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INTERMITTENT OSCILLATION AND TANGENTIAL GROWTH OF FUNCTIONS WITH RESPECT TO NAGEL-STEIN REGIONS ON A HALF-SPACE

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1. Introduction

Let n be a positive integer and denote the upper half-space $\mathbf{R}^n \times (0, \infty)$ in $\mathbf{R}^{n+1}$ by $\mathbf{R}_+^{n+1}$. The boundary $\partial\mathbf{R}_+^{n+1}$ of $\mathbf{R}_+^{n+1}$ will be identified, in the usual way, with $\mathbf{R}^n$. In the discussion below, we shall transpose (without explicit mention) results originally stated relative to the unit disk and its circumference in the complex plane, to the upper half-space $\mathbf{R}_+^{n+1}$ and its boundary $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$.

In 1968, Å. Samuelsson [Sa] studied, for $n = 1$, the generalized derivatives of positive, Borel measures μ defined on $\mathbf{R}^n$ in relation to growth along the normal $N_x = \{(x,t): 0 < t < \infty\}$ of the positive harmonic functions $\mathscr{K}\mu$ associated by means of the Poisson integral formula,

$$(1) \qquad \mathscr{K}\mu(x,t) = \int_{\mathbf{R}^n} K(x,t,z)\, d\mu(z), \qquad (x,t) \in \mathbf{R}_+^{n+1}.$$

Here, $K(x,t,z)$, for $x, z \in \mathbf{R}^n$ and $t \in (0,\infty)$, denotes the Poisson kernel for the upper half-space $\mathbf{R}_+^{n+1}$, and the measures μ satisfy the usual integral condition required for the convergence of the Poisson integral. Among other things, Samuelsson considered generalized upper symmetric derivatives of μ with respect to functions such as $\omega(t) = t^\beta$, $0 < \beta < 1$, at a point $x \in \mathbf{R}$, defined by

$$\overline{D}_\omega\mu(x) = \limsup \frac{|\mu(I)|}{\omega(|I|)},$$

where the intervals I are centered at x and the limit superior is taken as their lengths $|I|$ converge to 0. When $\overline{D}_\omega\mu(x)$ is positive, there is a sequence of intervals $\{I_j\}$ centered at x such that $|I_j| \to 0$ and $\mu(I_j)$ is (at least) of the order of $\omega(|I_j|)$. One may describe this roughly by saying that μ has

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intermittent mass concentration of the order of ω at x. Samuelsson demonstrated that this is the case when there is intermittent growth of the corresponding positive harmonic function $\mathscr{K}\mu$ along the normal N_x of the order of $\omega(t)/t$ as $t \to 0^+$. By showing that the set of points where μ has intermittent concentration of the order of ω can have at most a certain Hausdorff measure, he was able to deduce the analogous statement concerning the set of normal lines N_x along which $\mathscr{K}\mu$ can grow intermittently at the rate $\omega(t)/t$.

The work of Samuelsson was followed up by the first author in [Be1] and [Be2] (still with $n = 1$). In [Be1], the results concerning the generalized derivatives of measures μ and the boundary behavior of the positive harmonic functions $\mathscr{K}\mu$ were expanded to a broader context in which the measures were replaced by functions and the harmonic functions were defined in terms of a formal integration by parts of the Poisson integral formula. The function $\mu: \mathbf{R} \to \mathbf{R}$ was assumed to have bounded generalized variation of the type studied by J. Musielak and W. Orlicz [MO]. (See also [Av] for related references.) For such a function μ, an analogous notion of "intermittent oscillation" replaces that of "intermittent mass concentration", where $\mu(I)$ is understood to mean $\mu(y) - \mu(x)$ for x and y the endpoints of I. The standard setting of measures and positive harmonic functions was reconsidered in [Be2], and it was shown that the ideas developed in [Sa] (and in [Be1]) led in most cases to a precise characterization of the exceptional set of points x where a measure can have a prescribed intermittent concentration of mass and a positive harmonic function can have a prescribed intermittent growth along the normal N_x.

In 1984, A. Nagel and E.M. Stein [NS] gave a different sort of generalization of the standard differentiation theory in connection with an improved Fatou theorem (for $n \geq 1$). They introduced a maximal function along with an associated derivative in terms of translates of an "approach set" more general than the usual nontangential approach set. This set determines a family of balls which is used to define the maximal function and the derivative. The hallmark of the Nagel-Stein generalization is the "cross-sectional measure" condition (see §2) satisfied by the approach set. The derivative is shown to exist a.e. (and agrees with the classical symmetric derivative a.e.). These results are then used to establish the existence of boundary limits of positive harmonic functions on the half-space for (Lebesgue) almost all translates of a boundary approach region also satisfying a cross-sectional measure condition. The paper [NS] was influential in many papers that followed. In the discussion below, we shall restrict our attention only to those which are more closely related to the present work. We refer the reader to [AN], [CDS], [Su1], [Su2], and [Su3] for other interesting developments.

The Nagel-Stein results were extended in joint work of the second author with B.A. Mair [MS] to the case where the kernel $K(x, t, z)$ in (1) is more general. Aside from the Poisson kernel, which gives rise to harmonic functions, other examples give solutions to certain parabolic equations on $\mathbf{R}_+^{n+1}$, and solutions to the heat equation on the right half space. (See [MS] for a discussion of these examples in greater detail.) Each such kernel $K(x, t, z)$ has associated with it a pseudo-distance d (see §2). As before, an approach set Ω is required to satisfy a cross-sectional measure condition, this time given in terms of the pseudo-distance d. Examples of such sets Ω which contain a sequence having an arbitrary degree of tangency were also constructed in [MS].

A related generalization was given in a paper of J. Sueiro [Su1]. He proved a Fatou theorem for Poisson-Szegö integrals of L^p functions on the boundary of a generalized half-plane in $\mathbf{C}^n$. Again, a cross-sectional measure condition appears in the description of the approach sets.

Subsequent generalizations were carried out by the second author with Mair and S. Philipp in [MPS1] and [MPS2]. There, functions were considered of the form (1) with $\mathbf{R}^n$ replaced by a space X of homogeneous type having a group structure, and Lebesgue measure replaced by a measure σ associated with X. Nagel-Stein type differentiation and Fatou theorems along with converse results (in an even more general setting) were given. In [MPS1], it was also shown that for any function, the existence of "standard" limits (defined in terms of translates of the standard set $\Omega = \{(y, s)\colon d(0, y) < as\}$ for $a > 0$) a.e. $[\sigma]$ in a measurable set E in the boundary implies the existence of corresponding Nagel-Stein type limits a.e. $[\sigma]$ in E. We note, however, in this result "a.e." means almost everywhere with respect to the specific measure σ (e.g., where σ is Lebesgue measure), and not for a more general Hausdorff measure or content. Here, we shall concern ourselves with obtaining results which involve Nagel-Stein type approach sets and thinner exceptional sets.

The purpose of this paper is to join together the ideas stemming from the work of Samuelsson on intermittent concentration and growth, with those evolving from the work of Nagel and Stein on more general approach sets. The focus of our study will be on intermittent concentration of measures, intermittent oscillation of functions of bounded generalized variation, and intermittent boundary growth of functions given in the form (1) (where μ is a suitable measure). Maximal functions, "lim sup" maximal functions, and generalized upper derivatives are defined with respect to "test-functions" $\omega(t)$ and "admissible" approach sets Ω. The test functions are given in the form $\omega(t) = |B_d(0, t)|^\beta$, where $B_d(x, t)$ denotes the open ball (determined by the pseudo-distance) of radius t around the point x, and 0 denotes the origin. They are the functions in relation to which the mass concentration, oscillation, and growth are measured. The admissible approach sets Ω satisfy a Nagel-Stein type cross-sectional measure condition and determine the family of balls used in the definition of the maximal functions. For the results concerning intermittent oscillation, we shall concentrate our attention on classes of functions $u\colon \mathbf{R}_+^{n+1} \to \mathbf{R}$ of bounded γ-variation which generalize those studied by N. Wiener [Wi] and others. (See also references 6–12 in [Av], and [Ge], in particular.) Paradigms for such functions u arise in the case $n = 1$ from any function μ: $\mathbf{R} \to \mathbf{R}$ of bounded γ-variation in the sense of Wiener by defining $u(x, t) = \mu(x + t) - \mu(x - t)$, and, from any positive, Borel measure μ on $\mathbf{R}^n$ (the case $\gamma = 1$), when $u(x, t) = \mu[B_d(x, t)]$. The lim sup maximal function mirrors a corresponding generalized upper derivative of a function μ defined on the boundary $\mathbf{R}^n$ when u arises from such a function. We shall apply our results in the special case of measures (when $\gamma = 1$) to obtain results concerning the intermittent boundary growth of a variety of classes of functions given in the form (1).

The paper is organized as follows. In §2 we develop the basic setting, giving background definitions and facts. It is divided into the five subsections: *Pseudo-Distance*, *Approach Sets*, *Content and Hausdorff Measure*, *Generalized Variation*, and *Maximal Functions and Generalized Derivatives*. In §3, we study the Nagel-Stein type approach sets Ω, which we refer to as "admissible sets". In that section are included several results which may be of independent interest. For example, Lemma 2 asserts (roughly) that any admissible set Ω is contained in an admissible set generated by placing standard regions over a countable set. Furthermore, each cross-section is contained in the union of the cross sections of the standard regions above a finite set of points taken from the countable set, the cardinality being uniformly bounded regardless of the height of the cross-section. In §4 we turn to the intermittent oscillation of functions of bounded γ-variation and prove the first two of our central results, Theorems 1 and 2. Theorem 1 gives a weak-type inequality satisfied by the maximal function of u. The measuring function is "δ-content", where δ is defined in a natural way in terms of the parameters associated with the tangentiality of the approach set, the test function against which the mass concentration or oscillation is measured, and the class of functions of bounded generalized variation under consideration. Theorem 1 leads to corresponding content and Hausdorff measure conditions on the size of the sets of points where functions μ, defined on the boundary, can have intermittent oscillation of a prescribed type. Theorem 2 gives a stronger result for the intermittent concentration of measures than follows immediately from Theorem 1 in the case $\gamma = 1$. Our third main result, Theorem 3, is the application of Theorem 2 to the intermittent growth of functions of the form (1). This is the subject of §5. Converse results, demonstrating that the conditions on the maximum size of the exceptional sets in Theorems 1, 2, and 3 are best possible, are given in §6.

