks and §3.2.2 prove that $K\omega \in \mathscr{K}_{\bar{p}}^{*-1}(I \times A)$. Let $I' \times C \subset I \times A$ be a compact containing supp(ω) and satisfying supp$(e) \subset I'$. A straightforward calculation shows that supp$(K\omega) \subset I' \times C$.

(d) It is proved in [1, page 38]. ■

Since $\oint e_*$ is the identity on $\mathscr{K}_{c,\bar{p}}^*(A)$ we get from the above lemma:

PROPOSITION 4.2.5. *The operator e_* induces an isomorphism $H^*(\mathscr{K}_{c,\bar{p}}(A)) \cong H^{*+1}(\mathscr{K}_{c,\bar{p}}(I \times A))$.*

For the cone cA, with the notations of §4.2.4 and $(u, v) = (0, 1)$ we obtain the following:

PROPOSITION 4.2.6. *If A is compact then the operator e_* induces an isomorphism*

$$H^i\big(\mathscr{K}_{c,\bar{p}}^*(cA)\big) \cong \begin{cases} H^i\big(\mathscr{K}_{\bar{p}}^*(A)\big) & \text{if } i \geq \bar{p}(\text{vertex of } cA) + 2 \\ 0 & \text{if } i \leq \bar{p}(\text{vertex of } cA) + 1 \end{cases}$$

Proof. First of all, we calculate the cohomology of the quotient complex $\mathscr{K}_{\bar{p}}^*(cA)/\mathscr{K}_{c,\bar{p}}^*(cA)$. This complex is isomorphic by restriction to

$$\mathscr{K}_{\bar{p}}^*(A \times]0,1[)/\mathscr{L}^*,$$

where

$$\mathscr{L}^* = \{\omega \in \mathscr{K}_{\bar{p}}^*(A \times]0,1[)/\text{supp}(\omega) \subset A \times]0,\epsilon[\text{ for some } \varepsilon < 1\};$$

the inverse is given by (class of ω) $\mapsto$ (class of $(\int_0^- pr_1^* e) \cdot \omega$).

We claim that $\mathscr{L}^*$ is acyclic. In fact, for each cycle ω of $\mathscr{L}^*$ we have the formula

$$\omega = \pm d \int_\varepsilon^- \omega \quad \text{with} \int_\varepsilon^- \omega \in \mathscr{K}_{\bar{p}}^*(A \times]0,1[).$$

Since supp$(\int_\varepsilon^- \omega) \subset A \times]0, \varepsilon[$, we get the claim. Consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathscr{K}_{c,\bar{p}}^*(cA) & \longrightarrow & \mathscr{K}_{\bar{p}}^*(cA) & \xrightarrow{\tau_1} & \mathscr{K}_{\bar{p}}^*(cA)/\mathscr{K}_{c,\bar{p}}^*(cA) \longrightarrow 0 \\ & & & & \uparrow{\scriptstyle \tau_2} \\ & & & & \mathscr{K}_{\bar{p}}^*(A) \end{array}$$

where τ_1 is the projection and τ_2 is defined by ($\eta \mapsto$ (class of $(\int_0^- pr_1^* e) \cdot$

$pr_2^*\eta)$). The above calculations and Proposition 3.2.1 show that τ_2 is a quasi-isomorphism.

Under this quasi-isomorphism the connecting homomorphism of the associated long sequence becomes

$$\delta\colon H^*\big(\mathcal{K}_{\bar{p}}(A)\big) \to H^{*+1}\big(\mathcal{K}_{c,\bar{p}}(cA)\big),$$

defined by $\delta[\eta] = [e_*(\eta)]$; and τ_1 becomes J^*, where $J\colon A - \Sigma \to (A - \Sigma) \times]0,1[$ is defined by $J(a) = (a, \frac{1}{2})$. Now the result follows from Proposition 3.2.2. ■

We have arrived to the Poincaré Duality.

THEOREM (POINCARÉ DUALITY) 4.2.7. *Let A be an orientable stratified space and let $(\bar{p}, \bar{q})$ be two dual perversities. The Poincaré pairing $\int$: $H^*(\mathcal{K}_{\bar{q}}(A)) \otimes H^*(\mathcal{K}_{c,\bar{p}}(A)) \to \mathbf{R}$ is nondegenerate.*

Proof. It suffices to show that the map $\int\colon H^*(\mathcal{K}_{\bar{q}}(A)) \to \mathrm{Hom}(H^*(\mathcal{K}_{c,\bar{p}}(A)); \mathbf{R})$, defined by

$$\left([\omega] \mapsto \left([\eta] \mapsto \int_{A-\Sigma} \omega \wedge \eta\right)\right),$$

is an isomorphism (see [1, page 44]). To see this we follow the same procedure as in 2.3.7, taking into account the following facts:

If the depth of A is 1 the Theorem is the usual Poincaré Duality for orientable manifolds.

