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A PRESERVATION PRINCIPLE OF EXTREMAL MAPPINGS NEAR A STRONGLY PSEUDOCONVEX POINT AND ITS APPLICATIONS

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Introduction

Let D be a bounded domain in $\mathbf{C}^n$ and Δ the unit disc in $\mathbf{C}^1$. An extremal mapping (respectively, a complex geodesic) ϕ of D is a holomorphic mapping from Δ to D such that the Kobayashi metric of D at $\phi(0)$ and in the direction $\phi'(0)$ (respectively, the Kobayashi distance between any two points on $\phi(\Delta)$) is realized by ϕ. An obvious fact is that all complex geodesics are extremal.

In the one dimensional case, ϕ is extremal if and only if ϕ gives a covering mapping from Δ to D. In 1981, Lempert [Lm1] systematically studied the extremal mappings of a strongly convex domain. He proved that every extremal mapping of a C^k-strongly convex domain is actually a complex geodesic and admits a C^{k-2}-smooth extension up to the boundary ($k > 2$). As applications, he obtained the precise form of Fefferman's mapping extension theorem and the solutions of some types of Monge-Ampere equations [Lm1], [Lm2]. In [RW], some of Lempert's results were generalized to bounded convex domains. For non-convex domains, the abstract nature of the Kobayashi metric makes things more subtle. A simple investigation of the covering mappings of an annulus indicates immediately that (1) the extremal mappings may not be complex geodesics anymore and (2) the boundary behavior of extremal mappings may be very complicated although the domain is analytic and strongly pseudoconvex. In 1983, Poletskii [P] showed that the extremal mappings of a ρ-pseudoconvex domain must be almost proper and satisfy the Euler-Lagrange equations if the domain has in addition C^1 boundary. This result gives a very strong restriction for a holomorphic mapping to be extremal, and was later on used in some papers to characterize such mappings.

One purpose of this note is to present a preservation principle for extremal mappings near a C^3-strongly pseudoconvex point. Roughly speaking, we show that an extremal mapping with the initial point close to the bottom of a strongly pseudoconvex hole and with the initial velocity almost parallel to the

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bottom tangent space must stay completely in the hole. In other words, we show that extremal mappings can only wind around the strongly pseudoconvex boundary in the complex normal directions. As an application of this principle, we obtain the Lipschitz-1 continuity of a complex geodesic near a C^3-strong pseudoconvex point. This improves, in some sense, a result in [Lm1] and [Ab2], where only Hölder-1/2 continuity was obtained for the complex geodesics of a C^2-strongly pseudoconvex domain. Another purpose of this paper is to study the boundary version of a uniqueness theorem for holomorphic self-mappings, by making use of iteration theory and the aforementioned regularity results of complex geodesics, especially the Lipschitz-1 continuity of an arbitrary holomorphic retract near a strongly pseudoconvex point.

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1. Statement of theorems and some related observations

We first introduce the following notation.

Let D be a bounded domain in $\mathbf{C}^n$ with p a C^2-smooth point. We recall that for every $z \in D$, close enough to p, there is a unique point on ∂D, denoted by $\pi(z)$, so that distance$(z, \pi(z)) = \delta(z)$, the distance between z and the boundary of D. In this case, for every $\xi \in T^{(1,0)}D$, we use $(\xi)_{tn}$ and $(\xi)_{nr}$ to denote the complex tangential and complex normal components of ξ at $\pi(z)$, respectively.

For a bounded pseudoconvex domain D in $\mathbf{C}^n$, we say that it has a Stein neighborhood basis if there exists a sequence of bounded pseudoconvex domains $\{D_v\}$ in $\mathbf{C}^n$ so that $D_1 \supset\supset D_2 \supset\supset D_3 \supset\supset \cdots \supset\supset D$ and $\cap_v D_v = D$. It is well known that every bounded domain defined by a C^1 plurisubharmonic function (in particular, every bounded C^2-strongly pseudoconvex domain) has a Stein neighborhood basis.