2. Preliminaries

In this section we shall develop the background necessary in the rest of the paper. We divide it into five subsections: *Pseudo-Distance*, *Approach Sets*, *Content and Hausdorff Measure*, *Generalized Variation*, and *Maximal Functions and Generalized Derivatives*.

Pseudo-distance

Let n be a positive integer. For E a measurable subset of $\mathbf{R}^n$ we denote the Lebesgue measure of E by $|E|$. A *translation invariant pseudo-distance* on $\mathbf{R}^n$ is defined to be a function d: $\mathbf{R}^n \times \mathbf{R}^n \to [0, \infty)$ together with a constant $K \in [1, \infty)$ and a function τ: $(0, \infty) \to (0, \infty)$, such that for all points $x, y, z \in \mathbf{R}^n$, we have

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x + z, y + z) = d(x, y)$,

(iv) $d(x, z) \leq K\,[d(x, y) + d(y, z)]$,

(v) $\{B_d(x, t)\colon t > 0\}$ forms a base of neighborhoods of x (in the Euclidean topology), where

$$B_d(x, t) = \{y \in \mathbf{R}^n\colon d(y, x) < t\},$$

and,

(vi) $|B_d(0, st)| \leq \tau(s)|B_d(0, t)|$ for all $s, t > 0$.

The constants K and $\tau(2)$ are greater than or equal to 1 and are called the *constants of the pseudo-distance*. For more background see [CW].

We shall add one more assumption; that is, there exist positive constants c_1, c_2, and r, such that

(vii) $c_1 t^r < |B_d(0, t)| < c_2 t^r$ for all $t > 0$.

Observe that condition (vii) implies that (vi) holds with $\tau(s) = (c_2/c_1)s^r$.

Terminology. Any function d satisfying (i)–(vii) will be called an *allowed pseudo-distance*.

Examples of allowed pseudo-distances are $d(x, y) = \Sigma|x_i - y_i|^{\alpha_i}$, where $\{\alpha_i\colon i = 1, \ldots, n\}$ is a set of positive real numbers. Specific choices allow us to apply our results to various classes of functions defined as in (1). Some examples are $\alpha_i = 1$ for all i (positive harmonic functions), $\alpha_i = 2$ for all i (solutions of a wide variety of parabolic equations), and $\alpha_i = 2$ for $i = 1, \ldots, n - 1$, $\alpha_n = 1/2$ (solutions of the heat equation as they approach a vertical boundary). These examples are described in detail in [MS].

Suppose now that $\alpha \in (0, \infty)$. One verifies without difficulty that d^α is an allowed pseudo-distance. In fact, (i)–(iii) and (v) are immediate. Condition (iv) holds with K replaced by K^α or $2^{\alpha-1}K^\alpha$ according as $\alpha \in (0, 1]$ or $\alpha \in (1, \infty)$. Condition (vii) evidently holds with t^r replaced by $t^{r/\alpha}$. In the sequel, we shall use the pseudo-distance d^α, for $\alpha > 1$, to define approach sets which, roughly speaking, have α times the order of contact of standard approach sets defined in terms of d. (This will be discussed in detail in the next subsection.) Certain modifications and conventions in notation will be adopted in this connection. The notation $B_\alpha(x,t)$ will be used in place of $B_d(x,t)$ when the ball is defined in terms of d^α instead of d. More generally, "α" will be used in place of "d" in other related notation in which the latter appears explicitly.

The following result will be used frequently.

COVERING THEOREM [CW, Théorème 1.2, p. 69]. *Let E be a bounded subset of $\mathbf{R}^n$ and let*

$$\{B_d[x, r(x)] : x \in E\}$$

be a covering of E. Then there exists a sequence of disjoint d-balls $B_d[x_i, r(x_i)]$ taken from the covering such that $\{B_d[x_i, kr(x_i)]\}$ covers E. The constant k depends only on the constants of the pseudo-distance d.

Throughout the remainder of the paper, d will always denote an allowed pseudo-distance. The notation K, τ, c_1, c_2, r, and k given in this subsection will be reserved for use in connection with the pseudo-distance d. On occasions when two pseudo-distances are being considered, we shall sometimes use subscripts to avoid ambiguity.

Approach sets

We will consider subsets of $\mathbf{R}^{n+1}_+$. The term approach set will refer to any such subset Ω having the origin as its only limit point in $\mathbb{R}^n$. By translation, we can view $(x,0) + \Omega$ as an approach set to $x \in \mathbf{R}^n$.

For $x \in \mathbf{R}^n$, $t \geq 0$, and $a > 0$, define

$$(d,a)\text{-}S(x,t) = \{(y,s) \in \mathbf{R}^{n+1}_+ : d(x,y) < a(s-t)\}.$$

We call this the *d-standard region having aperture a and vertex (x,t)*. If $a = 1$, then we simply write d-$S(x,t)$. Clearly d-$S(0,0)$ is an approach set. As indicated before, when d is replaced by d^α in the above definition, we shall replace d in the notation with α.

Observe that in the classical setting where d is the Euclidean distance, a d-standard region is just a cone above its vertex. When $\alpha > 1$, the α-standard region is tangential to the horizontal hyperplane containing its vertex, with the tangentiality increasing with α.

For Ω any subset of $\mathbf{R}^{n+1}_+$ and $t > 0$, we call

$$\Omega(t) = \{x \in \mathbf{R}^n : (x,t) \in \Omega\}$$

the *t-section* of Ω. For $a > 0$, define

$$\Omega_{d,a} = \bigcup\{(d,a)\text{-}S(x,t) : (x,t) \in \Omega\}.$$

In case $a = 1$, we simply write Ω_d. Note that

$$\Omega_{d,a}(t) = \bigcup\{B_d[y, a(t-s)] : (y,s) \in \Omega,\ s < t\}.$$

We say that an approach set Ω is *(d,a)-admissible* provided
(i) $\Omega(t)$ is a bounded, measurable set for each $t > 0$, and
(ii) there is a positive constant $A_{d,a}$ such that

$$|\Omega_{d,a}(t)| \leq A_{d,a}|B_d(0,t)|.$$

Note that (ii) says that the t-section through $\Omega_{d,a}$ has measure at most a fixed multiple of the measure of the t-section of the standard region with vertex at the origin. This is sometimes called the "cross-sectional measure" condition.

LEMMA 1. *If Ω is (d,a)-admissible, then Ω is (d,b)-admissible for all $b > 0$.*

The proof follows easily by applying the Covering Theorem to the t-sections, $\Omega_{d,b}(t)$.

DEFINITION. Let Ω be a subset of $\mathbf{R}^{n+1}_+$. We say that Ω is *d-admissible* if Ω is (d,a)-admissible for some (hence, by the preceding lemma, for every) $a > 0$.

By the triangle inequality (condition (iv) in the definition of an allowed pseudo-distance), we see that $(\Omega_{d,a})_{d,a} \subseteq \Omega_{d,Ka}$. The preceding lemma then implies that if Ω is d-admissible, then $\Omega_{d,a}$ is d-admissible as well.

The d-admissible sets were first studied with the usual Euclidean distance in [NS]. They were then studied with a general pseudo-distance in [MS] and [Su1]. We shall obtain some more properties of d-admissible sets in §3.

Clearly d-standard regions are d-admissible. Examples of d-admissible sets containing sequences converging to the origin having arbitrary degree of tangency can be constructed using techniques developed in any of the references quoted in the previous paragraph.

In the sequel, we shall be primarily interested in using α-admissible sets for our approach sets.

Content and Hausdorff measure

We now recall the definitions of content and Hausdorff measure for the cases under consideration here. Recall we have fixed c_1, c_2, and r such that (vii) in the definition of an allowed pseudo-distance holds.

DEFINITION. Let E be any subset of $\mathbf{R}^n$ and $0 < \delta \leq r$. The *δ-content* of E is given by

$$\mathscr{C}_\delta(E) = \inf \sum_{B \in \mathscr{O}} |B|^{\delta/r},$$

where the infimum is taken over all countable covers $\mathscr{O} = \{B\}$ of E by open d-balls. If E is an analytic set, the *δ-dimensional Hausdorff measure* of E is given by

$$\mathscr{H}_\delta(E) = \lim_{s \to 0^+} \mathscr{C}_\delta^{(s)}(E),$$

where $\mathscr{C}_\delta^{(s)}(E)$ is defined similarly to $\mathscr{C}_\delta$ with the additional stipulation that the balls in the cover 0 all have radius no greater than s.

For general background concerning Hausdorff measures and content, see [Cn; §II], [Fr], [Ts; Chapter 3, §4], or [Ro]. The following collects some of these facts.

PROPOSITION 1. *Let $\mathscr{C}_\delta$ and $\mathscr{H}_\delta$ be the content and Hausdorff measure defined on $\mathbf{R}^n$ as above. Let E be a subset of $\mathbf{R}^n$. Then the following assertions are valid.*

(i) *$\mathscr{C}_\delta$ is an outer measure that is subadditive and outer regular in the sense that for each $\varepsilon > 0$, there exists an open set $0 \supset E$ such that $\mathscr{C}_\delta(0) < \mathscr{C}_\delta(E) + \varepsilon$.*

(ii) *$\mathscr{H}_\delta$ is an inner regular (infinite) measure with respect to which every analytic set is measurable.*

(iii) *A set E has $\mathscr{H}_\delta(E) = 0$ if and only if $\mathscr{C}_\delta(E) = 0$. (This follows from the Covering Theorem and the subadditivity of $t \to t^{\delta/r}$.)*

(iv) *$\mathscr{H}_\delta(E) = 0$ for a Borel set E if and only if there does not exist a nontrivial, positive, Borel measure ν with support on E such that for every d-ball $B_d(x, t)$, we have $\nu[B_d(x, t)] \leq |B_d(0, t)|^{\delta/r}$ (Frostman* [Fr]).