The diagram

$$\begin{array}{ccc}
\mathcal{K}_{\bar{q}}^*(\mathbf{R}^i \times cL) & \xrightarrow{\int} & \mathrm{Hom}(\mathcal{K}_{c,\bar{p}}^{n-*}(\mathbf{R}^i \times cL); \mathbf{R}) \\
\uparrow{\scriptstyle pr_2^*} & & \downarrow{\scriptstyle e^*} \\
\mathcal{K}_{\bar{q}}^*(cL) & \xrightarrow{\int} & \mathrm{Hom}(\mathcal{K}_{c,\bar{p}}^{n-i-*}(cL); \mathbf{R})
\end{array}$$

is commutative (see below) and the operators pr_2^* and e^* are quasi-isomorphisms (see §3.2.1) and §4.2.5). Here

$$e^*(F)(\eta) = F(pr_1^* e \wedge pr_2^*\eta),$$

with

$$pr_1\colon \mathbf{R}^i \times (L - \Sigma(L)) \times]0,1[\to \mathbf{R}^i$$

and

$$pr_2\colon \mathbf{R}^i \times (L - \Sigma(L)) \times]0,1[\to (L - \Sigma(L)) \times]0,1[$$

the canonical projections, and $e = f(x_1) \cdots f(x_i)\, dx_1 \wedge \cdots \wedge dx_i$ is a compactly supported 1-form on $\mathbf{R}^i$ with total integral 1.

The commutativity of the diagram comes from the identity

$$\int_{\mathbf{R}^i \times (L - \Sigma(L)) \times]0,1[} pr_2^* \omega \wedge pr_2^* \eta \wedge pr_1^* e = \int_{(L - \Sigma(L)) \times]0,1[} \omega \wedge \eta$$

for each $\omega \in \mathscr{K}_{\bar{q}}^*(cL)$ and $\eta \in \mathscr{K}_{c,\bar{p}}^{n-i-*}(cL)$.

$H^j(\mathscr{K}_{\bar{q}}^*(cL)) \cong H^{n-i-j}(\mathscr{K}_{c,\bar{p}}^*(cL)) \cong 0$ for $j \geq \bar{q}(\text{vertex of } cL) + 1$ (see Propositions 3.2.2 and 4.2.6).

The diagram

$$\begin{array}{ccc} H^j(\mathscr{K}_{\bar{q}}^*(cL)) & \xrightarrow{\int} & \operatorname{Hom}(H^{n-i-j}(\mathscr{K}_{c,\bar{p}}^*(cL)); \mathbf{R}) \\ {\scriptstyle pr_2^*}\uparrow & & \downarrow{\scriptstyle e^*} \\ H^j(\mathscr{K}_{\bar{q}}^*(L)) & \xrightarrow{\int} & \operatorname{Hom}(H^{n-1-i-j}(\mathscr{K}_{\bar{p}}^*(L)); \mathbf{R}) \end{array}$$

is commutative (see below) and the operators pr_2^* and e^* are quasi-isomorphisms (see Propositions 3.2.2 and 4.2.6) for $j \leq \bar{q}(\text{vertex of } cL)$. Here

$$e^*(F)(\eta) = F(pr_1^* e \wedge pr_2^* \eta),$$

with

$$pr_1\colon (L - \Sigma(L)) \times]0,1[\to]0,1[$$

and

$$pr_2\colon (L - \Sigma(L)) \times]0,1[\to L - \Sigma(L)$$

the canonical projections, and e is a compactly supported i-form on $]0,1[$ with total integral 1.

The commutativity of the diagram comes from the identity

$$\int_{(L - \Sigma(L)) \times]0,1[} pr_2^* \omega \wedge pr_2^* \eta \wedge pr_1^* e = \int_{L - \Sigma(L)} \omega \wedge \eta,$$

which holds for each $\omega \in \mathscr{K}_{\bar{q}}^j(L)$ and $\eta \in \mathscr{K}_{\bar{p}}^{n-i-1-j}(L)$. ∎

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