THEOREM 1. *Let $D \subset\subset \mathbf{C}^n$ $(n > 1)$ be either a pseudoconvex domain with a Stein neighborhood basis or a pseudoconvex domain with C^∞ boundary. Suppose that $p \in \partial D$ is a strongly pseudoconvex point of D with at least C^3 smoothness. Then for every open neighborhood V of p, there is a positive number ε such that for each extremal mapping ϕ of D, when $|\phi(0)$-$p| < \varepsilon$ and $|(\phi'(0))_{nr}| < \varepsilon|(\phi'(0))_{tn}|$, then ϕ is the complex geodesic of D and $\phi(D) \subset V$.*

COROLLARY 1. *Let $D \subset \mathbf{C}^n$ be a C^3 strongly pseudoconvex domain. For every $\varepsilon > 0$, there is an $\eta > 0$ such that if ϕ is an infinitesimal complex* *geodesic with* $\delta(\phi(\tau))) < \eta$ *and* $|\phi'_{nr}(\tau)| < \eta|\phi'_{tn}(\tau)|$ *for some* $\tau \in \Delta$, *then the diameter of* $\phi(\Delta)$ *is less than* ε. *Here, we recall that a holomorphic mapping from* Δ *to* D *is said to be an infinitesimal complex geodesic of* D *if it realizes the Kobayashi metric of* D *at any pair* $(\phi(\tau), \phi'(\tau))$ *with* $\tau \in \Delta$.

COROLLARY 2. *Let* D *and* p *as in Theorem* 1. *Suppose that* ϕ *is a complex geodesic of* D. *If there is a sequence* $\{\tau_k\} \subset \Delta$ *converging to* 1, *such that* $\phi(\tau_k) \to p$, *then* ϕ' *is bounded near* $1 \in \partial\Delta$. *Thus* ϕ *admits a Lipschitz*-1 *continuous extension near* 1.

THEOREM 2. *Let* $D \subset\subset \mathbf{C}^2$ *be either a simply connected taut domain with a Stein neighborhood basis or a simply connected pseudoconvex domain with* C^∞ *boundary. Suppose that* $p \in \partial D$ *is a strongly pseudoconvex point with at least* C^3 *smoothness. If* f *is a non-identical holomorphic self-mapping of* D *so that*

$$f(z) = z + o\big((z-p)^k\big) \quad \text{as } z \to p,$$

then:

(1) $k < 3$.

(2) *If* $k = 1$, *then either* $\{f^m\}$ *converges compactly to* p *or the fixed point set of* f *is a one dimensional holomorphic retract passing through* p. *In case* D *is not biholomorphic to the ball*, f *cannot be an automorphism*.

(3) *If* $k = 2$, *then* f *can not be an automorphism of* D *and* $\{f^m\}$ *must converge compactly to* p.

COROLLARY 3. *Let* D *and* p *be as in Theorem* 2. *Suppose that* $f \in \mathrm{Hol}(D, D)$ *is such that* $f(z_0) = z_0$ *for some* $z_0 \in D$ *and* $f(z) = z + o((z-p)^2)$ *as* $z \to p$. *Then* $f \equiv \mathrm{id}$.

We do not know whether the C^2-smoothness at p suffices for Theorem 1. However, the strong pseudoconvexity assumption for p is necessary as the following example demonstrates.

Example 1. Let E_k be the egg domain defined by $\mathrm{E_k} = \{(\mathrm{z_1}, \mathrm{z_2})\colon |\mathrm{z_1}|^2 + |\mathrm{z_2}|^{2\mathrm{k}} < 1\}$ for $k \geq 2$ and let $p = (1, 0)$ (we note that when $k = 1$, E_1 reduces to the unit 2-ball). Then E_k is an analytic strictly convex domain with the boundary point p of type $2k$, where by a strictly convex domain we mean a convex domain whose boundary contains no segment. For each $j > 1$, choose a biholomorphism σ_j from Δ to the disk $\{z \in \mathbf{C}^1\colon |z|^2 + |1 - z|^2 < 1\}$ with $\sigma_j(0) = 1 - 1/j$ and $\sigma_j(1) = 1$. We then claim that $\phi_j \triangleq (\sigma_j, (1 - \sigma_j)^{1/k})$ is a complex geodesic of E_k for each j. In fact, this follows easily from the following two facts and the monotonicity property of the Kobayashi distance.

(i) ϕ_j is a proper holomorphic mapping from Δ to E_k.

(ii) Let π be the standard proper covering mapping from E_k to the unit 2-ball B_2 ($= E_1$), i.e., $\pi(z_1, z_2) = (z_1, z_2^k)$. Then $\pi \circ \phi_j$ is a complex geodesic of B_2, for $\pi \circ \phi_j$ is a biholomorphism from Δ to the intersection of B_2 with the complex line $w_1 + w_2 = 1$ (see [Ab1] for more on this matter).