Generalized variation

We define next a notion of generalized variation which extends that considered by Wiener. This very general notion of variation may be viewed as a convenient formalism in which to give the proofs. Our primary interest is in applications to measures, functions of generalized variation in the more classical sense of Wiener, and the boundary behavior of integrals of the form

(1) defined relative to measures. In the remainder of this section, we will be considering functions $u\colon \mathbf{R}_+^{n+1} \to \mathbf{R}$.

DEFINITION. Let $\gamma \geq 1$. The *γ-variation* $\|u\| = \|u\|_\gamma$ of u is defined as

$$\sup \sqrt[\gamma]{\Sigma |u(x_j, t_j)|^\gamma}$$

where the supremum is taken over all sequences $\{(x_j, t_j)\}$ in $\mathbf{R}_+^{n+1}$ such that $\{B_d(x_j, t_j)\}$ is a mutually disjoint family of balls. In addition, we denote the class of all functions u of finite γ-variation by γBV, so that

$$\gamma BV = \{u\colon \|u\|_\gamma < \infty\}.$$

In the sequel, we shall assume $\gamma \geq 1$ is fixed, and sometimes suppress reference to this parameter by writing $\|u\|$ for $\|u\|_\gamma$. In case $\mu\colon \mathbf{R} \to \mathbf{R}$ and $u(x,t)$ is equal to $\operatorname{diam}(\mu[B_d(x,t)])$, $\mu(x+t) - \mu(x)$, or $\mu(x) - \mu(x-t)$, the class of functions μ for which $u \in \gamma BV$ is the class of functions that Wiener considered to be of bounded γ-variation. It is easy to show that any such function μ is regulated; that is, μ has both one-sided limits at each point. In particular, μ is bounded on any bounded interval with at most countably many discontinuities, each of these being a jump discontinuity.

Maximal functions and generalized derivatives

We assume that $\alpha \geq 1$, $0 < \beta < +\infty$, u is a real-valued function on $\mathbf{R}_+^{n+1}$, and Ω is an approach set as described in the second subsection.

DEFINITION. For each $x \in \mathbf{R}^n$, define the *maximal function*

$$\mathscr{M}_{\Omega,\beta} u(x) = \sup\left\{ \frac{|u(x+y,t)|}{|B_d(0,t)|^\beta} : (y,t) \in \Omega,\, t < 1 \right\}.$$

Similarly, we define the *lim sup maximal function*

$$\overline{\mathscr{M}}_{\Omega,\beta} u(x) = \limsup_{t\to 0} \left\{ \frac{|u(x+y,t)|}{|B_d(0,t)|^\beta} : (y,t) \in \Omega \right\},$$

Notation. (i) Let $\mu\colon \mathbf{R}^n \to \mathbf{R}$. Then $\overline{D}_{\Omega,\beta}\mu(x)$ denotes $\overline{\mathscr{M}}_{\Omega,\beta} u(x)$ when

$$u(x,t) = \operatorname{diam}(\mu[B_d(x,t)]).$$

In case $n = 1$, we use the notation $\overline{D}^r_{\Omega,\beta}\mu(x)$ and $\overline{D}^l_{\Omega,\beta}\mu(x)$ when $u(x,t)$ is equal to $\mu(x+t) - \mu(x)$ and $\mu(x) - \mu(x-t)$, respectively.

(ii) Let μ be the difference of two positive Borel measures on $\mathbf{R}^n$. Then we use the notation $\overline{D}_{\Omega,\beta}\mu(x)$ in connection with the function $u(x,t) = \mu[B_d(x,t)]$. In case $n = 1$, we write $\overline{D}^r_{\Omega,\beta}\mu(x)$ and $\overline{D}^l_{\Omega,\beta}\mu(x)$ when $u(x,t)$ is equal to $\mu[B_d(x,t) \cap \{y\colon y > x\}]$ and $\mu[B_d(x,t) \cap \{y\colon y < x\}]$, respectively.

(iii) In case Ω is the α-standard region (α, a)-$S(0,0)$, we replace the subscript Ω in the notation defined above with α. (Specific reference to a is suppressed.)

For $\mu\colon \mathbf{R}^n \to \mathbf{R}$, the upper derivatives $\overline{D}_{\alpha,\beta}\mu(x)$, $\overline{D}^r_{\alpha,\beta}\mu(x)$, and $\overline{D}^l_{\alpha,\beta}\mu(x)$ were considered in [Be1] in the case $n = \alpha = 1$ and $0 < \beta < 1$.

If μ is a positive, Borel measure on $\mathbf{R}$ and $d(x,y) = |x - y|$ is the usual metric on $\mathbf{R}$, then $\overline{D}_{\alpha,\beta}\mu(x)$, $\overline{D}^r_{\alpha,\beta}\mu(x)$, and $\overline{D}^l_{\alpha,\beta}\mu(x)$ denote the generalized upper symmetric, right-hand, and left-hand derivatives (with respect to t^β) considered by Samuelsson in [Sa], and by the first author in [Be2] in the case $n = \alpha = 1$.

We note that the notation established above conflicts with that used in [Be1] since there, the analogous notation would be given, for example, as $\overline{AD}_{\Omega,\beta}\mu(x)$ in place of $\overline{D}_{\Omega,\beta}\mu(x)$, to underline the point that these are "absolute" lim sup maximal functions.

3. Admissible sets

In this section we develop some more properties of d-admissible sets, and in particular, α-admissible sets. Recall that the latter arise when the allowed pseudo-distance d is replaced by d^α. Proposition 2 is a technical but very useful description of α-admissible sets which will enable us in §4 to overcome the nontrivial problem of transposing the Nagel-Stein machinery to the current context. It relies on Lemma 2 which we believe represents a new description of admissible sets which will be a helpful tool in future results.

Recall that we have fixed the notation K, c_1, c_2, τ, and r in relation to the allowed pseudo-distance d, and k is the constant associated by the Covering Theorem.

Definition. Let $\alpha \geq 1$ and $0 < \beta \leq 1/\alpha$. Let Ω be an approach set such that $\Omega(t)$ is a bounded, measurable set for each $t > 0$. Let $a > 0$. We say that Ω is *(α, β, a)-admissible* if there exists a positive constant A such that, for all $t > 0$, the set $\Omega_{\alpha,a}(t)$ can be covered by d-balls $\{B_d(x_i, t_i)\}$ and

$$\sum |B_d(x_i, t_i)|^{\alpha\beta} \leq A|B_d(0,t)|^\beta.$$

Notice that if Ω is (α, β, a)-admissible, then from the definition and the subadditivity of the function $t \to t^{\alpha\beta}$, we have

$$|\Omega_{\alpha,a}(t)|^{\alpha\beta} \leq \left(\sum |B_d(x_i, t_i)|\right)^{\alpha\beta} \leq A|B_d(0,t)|^\beta.$$

Thus,

$$|\Omega_{\alpha,a}(t)| \le A^{1/(\alpha\beta)}|B_d(0,t)|^{1/\alpha} \le A^{1/(\alpha\beta)}\left(\frac{c_2^{1/\alpha}}{c_1}\right)|B_\alpha(0,t)|,$$

which implies that Ω is α-admissible. The following result gives the converse.

PROPOSITION 2. *Let Ω be a subset of $\mathbf{R}_+^{n+1}$. Let $\alpha \ge 1$. Then Ω is α-admissible if and only if for some $a > 0$ and every $\beta \in (0, 1/\alpha]$, the set Ω is (α, β, a)-admissible.*

We shall use the following lemma in the proof.

LEMMA 2. *Let Ω be a d-admissible set. Fix $a > 0$. Then there exists a countable d-admissible set Ω', such that $\Omega'_{d,a}$ contains $\Omega_{d,a}$ and satisfies the following property: there exists a positive integer M such that for all $t > 0$, the set $\Omega'_{d,Ka}(t)$ is covered by*

$$\left(\bigcup(d, kKa)\text{-}S(z_j)\right)(t),$$

where the union is taken over a subset of $\{z_j\}$ of Ω' having cardinality less than M.

Proof of Lemma 2. For $z = (x,t) \in \Omega_{d,a}$, let R^z be the "rectangle" about the point (x,t) defined by

$$R^z = \left\{(w,s) \in \mathbf{R}_+^{n+1}\colon d(w,x) < at/4 \quad \text{and} \quad |s-t| < t/4\right\}.$$

We claim that for all points $(w,s) \in R^z$ we have $R^z \subseteq (d, Ka)\text{-}S(w, s/5)$. To show this, let $(u,q) \in R^z$. Using the triangle inequality for d, we have

$$\begin{aligned} d(u,w) &\le K[d(u,x) + d(x,w)] \le Ka\left(\frac{t}{4} + \frac{t}{4}\right) \\ &= Ka\left(\frac{3t}{4} - \frac{t}{4}\right) \\ &< Ka\left(q - \frac{s}{5}\right). \end{aligned}$$

The claim is thereby verified.

Let

$$\mathscr{F} = \left\{\omega\colon \omega \subset \Omega_{d,a},\, z_1, z_2 \in \omega,\, z_1 \ne z_2 \Rightarrow z_1 \notin R^{z_2} \quad \text{and} \quad z_2 \notin R^{z_1}\right\}.$$

This is a nonempty collection of subsets of $\Omega_{d,a}$ which are at most countably infinite. If we order $\mathscr{F}$ by inclusion, a straightforward argument using Zorn's lemma implies that $\mathscr{F}$ has a maximal element. Denote it by $\{z_j\}$ where $z_j = (x_j, t_j)$ for each j. Thus, by maximality,

$$\forall z \in \Omega_{d,a}\ \exists j \quad \text{such that either } z \in R^{z_j} \quad \text{or} \quad z_j \in R^{z}. \tag{2}$$

It follows from the claim in the previous paragraph that

$$\Omega_{d,a} \subset \bigcup_j (d, Ka)\text{-}S(x_j, t_j/5).$$

Let Ω' denote the sequence $\{(x_j, t_j/5)\}$. We shall show that Ω' satisfies the conclusion of the lemma.

We have already seen that $\Omega'_{d,Ka}$ contains $\Omega_{d,a}$. We show now that Ω' is d-admissible. Note that if $s \geq 2t_j$, then

$$(d,a)\text{-}S(x_j, t_j/5)(s) \subset (d, 9a/5) - S(x_j, t_j)(s).$$

Let $t > 0$. Then

$$\begin{aligned}
|\Omega'_{d,a}(t)| &= \left| \cup\left\{(d,a)\text{-}S(x_j, t_j/5)(t) \colon t_j/5 < t\right\}\right| \\
&\leq \left| \cup\left\{(d,a)\text{-}S(x_j, t_j/5)(10t) \colon t_j/5 < t\right\}\right| \\
&\leq \left| \cup\left\{(d, 9a/5)\text{-}S(x_j, t_j)(10t)\right\}\right| \\
&\leq |(\Omega_{d,a})_{d,9a/5}(10t)| \\
&\leq |\Omega_{d,9Ka/5}(10t)| \\
&\leq A_{d,9Ka/5}\tau(10)|B_d(0,t)|,
\end{aligned}$$

where $A_{d,9Ka/5}$ is defined as in (ii) of the definition of $(d, 9Ka/5)$-admissible. Thus Ω' is d-admissible.

By Lemma 1, there exists a positive constant D such that

$$|\Omega'_{d,Ka}(t)| \leq D|B_d(0,t)|, \ t > 0. \tag{3}$$

Choose $\varepsilon, \delta > 0$ such that $(1+\delta)/(1-\varepsilon) < 1 + 5/(8K^2)$ and $\varepsilon < 1/5$. Fix $t > 0$. Let

$$J_1 = \left\{j \colon 0 < t_j/5 \leq (1-\varepsilon)t\right\}$$

and

$$J_2 = \left\{j \colon (1-\varepsilon)t < t_j/5 < t\right\}.$$

For $j \in J_1 \cup J_2$, let $B_j = B_d(x_j, Ka(t - t_j/5))$. Observe that

$$\Omega'_{d,Ka}(t) = \bigcup\{B_j : j \in J_1 \cup J_2\}.$$

Consider first $\{B_j : j \in J_1\}$. By the Covering Theorem, there exists a finite subcollection $\{B_j : j \in J_1'\}$ of mutually disjoint balls such that

$$\bigcup\{B_j : j \in J_1\} \subset \bigcup\{B_j'' : j \in J_1'\},$$

where $B_j'' = B_d[x_j, kKa(t - t_j/5)]$. Since $t_j \leq 5(1 - \varepsilon)t$, each B_j for $j \in J_1'$ contributes to the t-section a measure of at least $|B_d(x_j, Ka\varepsilon t)|$. Thus if $\# J_1'$ denotes the cardinality of J_1', then

$$\begin{aligned} D|B_d(0,t)| \geq |\Omega'_{d,Ka}(t)| &\geq \# J_1' |B_d(x_j, Ka\varepsilon t)| \\ &\geq \# J_1'[\tau(1/(Ka\varepsilon))]^{-1}|B_d(0,t)| \end{aligned}$$

so that

$$\# J_1' \leq D\tau[1/(Ka\varepsilon)].$$

Consider now $\{B_j : j \in J_2\}$. Let $i, j \in J_2$. Then

$$|t_i - t_j| \leq 5t\varepsilon < 5t(1 - \varepsilon)/4 \leq t_j/4.$$

Since $(x_i, t_i) \notin R^{(x_j, t_j)}$, we have

$$d(x_i, x_j) \geq at_j/4 \geq 5a(1 - \varepsilon)t/4.$$

Let the lowest point of intersection of the closures of the sets (d, Ka)-$S(x_i, t_i/5)$ and (d, Ka)-$S(x_j, t_j/5)$ be denoted by (y, s). Then

$$\begin{aligned} 5a(1 - \varepsilon)t/4 \leq d(x_i, x_j) &\leq K\big(d(x_i, y) + d(x_j, y)\big) \\ &\leq K^2a\big((s - t_i/5) + (s - t_j/5)\big) \\ &\leq 2K^2a(s - (1 - \varepsilon)t), \end{aligned}$$

so that

$$s \geq (1 - \varepsilon)t[1 + 5/(8K^2)] > (1 + \delta)t.$$

Thus for all $j \in J_2$, the d-balls $B_d[x_j, Ka((1 + \delta)t - t_j/5)] \supset B_d(x_j, Ka\delta t)$

are disjoint and are contained in $\Omega'_{d,Ka}[(1+\delta)t]$. It follows that

$$\begin{aligned}\#J_2[\tau(1/(Ka\delta))]^{-1}|B_d(x_j,t)| &\le \#J_2\cdot|B_d(x_j,Ka\delta t)|\\ &\le |\Omega'_{d,Ka}[(1+\delta)t]|\\ &\le D|B_d[0,(1+\delta)t]|\\ &\le D\tau(1+\delta)|B_d(0,t)|,\end{aligned}$$

and so

$$\#J_2 \le D\tau(1+\delta)\tau[1/(Ka\delta)].$$

This completes the proof. ∎

Proof of Proposition 2. We prove only necessity since sufficiency was demonstrated following the definition of (α,β,a)-admissibility. Suppose Ω is α-admissible and let $\beta\in(0,1/\alpha]$. Let Ω' and M be as in Lemma 2 relative to Ω, a, and the pseudo-distance d^α. In the following argument, constants K_α, k_α and the function τ_α are those associated with d^α. Observe that the constants c_1 and c_2 of (vii) in the definition of an allowed pseudo-distance are the same for d and d^α. Let $t>0$. Then there exists a set of less than M elements of Ω', which we denote by $\{(x_j,t_j)\}$, such that

$$\begin{aligned}\Omega_{\alpha,a}(t)\subset\Omega'_{\alpha,K_\alpha a}(t) &\subset \bigcup B_\alpha[x_j,k_\alpha K_\alpha a(t-t_j)]\\ &= \bigcup B_d\left[x_j,\left(k_\alpha K_\alpha a(t-t_j)\right)^{1/\alpha}\right],\end{aligned}$$

and

$$\sum|B_\alpha[x_j,k_\alpha K_\alpha a(t-t_j)]|^{\alpha\beta} \le M\left(c_2c_1^{-1/\alpha}\tau_\alpha(k_\alpha K_\alpha a)\right)^{\alpha\beta}|B_d(0,t)|^\beta.$$

This completes the proof. ∎

We note in passing that Lemma 2 can be used to prove the following result. For brevity, we omit the proof.

Proposition 3. *Suppose that $\delta>0$ and d and e are pseudo-distances on $\mathbf{R}^n$ such that $e(x,y)\le d(x,y)$ whenever $x,y\in\mathbf{R}^n$ and $d(x,y)\le\delta$. Let Ω be a subset of $\mathbf{R}^{n+1}_+$ such that its projection onto $\mathbf{R}^n$ is a bounded set. If Ω is d-admissible, then Ω is e-admissible.*

4. Intermittent oscillation of functions of finite γ-variation

In this section we prove our principal result, Theorem 1, in which we combine some of the ideas of Samuelsson on growth with those of Nagel and Stein on admissible approach sets. The extension of the Nagel-Stein machinery to this context does not seem to us to be routine. It makes great use of Proposition 2 of §3.

We assume $\alpha, \gamma \geq 1$ and $0 < \beta < 1$. The significance of the parameters is that α determines the tangentiality of the approach sets, γ the class of functions of bounded generalized γ-variation under consideration, and β the test-function against which intermittent oscillation is measured. We continue to assume that K, τ, c_1, c_2, r, and k are fixed relative to the pseudo-distance d as in §2.

Due to Proposition 4 in §6, we are primarily interested in the case where $\alpha\beta\gamma \leq 1$. In order to state subsequent results, we establish the following terminology. A function u: $\mathbf{R}_+^{n+1} \to \mathbf{R}$ is said to *vanish eventually in the strip* $\mathbf{R}^n \times (0,1)$ provided there exists a d-ball B such that u vanishes on $(\mathbf{R}^n \setminus B) \times (0,1)$. Since our results are local in nature, there is no loss of generality in restricting to such functions.

The principal result of this paper is the following.

THEOREM 1. *Let α, β, γ satisfy the conditions $\alpha, \gamma \geq 1$ and $0 < \beta \leq 1/(\alpha\gamma)$. Put $\delta = \alpha\beta\gamma r$. Let Ω be an α-admissible subset of $\mathbf{R}_+^{n+1}$. Then there exists a positive constant C such that, for every function u: $\mathbf{R}_+^{n+1} \to \mathbf{R}$ which vanishes eventually in the strip $\mathbf{R}^n \times (0,1)$ and all numbers $\lambda > 0$, we have*

$$(4) \qquad \mathscr{C}_\delta\left[\left\{x \in \mathbf{R}^n : \mathscr{M}_{\Omega,\beta} u(x) > \lambda\right\}\right] \leq C \frac{\|u\|^\gamma}{\lambda^\gamma}.$$

Hence, if u is an element of γBV, then

$$(5) \qquad \mathscr{H}_\delta\left[\left\{x \in \mathbf{R}^n : \mathscr{M}_{\Omega,\beta} u(x) = \infty\right\}\right] = 0.$$

Note that (4) and (5) hold when $\mathscr{M}$ is replaced by $\overline{\mathscr{M}}$, since $\overline{\mathscr{M}}_{\Omega,\beta} u \leq \mathscr{M}_{\Omega,\beta} u$. With this replacement made, the theorem can be understood in terms of the intermittent oscillation of μ, where μ is either a real-valued function or a measure defined on the boundary $\mathbf{R}^n$. In this case, Theorem 1 holds with $\mathscr{M}_{\Omega,\beta} u$ replaced by $\overline{D}_{\Omega,\beta}\mu$. (See notation at the end of §2.)