Meanwhile, it is easy to see that $\phi_j(0) \to p$ and

$$\left((\phi_j')_{jr}\right)/((\phi_n')_{tn})(0) \sim k\left(1 - \sigma_j(0)\right)^{1-1/k} \to 0.$$

However, $\phi_j(\Delta)$ does not reduce to p as $j \to \infty$. By the way, this example also shows the importance of the strong pseudoconvexity in Corollary 1 and Corollary 2.

Corollary 2 is obviously false for the general extremal mappings even in the one dimensional case. Actually, all universal covering mappings of the annulus are infinitesimal complex geodesics, but are not continuous up to the boundary.

Theorem 2 can be viewed as a boundary version of the classical Cartan theorem. The case (1) is the local version of the Burns-Krantz theorem (see [BK] and [H]). For the disk in $\mathbf{C}^1$, as noted in [Lm3], the exponent in Corollary 3 can be reduced to just 1. However, the following examples show that the situation in the higher dimensional case is different and our result is actually quite sharp:

Example 2. Let

$$\sigma(z_1, z_2) = \left(\frac{(1-2i)z_1 - 1}{z_1 - 1 - 2i}, \frac{-2iz_2}{z_1 - 1 - 2i}\right) \text{ for } (z_1, z_2) \in B_2.$$

Then $\sigma \in \mathrm{Aut}(B_2)$ with $\sigma(p) = \sigma'(p) = 1$, where $p = (,1\,0)$. But $\sigma \neq \mathrm{id}$.

Example 3. Let D be a bounded strongly pseudoconvex domain defined by

$$D = \left\{(z_1, z_2) \in \mathbf{C}^2 \colon |z_1|^2 + h\left(|z_2|\right) < 1\right\}$$

for some smoothly increasing function $h(\cdot)$ with $h(0) = 0$. Denote by p the boundary point $(1, 0)$. Define $f(z_1, z_2) = (z_1, z_1 z_2)$. Then f fixes the holomorphic retract of D: $\{(z_1, 0): |z_1| < 1\}$, and $f(z) = z + o(|z-p|)$ as $z \to p$, but $f \neq \mathrm{id}$.

Example 4. Let $B_2 = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit 2-ball and let $p = (1, 0)$. For every $a > 0$, define a holomorphic mapping f_a from D to $\mathbf{C}^2$ by

$$f_a(z_1, z_2) = \left(\frac{z_1 + a(1-z_1)^2}{1 + a(1-z_1)^2}, \frac{z_2}{1 + a(1-z_1)^2} \right).$$

Then it is easy to check that f_a is a self-mapping and $f_a(z) = z + O(|z-p|^3)$ as $z \to p$. By Theorem 2, $\{f^k\}$ converges compactly to p.

2. Proof of Theorem 1

The purpose of this section is to present the proofs of Theorem 1, Corollary 1, and Corollary 2. Our idea is to make use of the C^k-version of the reflection principle to get the uniform Hölder continuity of the differentials of a sort of 'normalized' complex geodesics on strongly convex domains. We then apply it with the Fornaess embedding theorem and the Graham estimates of the Kobayashi metric to obtain our results.

In what follows, we fix the symbol $\langle \cdot , \cdot \rangle$ for the standard Hermitian inner product in $\mathbf{C}^n$ and the symbol $|\cdot|$ for the corresponding euclidean norm. For two domains D_1 and D_2, $\mathrm{Hol}(D_1, D_2)$ stands for the set of all holomorphic mappings from D_1 to D_2. When $f \in \mathrm{Hol}(D_1, D_2)$ with $D_1 = D_2$, we denote by f^m the m^{th}-iterate of f defined inductively by $f^1 = f, \ldots, f^m = f \circ f^{m-1}$.

For a bounded domain D in $\mathbf{C}^n$, denote by K_D the Kobayashi distance and by κ_D the Kobayashi metric of D (see [Kr2] for the definitions). We recall that $\phi \in \mathrm{Hol}(\Delta, D)$ is said to be a complex geodesic (respectively, an extremal mapping) of D if

$$K_D(\phi(\tau_1), \phi(\tau_2)) = K_\Delta(\tau_1, \tau_2)$$

for every pair $\tau_1, \tau_2 \in \Delta$ [Ve] (respectively, $\kappa_D(\phi(0), \phi'(0)) = \kappa_\Delta(0, 1) = 1$).