We begin the proof with a special case.

LEMMA 3. *Let $a > 0$. Assume α, β, γ, and δ are as stated in Theorem 1. Then there exists a positive constant C such that for every function u: $\mathbf{R}_+^{n+1} \to \mathbf{R}$*

as in the theorem, we have

$$\mathscr{C}_\delta\left[\left\{x \in \mathbf{R}^n : \mathscr{M}_{\alpha,\beta}u(x) > \lambda\right\}\right] \le C\frac{\|u\|^\gamma}{\lambda^\gamma}. \tag{6}$$

Proof. For simplicity, we assume that $a = 1$. Let $H = \{x \in \mathbf{R}^n : \mathscr{M}_{\alpha,\beta}u(x) > \lambda\}$. Note that our assumption on u implies that H is bounded. For each point $x \in H$, there exists (y_x, t_x) such that $d(y_x, 0)^\alpha < t_x < 1$ and $u(x + y_x, t_x) > \lambda |B_d(0, t_x)|^\beta$. Now

$$\left\{B_d\left[x, K(d(y_x, 0) + t_x)\right] : x \in H\right\}$$

is a covering of H by open d-balls. By the Covering Theorem, we may choose a finite, pairwise disjoint subfamily of balls

$$\{B_i\} = \left\{B_d\left[x_i, K(d(y_i, 0) + t_i)\right]\right\}$$

such that

$$\{B_i'\} = \left\{B_d\left[x_i, kK(d(y_i, 0) + t_i)\right]\right\}$$

covers H. Since $B_d(x_i + y_i, t_i) \subset B_i$, the family $\{B_d(x_i + y_i, t_i)\}$ is pairwise disjoint. Thus

$$\begin{aligned}
\mathscr{C}_\delta(H) &\le \sum |B_i'|^{\alpha\beta\gamma} \le \sum \left|B_d\left(x_i, kK2t_i^{1/\alpha}\right)\right|^{\alpha\beta\gamma} \\
&\le \left(\tau(2kK)\left(c_2 c_1^{-1/\alpha}\right)\right)^{\alpha\beta\gamma} \sum |B_d(0, t_i)|^{\beta\gamma} \\
&\le \left(\tau(2kK)\left(c_2 c_1^{-1/\alpha}\right)\right)^{\alpha\beta\gamma} \lambda^{-\gamma} \sum u(x_i + y_i, t_i)^\gamma \\
&\le \left(\tau(2kK)\left(c_2 c_1^{-1/\alpha}\right)\right)^{\alpha\beta\gamma} \lambda^{-\gamma} \|u\|^\gamma.
\end{aligned}$$

This gives the required conclusion. ∎

The following is a general result concerning content for the present setting.

LEMMA 4. *Let $0 < \delta \le r$. There exists a positive constant c depending only on the constants of the pseudo-distance such that, for every $\varepsilon > 0$ and bounded subset H of $\mathbf{R}^n$, there is a sequence of d-balls $\{B_d(x_i, r_i)\}$ such that*

(i) $\sum |B_d(x_i, r_i)|^{\delta/r} < \mathscr{C}_\delta(H) + \varepsilon$,
(ii) $H \subset \bigcup B_d(x_i, cr_i)$, *and*
(iii) *for each i, we have $B_d(x_i, cr_i) \cap (\mathbf{R}^n \setminus H) \neq \varnothing$.*

Proof. Let $\varepsilon > 0$ and choose $c' > 2(K + K^3)$. By the outer regularity of $\mathscr{C}_\delta$, there exists an open set O containing H such that $\mathscr{C}_\delta(O) < \mathscr{C}_\delta(H) + \varepsilon/2$.

For each $y \in O$, let $s(y) = (2c')^{-1}d(y, \mathbf{R}^n \setminus O)$, where $d(y, \mathbf{R}^n \setminus O)$ denotes the distance of y to the complement of O. Note that for each $y \in O$, the ball $B_d[y, c's(y)]$ is contained in O. Then by the Covering Theorem, there exists a sequence of mutually disjoint d-balls $\{B_d(y_i, c's_i)\}$ (where we denote $s(y_i)$ by s_i) such that $O \subset \bigcup B_d(y_i, kc's_i)$. Notice that

$$B_d(y_i, 3c's_i) \cap (\mathbf{R}^n \setminus O) \neq \varnothing \quad \text{for each } i. \tag{7}$$

We shall construct the sequence $\{B_d(x_i, r_i)\}$ by taking a subset of $\{B_d(y_i, s_i)\}$ and another collection of balls $\{B_d(z_i, t_i)\}$ which will be defined after some preparatory discussion. Suppose that a ball $B_d(w, t)$ intersects each of two balls $B_d(y_i, s_i)$ and $B_d(y_j, s_j)$ for $i \neq j$. We assume, without loss of generality, that $s_i \geq s_j$. Select

$$w_i \in B_d(w, t) \cap B_d(y_i, s_i) \quad \text{and} \quad w_j \in B_d(w, t) \cap B_d(y_j, s_j).$$

Since $B(y_i, c's_i) \cap B(y_j, c's_j) = \varnothing$, we have

$$\begin{aligned} 2(K + K^3)s_i &< c's_i < d(y_i, y_j) \\ &\leq K\left[d(y_i, w_i) + K\left[d(w_i, w) + K\left[d(w, w_j) + d(w_j, y_j)\right]\right]\right] \\ &< (K + K^3)s_i + K^2(1 + K)t, \end{aligned}$$

so that

$$t > \frac{(K + K^3)s_i}{K^2(1 + K)} > \frac{s_i}{K},$$

using the fact that $K \geq 1$. From this we get

$$d(y_i, w) \leq K\left[d(y_i, w_i) + d(w_i, w)\right] < K(s_i + t) < K(K + 1)t.$$

It follows from the triangle inequality that

$$B_d(y_i, c'ks_i) \subset B_d\left[w, K^2(c'k + K + 1)t\right] \tag{8a}$$

and

$$B_d(y_i, 3c's_i) \subset B_d\left[w, K^2(3c' + K + 1)t\right]. \tag{8b)}$$

We shall now define the collection of balls $\{B_d(z_i, t_i)\}$. Let $U = \bigcup B_d(y_i, s_i) \subseteq O$. By the outer regularity of $\mathscr{C}_\delta$, we can select the sequence $\{B_d(z_i, t_i)\}$ of d-balls which covers U such that $\sum |B_d(z_i, t_i)|^{\delta/r} < \mathscr{C}_\delta(U) + \varepsilon/2$.

We select the sequence $\{B_d(x_i, r_i)\}$ from the collections $\{B_d(y_i, s_i)\}$ and $\{B_d(z_i, t_i)\}$ as follows. Of the collection $\{B_d(y_i, s_i)\}$, we keep only those balls $B_d(y_i, s_i)$ which are covered by a subset of $\{B_d(z_i, t_i)\}$, none of whose elements meet $B_d(y_j, s_j)$ for any $j \neq i$. Of the collection $\{B_d(z_i, t_i)\}$, we keep only those elements which meet at least two of the d-balls of $\{B_d(y_i, s_i)\}$. We denote the resulting set of d-balls which we keep by $\{B_d(x_i, r_i)\}$. Notice that if $B_d(y_i, s_i) \subset \bigcup B_d(z_{i_j}, t_{i_j})$, then since $0 < \delta/r \le 1$, we have

$$\sum \left|B_d(z_{i_j}, t_{i_j})\right|^{\delta/r} \ge \left(\sum \left|B_d(z_{i_j}, t_{i_j})\right|\right)^{\delta/r} \ge |B_d(y_i, s_i)|^{\delta/r}.$$

It follows that

$$\begin{aligned}\sum |B_d(x_i, r_i)|^{\delta/r} &\le \sum |B_d(z_i, t_i)|^{\delta/r} < \mathscr{C}_\delta(U) + \varepsilon/2 \\ &< \mathscr{C}_\delta(O) + \varepsilon/2 < \mathscr{C}_\delta(H) + \varepsilon.\end{aligned}$$

Let c be the maximum of $K^2(3c' + K + 1)$ and $K^2(c'k + K + 1)$. It follows from (8) that

$$H \subset O \subset \bigcup B_d(y_i, c'ks_i) \subset \bigcup B_d(x_i, cr_i).$$

We also conclude from (8) that for each i there exists j such that

$$B_d(x_i, cr_i) \cap (\mathbf{R}^n \setminus O) \supset B_d(y_j, 3c's_j) \cap (\mathbf{R}^n \setminus O),$$

which by (7) is nonempty. This completes the proof. ∎

Lemma 4 and Proposition 2 allow us to modify the technique by Sueiro in [Su2] to obtain the proof of Theorem 1. Lemma 4 replaces Sueiro's use of the Whitney decomposition.

Proof of Theorem 1. Let $\varepsilon > 0$ and $H = \{x \in \mathbf{R}^n : \mathscr{M}_{\alpha,\beta} u(x) > \lambda\}$. By our assumption on u, the set H is bounded. By Lemma 4, there exists a positive constant c depending only on the constants of the pseudo-distance, and a sequence of d-balls $\{B_d(x_i, r_i^{1/\alpha})\}$ such that

$$\sum \left|B_d(x_i, r_i^{1/\alpha})\right|^{\alpha\beta\gamma} < \mathscr{C}_\delta(H) + \varepsilon, \tag{9}$$

the collection $\{B_d(x_i, cr_i^{1/\alpha})\}$ covers H, and for each i, the ball $B(x_i, cr_i^{1/\alpha})$ has nonempty intersection with $\mathbf{R}^n \setminus H$.

Let $\mathscr{M}_{\Omega,\beta} u(x) > \lambda$. Then there exists a point $(y, t) \in \Omega$ such that

$$u(x + y, t) > \lambda |B_d(0, t)|^\beta.$$

For each point $z \in B_d(x+y, t^{1/\alpha})$, we have

$$\mathscr{M}_{\alpha,\beta}u(z) \geq \frac{u[z+(x+y-z),t]}{|B_d(0,t)|^\beta} = \frac{u(x+y,t)}{|B_d(0,t)|^\beta} > \lambda,$$

so that

$$B_d(x+y, t^{1/\alpha}) \subset H. \tag{10}$$

In particular, $x+y \in B_d(x_i, cr_i^{1/\alpha})$ for some i. If $d(z,x_i)^\alpha < c^\alpha r_i$, then

$$d(z,x+y)^\alpha \leq 2^{\alpha-1}K^\alpha\big(d(z,x_i)^\alpha + d(x_i,x+y)^\alpha\big) < 2^\alpha K^\alpha c^\alpha r_i.$$

Thus $B_d(x_i, cr_i^{1/\alpha})$ is a subset of $B_d(x+y, 2Kcr_i^{1/\alpha})$ but, according to (10) it is not a subset of $B_d(x+y, t^{1/\alpha})$. It follows that $t < 2^\alpha K^\alpha c^\alpha r_i$. Thus

$$d(x_i - x, y)^\alpha < c^\alpha r_i < (c^\alpha + 2^\alpha K^\alpha c^\alpha) r_i - t.$$

This implies that $(x_i - x, (1+2^\alpha K^\alpha)c^\alpha r_i) \in \Omega_\alpha$. We have thus shown

$$\{x : \mathscr{M}_{\Omega,\beta}u(x) > \lambda\} \subset \bigcup_i \{x_i - \Omega_\alpha[(1+2^\alpha K^\alpha)c^\alpha r_i]\}.$$

By Proposition 2, Ω_α is $(\alpha, \beta\gamma, 1)$-admissible. Thus

$$\begin{aligned}\mathscr{C}_\delta(\{x : \mathscr{M}_{\Omega,\beta}u(x) > \lambda\}) &\leq c'\sum |B_d(0,r_i)|^{\beta\gamma}\\ &\leq c'\left(\frac{c_2^{1/\alpha}}{c_1}\right)^{\alpha\beta\gamma} \sum |B_d(0, r_i^{1/\alpha})|^{\alpha\beta\gamma},\end{aligned}$$

where c' depends only on α and the constants associated with the pseudo-distance d. Thus, by (9), there exists a positive constant c independent of λ, u, and ε, such that

$$\mathscr{C}_\delta(\{x : \mathscr{M}_{\Omega,\beta}u(x) > \lambda\}) \leq c(\mathscr{C}_\delta(H) + \varepsilon).$$

Since ε is arbitrary, (4) follows from this and Lemma 3. We obtain (5) immediately. ■

The next result concerning the "lim sup" maximal function is an immediate corollary to the proof of Lemma 3. We do not know if this holds in general for $\overline{\mathscr{M}}_{\Omega,\beta}$.

COROLLARY. *In the setting of Lemma* 3, *we have*

$$\mathscr{H}_\delta\Big(\Big\{x \in \mathbf{R}^n : \overline{\mathscr{M}}_{\alpha,\beta}u(x) > \lambda\Big\}\Big) \le c\frac{\|u\|^\gamma}{\lambda^\gamma}.$$

To prove the corollary, the same proof can be used as in Lemma 3 except that for an arbitrarily prescribed number $s > 0$, we make the requirement that $t_x \le s$ for each $x \in H$. This is possible because a "lim sup" appears in the definition of $\overline{\mathscr{M}}_{\alpha,\beta}$ instead of the "sup" of $\mathscr{M}_{\alpha,\beta}$.

COROLLARY. *Consider the setting of Theorem* 1. *Let* ν *be a Borel measure such that, for every d-ball* $B_d(x,t)$, $\nu[B_d(x,t)] \le |B_d(0,t)|^{\alpha\beta\gamma}$. *Then*

$$\nu\Big(\Big\{x \in \mathbf{R}^n : \mathscr{M}_{\Omega,\beta}u(x) > \lambda\Big\}\Big) \le C\frac{\|u\|^\gamma}{\lambda^\gamma}.$$

Proof. Let K be a compact subset of $\{x : \mathscr{M}_{\Omega,\beta}u(x) > \lambda\}$ and let $\varepsilon > 0$. There exists a sequence of d-balls $\{B_d(x_i, r_i)\}$ which covers K such that $\sum |B_d(x_i, r_i)|^{\alpha\beta\gamma} < \mathscr{C}_\delta(K) + \varepsilon$. By the theorem,

$$\nu(K) \le \sum \nu[B_d(x_i, r_i)] \le \sum |B_d(x_i, r_i)|^{\alpha\beta\gamma} < \mathscr{C}_\delta(K) + \varepsilon \le C\frac{\|u\|^\gamma}{\lambda^\gamma} + \varepsilon.$$

As ε is arbitrary, the result follows. ■

More can be said in the case that we are working with measures on $\mathbf{R}^n$ (so that $\gamma = 1$).

THEOREM 2. *Let* $\alpha > 1$ *and* $0 < \beta \le 1/\alpha$. *Let* Ω *be an* α*-admissible subset of* $\mathbf{R}^{n+1}_+$. *Then, for every Borel measure* μ *on* $\mathbf{R}^n$, *we have*

$$\mathscr{H}_{\alpha\beta r}\Big(\Big\{x \in \mathbf{R}^n : \overline{D}_{\Omega,\beta}\mu(x) > 0\Big\}\Big) = 0.$$

Proof. Fix $\lambda > 0$. Let

$$E(d\mu, \lambda) = \Big\{x \in \mathbf{R}^n : \overline{D}_{\Omega,\beta}\mu(x) > \lambda\Big\}.$$

It suffices to show that $\mathscr{H}_{\alpha\beta r}[E(d\mu,\lambda)] = 0$. Let $\varepsilon > 0$. Let K be a compact subset of $E(d\mu,\lambda)$ and let ν be a measure with support in K such that

$$\nu[B_d(x,t)] \le |B_d(x,t)|^{\alpha\beta}, \; x \in \mathbf{R}^n, \, t > 0. \tag{11}$$

By Frostman's Theorem (Proposition 1 (iv)), it is enough to show that $\|\nu\| < c\varepsilon$, where c depends only on α and β. Let $d\mu = g\,d\nu + d\omega$ be the

Jordan decomposition of μ with respect to ν, where $g \in L^1(d\nu)$ and ω is singular with respect to ν. Let f be a continuous function such that $\|f - g\|_{L^1(d\nu)} < \varepsilon\lambda$. Since $\alpha > 1$, it follows easily from (11) that $E(fd\nu, \lambda/2) = \varnothing$. With C as in the second corollary of Theorem 1, we get

$$(12) \quad \nu[E(g\,d\nu, \lambda/2)] \le \nu[E((f-g)\,d\nu, \lambda/2)] \le \frac{C}{\lambda}\|f - g\|_{L^1(d\nu)} \le C\varepsilon.$$

Since ω is singular with respect to ν, there exists a set A such that $\omega(\mathbf{R}^n \setminus A) = 0$ and $\nu(A) = 0$. Let $L \subset A \subset U$ where L is compact, U is open, $\nu(U) < \varepsilon$, and $\omega(A \setminus L) < \varepsilon\lambda$. Let $\nu = \nu_1 + \nu_2$ and $\omega = \omega_1 + \omega_2$ where $\nu_1 = \nu|_U$, $\nu_2 = \nu|_{\mathbf{R}^n \setminus U}$, $\omega_1 = \omega|_L$, $\omega_2 = \omega|_{A \setminus L}$. Clearly $(\mathbf{R}^n \setminus U) \cap E(d\omega_1, \lambda/4) = \varnothing$. Again by the second corollary of Theorem 1, we get

$$\begin{aligned}(13) \quad \nu[E(d\omega, \lambda/2)] &= \nu_1[E(d\omega, \lambda/2)] + \nu_2[E(d\omega, \lambda/2)] \\ &\le \nu(U) + \nu_2[E(d\omega_1, \lambda/4)] + \nu_2[E(d\omega_2, \lambda/4)] \\ &\le \nu(U) + \frac{C}{\lambda}\|\omega_2\| \\ &< (1 + C)\varepsilon.\end{aligned}$$

It follows from (12) and (13) that

$$\begin{aligned}\|\nu\| = \nu(K) &\le \nu[E(d\mu, \lambda)] \le \nu[E(g\,d\nu, \lambda/2)] + \nu[E(d\omega, \lambda/2)] \\ &< (1 + 2C)\varepsilon.\end{aligned}$$

This completes the proof. ∎

5. Intermittent growth of functions defined by integral representations

In this section we apply our results to study the intermittent growth of functions of the form (1), where μ is a regular Borel measure on $\mathbf{R}^n$. The kernel $K\colon \mathbf{R}^n \times (0, \infty) \times \mathbf{R}^n \to [0, \infty)$ is assumed to be measurable in the last variable and satisfy the following additional conditions:

(i) $\int K(x, t, y)\,dy \to 1$ continuously as $(x, t) \to (x_0, 0)$ for each $x_0 \in \mathbf{R}^n$.

(ii) For all points $(x, t) \in \mathbf{R}^{n+1}$, we have

$$K(x, t, y) \le \frac{\phi[d(x, y)/t]}{|B_d(0, t)|},$$

where ϕ is a bounded, decreasing, real-valued function on $[0, \infty)$ for which $\sum \tau(2^{i+1})\phi(2^i) < \infty$.

(iii) For each point $x_0 \in \mathbf{R}^n$, open set W containing x_0, and $T \in (0, \infty]$, there exists a point $(y_0, s_0) \in \mathbf{R}^n \times (0, T)$ and open sets $U \supset V \ni x_0$ such that $U \subset W$ for all $x \in V$, $y \in \mathbf{R}^n \setminus U$, and for all t sufficiently close to 0, we have $K(x, t, y) \leq \delta(t) K(y_0, s_0, y)$ where $\delta(t) \to 0$ as $t \to 0^+$.

(iv) There exists a positive constant c such that for all $x \in \mathbf{R}^n$ and $t > 0$, we have

$$\int_{B(x,t)} K(x, t, y)\, dy > c.$$

Functions of this form were studied in [MS] where certain almost everywhere limit results were established. Such functions include positive harmonic functions and a wide class of positive parabolic functions on the upper half space as well as positive solutions of the heat equation on the right half space. These are described in detail in [MS]. In the next result we apply the results of §4 to functions representable in the form (1) to obtain Samuelsson-type results with respect to generalized Stein-Nagel regions allowing greater tangencies.

For Ω an approach set and $\beta > 0$, define

$$\mathscr{K}_{\Omega,\beta}\mu(x_0) = \sup\left\{ |B_d(0,t)|^{1-\beta} \mathscr{K}\mu(x_0 + x, t) : (x,t) \in \Omega,\, t < 1 \right\},$$

and

$$\overline{\mathscr{K}}_{\Omega,\beta}\mu(x_0) = \limsup_{t \to 0} \left\{ |B_d(0,t)|^{1-\beta} \mathscr{K}\mu(x_0 + x, t) : (x,t) \in \Omega \right\}.$$

THEOREM 3. *Let Ω be any approach region in $\mathbf{R}^{n+1}_+$ such that $\Omega(s) \subset \Omega(t)$ if $s < t$. Then there exists a positive constant C such that for any measure μ on $\mathbf{R}^n$ and $\beta > 0$, we have $\mathscr{K}_{\Omega,\beta}\mu(x_0) \leq C\mathscr{M}_{\Omega,\beta}\mu(x_0)$ and $\overline{\mathscr{K}}_{\Omega,\beta}\mu(x_0) \leq C\overline{\mathscr{M}}_{\Omega,\beta}\mu(x_0)$ for each $x_0 \in \mathbf{R}^n$ where $\mathscr{M}_{\Omega,\beta}\mu(x_0) < \infty$.*

By making use of the results of §4, the proof is similar to the proof of Lemma 2.5 in [MS], and we omit it. We thus obtain analogues of Theorem 1 along with its corollaries, and Theorem 2 for $\mathscr{K}$ and $\overline{\mathscr{K}}$.

6. Converse results

In this section we prove results converse to those given in §5. Since our primary interest is in measures and functions defined as in (1), we restrict ourselves to $\gamma = 1$, although analogous results are easily deduced by inserting suitable powers of γ. We begin by showing why we have restricted ourselves to the case where $\alpha\beta \leq 1$.

PROPOSITION 4. *Let $\alpha > 1$, $\beta > 0$, and $\alpha\beta > 1$. If Ω is α-admissible but not $1/\beta$-admissible, then there exists a finite measure μ such that for almost all $y \in \mathbf{R}^n$, we have $\overline{\mathscr{K}}_{\Omega_\alpha,\beta}\mu(y) = \infty$.*

Note that if we take Ω to be the normal N_0, then Ω_α is the α-standard region α-$S_d(0,0)$. This is, of course, α-admissible but not $1/\beta$-admissible.

Proof. Without loss of generality, we may restrict y to lie in a bounded, measurable subset F of $\mathbf{R}^n$. Since Ω is not $1/\beta$-admissible, there exists a sequence $\{t_i\}$ decreasing to 0 such that $|\Omega_{1/\beta}(t_i)|/|B_d(0, t_i^\beta)| \to \infty$ as $i \to \infty$. Note that $\Omega_{1/\beta}(t_i) \subset \Omega_\alpha(t_i)$ if $t_i < 1$. By passing to a subsequence, we may assume that there exists a sequence of positive numbers $\{M_i\}$ converging to ∞ such that

$$(14) \qquad \sum M_i \frac{|B_d(0,t_i)|^\beta}{|\Omega_\alpha(t_i)|} < \infty.$$

Let $[[\cdot]]$ denote the greatest integer function. According to the proof of Theorem 1.2 of [MPS2], there exist a measurable subset H of F with $|F \setminus H| = 0$ and, for each i, a finite set of points $\{x_{ij}\colon j = 1, \dots, [[|\Omega_\alpha(t_i)|^{-1}]]\}$ with the following property: for every point $y \in H$, there is an infinite set of indices i and j such that y is contained in the translate $x_{ij} - \Omega_\alpha(t_i)$ of $\Omega_\alpha(t_i)$. On each ball $B_d(x_{ij}, t_i)$, place Lebesgue measure normalized so that the total mass is $M_i|B_d(0, t_i)|^\beta$. Let μ denote the sum of these measures. By (14), the measure μ is finite. It is straightforward to check that for every point $y \in H$, we have $\overline{\mathscr{K}}_{\Omega_\alpha,\beta}\mu(y) = \infty$ as required. ∎

In the next result, we see that the exceptional set of Theorem 1 is of the correct size.

THEOREM 4. *Let $\alpha \geq 1$ and $0 < \beta \leq 1/\alpha$. Let $\mathscr{H}_\delta(E) = 0$ where $\delta = \alpha\beta r$. Then there exists a finite measure μ such that $E \subseteq \{x \in \mathbf{R}^n\colon \overline{\mathscr{K}}_{\alpha,\beta}\mu(x) = \infty\}$.*

Proof. Let i be a positive integer. By the Covering Theorem, there exists a sequence of mutually disjoint d-balls, $\{B_d(x_{ij}, r_{ij})\}_j$, such that $E \subset \bigcup_j B_d(x_{ij}, kr_{ij})$, $r_{ij} < 1/i$, and $\sum_j |B_d(x_{ij}, r_{ij})|^{\alpha\beta} < 2^{-i}$. Let μ_i be the measure defined by

$$\mu_i = \sum_j (3/2)^i |B_d(x_{ij}, r_{ij})|^{\alpha(\beta-1)} \mu_{ij},$$

where μ_{ij} is the restriction of Lebesgue measure to $B_d(x_{ij}, k^\alpha r_{ij}^\alpha)$. Put $\mu = \sum \mu_i$. Then μ is a finite measure. Let $x \in E$. For each i there exists an index j such that $x \in B_d(x_{ij}, kr_{ij})$. Putting $y = x_{ij} - x$ and $t = k^\alpha r_{ij}^\alpha$, we have $d(y,0)^\alpha < t < 1/i$, and

$$\begin{aligned}
|B_d(0,t)|^{1-\beta}\mathscr{K}\mu(x+y,t) &= \left|B_d(0,k^\alpha r_{ij}^\alpha)\right|^{1-\beta}\mathscr{K}\mu(x_{ij},k^\alpha r_{ij}^\alpha)\\
&\geq \left|B_d(0,k^\alpha r_{ij}^\alpha)\right|^{1-\beta}\int K(x_{ij},k^\alpha r_{ij}^\alpha,y)\,d\mu_i(y)\\
&\geq |B_d(0,t)|^{1-\beta}(3/2)^i|B_d(x_{ij},r_{ij})|^{\alpha(\beta-1)}\\
&\quad\times\int_{B_d(x_{ij},t)} K(x_{ij},t,y)\,dy\\
&\geq c(3/2)^{i\gamma}\left(c_1c_2^{-\alpha}k^{\alpha r}\right)^{1-\beta},
\end{aligned}$$

where c is defined in condition (iv) of §5. It follows that

$$\overline{\mathscr{K}}_{\alpha,\beta}\mu(x) \geq (3/2)^i\left(c_1c_2^{-\alpha}k^{\alpha r}\right)^{(1-\beta\gamma)/\gamma}.$$

Letting $i \to \infty$, we see that $\overline{\mathscr{K}}_{\alpha,\beta}\mu(x) = \infty$. ■

In the next result we show that one can only obtain maximal inequalities as in the second corollary of Theorem 1 if the approach region is α-admissible.

THEOREM 5. *Let α and β satisfy $\alpha \geq 1$ and $0 < \beta \leq 1/\alpha$. Let Ω be an approach set that is not α-admissible. Then there exists a Borel measure ν and a sequence $\{\mu_i\}$ of finite measures such that for every d-ball $B_d(x,t)$, we have*

$$\nu[B(x,t)] \leq |B_d(x,t)|^{\alpha\beta}$$

and

$$\lim_{i\to\infty}\|\mu_i\|^{-1}\nu\left[\left\{x \in \mathbf{R}^n : \mathscr{K}_{\Omega_\alpha,\beta}\mu_i(x) > 1\right\}\right] = \infty.$$

Proof. Fix $x_0 \in \mathbf{R}^n$. It follows from the fact that Ω is not α-admissible that there exists a sequence $\{t_i\}$ decreasing to 0 and a collection of d-balls $\{B_{ij}\}$ such that, for every i, the family $\{B_{ij}\}_j$ is mutually disjoint and is contained in the set $x_0 - \Omega_\alpha(t_i)$,

$$|B_d(0,t_i)|^{-\beta}\left(\sum_j |B_{ij}|\right)^{\alpha\beta} > i^3,$$

and

$$\sum_j |B_{ij}| \geq \frac{1}{\tau(k)} |\Omega_\alpha(t_i)|,$$

where k is as in the Covering Theorem.

Let μ_i be the measure obtained by placing normalized Lebesgue measure on the ball $B_d(x_0, t_i)$ of total mass $c^{-1}|B_d(0, t_i)|^\beta$, c as in (iv) of §5. Then for every point $x \in x_0 - \Omega_\alpha(t_i)$, we have

$$\mathscr{K}_{\Omega_\alpha, \beta}\mu_i(x) \geq |B_d(0, t_i)|^{1-\beta} \int_{B_d(x_0, t_i)} K(x_0, t_i, y)\, d\mu_i(y) > 1.$$

Let ν_i be a multiple of Lebesgue measure on $\bigcup_j B_{ij}$ normalized so that $\|\nu_i\| = i^{-2}(\sum_j |B_{ij}|)^{\alpha\beta}$. Then for every d-ball $B_d(x, t)$,

$$\begin{aligned}
\nu_i[B_d(x, t)] &\leq i^{-2}\Big(\sum_j |B_{ij}|\Big)^{\alpha\beta - 1} \sum_j |B_d(x, t) \cap B_{ij}| \\
&\leq i^{-2}\Big(\sum_j |B_d(x, t) \cap B_{ij}|\Big)^{\alpha\beta} \\
&\leq i^{-2}|B_d(x, t)|^{\alpha\beta}.
\end{aligned}$$

Let $\nu = \Sigma(6/\pi^2)\nu_i$. Then $\nu[B_d(x, t)] \leq |B_d(x, t)|^{\alpha\beta}$. Finally,

$$\begin{aligned}
\|\mu_i\|^{-1}\nu\Big(\big\{x \in \mathbf{R}^n \colon \mathscr{K}_{\Omega_\alpha, \beta}\mu_i(x) > 1\big\}\Big) &\geq \|\mu_i\|^{-1}\nu_i\Big(\bigcup_j B_{ij}\Big) \\
&\geq ci^{-2}|B_d(0, t_i)|^{-\beta}\Big(\sum_j |B_{ij}|\Big)^{\alpha\beta} \\
&\geq ci.
\end{aligned}$$

This establishes the theorem. ■

In the following result, we let d denote the usual Euclidean distance on the real line. In case α, β, and n are all equal to 1, it was proved in [MPS2] for a general non-α-admissible set Ω. In the present context, we can prove the result in case the sections, $\Omega(t_j)$, are all intervals.

THEOREM 6. *Suppose that Ω is an open approach set in $\mathbf{R}^2_+$ bounded by the vertical axis $N_0 = \{(0, t)\colon t > 0\}$ and a nondecreasing curve in the first quadrant passing through the origin. Suppose that there exists a sequence $\{t_i\}$ decreasing to 0 such that $|\Omega(t_i)|/t_i^{1/\alpha} \to \infty$ as $i \to \infty$. Then, for all values of* *the parameters α and β with $\alpha \geq 1$ and $0 < \beta \leq 1/\alpha$, there exists a finite measure μ such that $\overline{\mathscr{K}}_{\Omega,\beta}\mu(x) = \infty$ on a set of positive $\alpha\beta$-Hausdorff measure.*

Proof. By assumption, for each positive integer i, the set $\Omega(t_i)$ is an interval. Let us denote its measure by Ω_i. Define $\Omega_0 = 1$. By passing to a subsequence, we may assume without loss of generality that there exists a sequence $\{M_i\} \to \infty$ as $i \to \infty$ such that

$$\sum M_i \Omega_i^{-\alpha\beta} t_i^{\beta} < \infty, \tag{15}$$

and $\Omega_i/\Omega_{i-1} \to 0$ as $i \to \infty$. To simplify notation, we shall assume that $\{(\Omega_{i-1}/\Omega_i)^{\alpha\beta}: i \geq 1\}$ is a sequence of integers. The changes necessary for the general case are easily made.

Partition the interval $[0, 1]$ by equally spaced points which divide it into intervals $I_{1,j}$, $j = 1, \ldots, \Omega_1^{-\alpha\beta}$. Divide each $I_{1,j}$ into two subintervals, the left one denoted by J_{1j} such that $|J_{1,j}| = \Omega_1$. Let $f_1(t)$ be the unique continuous function on $[0, 1]$ such that $f_1(0) = 0$ and $f_1'(t) = \Omega_1^{\alpha\beta-1}$ for all $t \in \bigcup J_{1,j}$, with f_1 constant otherwise.

Suppose now that for each $i \geq 2$, we have defied $\{I_{i-1,j}\}$, $\{J_{i-1,j}\}$, and $f_{i-1}(t)$, for $i = 1, \ldots, \Omega_{i-1}^{-\alpha\beta}$. For each i, partition $J_{i-1,j}$ so as to divide it into $(\Omega_{i-1}/\Omega_i)^{\alpha\beta}$ subintervals each of length $\Omega_{i-1}^{1-\alpha\beta}\Omega_i^{\alpha\beta}$. Call this collection of intervals $\{I_{i,j}: j = 1, \ldots, \Omega_i^{-\alpha\beta}\}$. Divide each $I_{i,j}$ into two subintervals, the left denoted by $J_{i,j}$ such that $|J_{i,j}| = \Omega_i$. Let f_i be the unique continuous function on $[0, 1]$ such that $f_i(0) = 0$ and $f_i'(t) = \Omega_i^{\alpha\beta-1}$ for all $t \in \bigcup_j J_{i,j}$, with f_i constant otherwise. Then $\{f_i\}$ is a nondecreasing sequence of nondecreasing functions. Let $f(t)$ denote the limit function.

For each pair of positive integers i, j, where $j = 1, \ldots, (\Omega_{i-1}/\Omega_i)^{\alpha\beta}$, define

$$g_{ji}(t) = \left(t - (j-1)\Omega_{i-1}^{1-\alpha\beta}\Omega_i^{\alpha\beta}\right)^{\alpha\beta} + (j-1)\Omega_i^{\alpha\beta}$$

for

$$(j-1)\Omega_{i-1}^{1-\alpha\beta}\Omega_i^{\alpha\beta} \leq t \leq j\Omega_{i-1}^{1-\alpha\beta}\Omega_i^{\alpha\beta}.$$

We claim that $t^{-\alpha\beta}g_{ji}(t)$ is bounded, independent of j and i. Elementary calculus shows that $t^{-\alpha\beta}g_{ji}(t)$ is nondecreasing. Thus

$$\begin{aligned}
\frac{g_{ji}(t)}{t^{\alpha\beta}} &\leq j^{-\alpha\beta}\left[1 + (j-1)\left(\frac{\Omega_i}{\Omega_{i-1}}\right)^{\alpha\beta(1-\alpha\beta)}\right] \\
&\leq \max\left\{x^{-\alpha\beta}\left[1 + (x-1)\left(\frac{\Omega_i}{\Omega_{i-1}}\right)^{\alpha\beta(1-\alpha\beta)}\right] : 1 \leq x \leq \left(\frac{\Omega_{i-1}}{\Omega_i}\right)^{\alpha\beta}\right\}.
\end{aligned}$$

Again by elementary calculus, this maximum occurs at $x = 1$, $x = (\Omega_{i-1}/\Omega_i)^{\alpha\beta}$, or, in case $\alpha\beta < 1$,

$$x = \left(\frac{\alpha\beta}{1-\alpha}\right)\left[\left(\frac{\Omega_{i-1}}{\Omega_i}\right)^{\alpha\beta(1-\alpha\beta)} - 1\right].$$

A simple computation shows the value of $t^{-\alpha\beta}g_{ik}(t)$ at each of these points is bounded by a constant depending only on α and β. This proves the claim.

We show now that $t^{-\alpha\beta}f_i(t)$ is bounded on $0 \le t \le 1$ independent of i. By symmetry, $t^{-\alpha\beta}f_i(t)$ is bounded on the interval $0 \le t \le \Omega_1^{\alpha\beta}$ if and only if $f_i(t)/g_{j1}(t)$ is bounded on the interval $(j-1)\Omega_1^{\alpha\beta} \le t \le j\Omega_1^{\alpha\beta}$. Since $t^{-\alpha\beta}g_{j1}$ is bounded independent of j on the latter interval, it thus suffices to show that $t^{-\alpha\beta}f_i(t)$ is bounded on the interval $0 \le t \le \Omega_1^{\alpha\beta}$. By a similar symmetry argument, it suffices to show that $t^{-\alpha\beta}f_i(t)$ is bounded on the interval $0 \le t \le \Omega_1^{1-\alpha\beta}\Omega_2^{\alpha\beta}$. Repeating this argument i times, we see it is enough to check that $t^{-\alpha\beta}f_i(t)$ is bounded on the interval $0 \le t \le \Omega_{i-1}^{1-\alpha\beta}\Omega_i^{\alpha\beta}$, and, of course, by construction and the concavity of the mapping $t \to t^{\alpha\beta}$, this is bounded by 1.

Let

$$E = \bigcap_i \bigcup_j \left\{J_{j,i}\colon j = 1,\dots,\Omega_i^{-\alpha\beta},\, i \ge 1\right\}.$$

Since $t^{-\alpha\beta}f(t)$ is bounded, we have by Frostman's theorem (Proposition 1 (iv)) that $\mathscr{H}_{\alpha\beta}(E) > 0$. Let μ_i be the measure obtained by placing the mass $M_i t_i^\beta$ uniformly along the interval of radius t_i centered at the right endpoint of each of the intervals $J_{i,j}$, $j = 1,\dots,\Omega_i^{-\alpha\beta}$. Let $\mu = \Sigma\mu_i$. Then by (15), μ is finite and, by construction and the fact that each Ω_i is an interval, $\mathscr{K}_{\Omega,\beta}\mu(x) = \infty$ at each point of E. ■

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