LEMMA 1. *Let $D_1 \subset D_2$ be two bounded domains in $\mathbf{C}^n$. If ϕ is a complex geodesic (respectively, an extremal mapping) of D_2 such that $\phi(\Delta) \subset D_1$, then ϕ is also a complex geodesic (respectively, an extremal mapping) of D_1.*

Proof. This follows immediately from the monotonicity property of the Kobayashi metric and the Kobayashi distance. □

Let D be a C^1-smoothly bounded domain in $\mathbf{C}^n$. Then for every $p \in \partial D$, we may define the outward unit normal vector of ∂D at p, denoted by $\nu(p)$. When $D \subset\subset \mathbf{C}^n$ is a bounded C^k-strongly convex domain ($k > 2$), Lempert in [Lm1] showed that a holomorphic mapping ϕ from Δ to D is an extremal mapping (or complex geodesic) of D if and only if it is proper and there exists a (unique) C^{k-2}-smooth function P_ϕ: $\partial\Delta \to \mathbf{R}^+$ so that the vector function $\xi P_\phi(\xi)\overline{\nu(\phi(\xi))}$, initially defined on $\partial\Delta$, can be extended to a holomorphic vector function $\tilde{\phi}$ on Δ (which is called the dual mapping of ϕ) with $\langle \phi', \tilde{\phi} \rangle \equiv 1$. The following lemma is an obvious consequence of this characterization:

LEMMA 2. *Let $D_1 \subset D_2$ be two bounded C^3-strongly convex domains in $\mathbf{C}^n$. Suppose that $\partial D_1 \cap \partial D_2$ is a piece of hypersurface. If $\phi \in \mathrm{Hol}(\Delta, D_1)$ is a complex geodesic of D_1 so that $\phi(\partial\Delta) \subset \partial D_2$, then ϕ is also a complex geodesic of D_2.* $\square$

In the next two lemmas, we assume D $\subset\subset \mathbf{C}^n$ to be a C^3-strongly convex domain.

For each $a > 0$, let $\mathscr{F}_a$ denote the set of all complex geodesics ϕ of D which satisfy $\delta(\phi(0)) \geq a$. From §7 of [Lm1], we see that there exist two positive constants C_0 and C_0', depending only on D and a, so that for every $\phi \in \mathscr{F}_a$, the following hold:

(1.a) $|\phi(\tau_1) - \phi(\tau_2)| < C_0|\tau_1 - \tau_2|^{1/2}$, $|\tilde{\phi}(\tau_1) - \tilde{\phi}(\tau_2)| < C_0|\tau_1 - \tau_2|^{1/2}$ for any $\tau_1, \tau_2 \in \Delta$;
(1.b) $C_0' < P_\phi < C_0$.

Starting with these properties, we now prove the following:

LEMMA 3. *There exist two positive constants R_1 and R_2, depending only on D and a, so that for every $\phi \in \mathscr{F}_a$, we have $R_1 < |\tilde{\phi}| < R_2$.*

Proof. We note that $\tilde{\phi} = \xi P_\phi(\xi)\overline{\nu(\phi(\xi))}$ for $\xi \in \partial\Delta$. Thus by applying the maximal principle to $|\tilde{\phi}|$, we see that R_2 can be chosen to be C_0 in (1.b). To obtain another inequality, we suppose not and seek a contradiction. Then there exist a sequence $\{\phi_n\} \subset \mathscr{F}_a$ and a sequence $\{\tau_n\} \subset \Delta$ which approaches some $\xi_0 \in \overline{\Delta}$, so that $\tilde{\phi}_n(\tau_n) \to 0$. By (1.a) and the Arzela-Ascoli theorem we can assume, without loss of generality, that $\{\phi_n\}$ converges uniformly to some $\phi \in \mathscr{F}_a$ and $\{\tilde{\phi}_n\}$ converges uniformly to some $\phi^* \in \mathrm{Hol}(\Delta, \mathbf{C}^n)$. Hence $P_{\phi_n} = (\xi^{-1}\tilde{\phi}_n, \nu(\phi_n(\xi)))$ converges uniformly to some positive continuous function P^* defined on $\partial\Delta$. Now, since

$$\phi^*(\xi) = \xi P^*(\xi)\overline{\nu(\phi(\xi))} \quad \text{and} \quad \tilde{\phi}(\xi) = \xi P_\phi(\xi)\overline{\nu(\phi(\xi))},$$

we see that

$$\phi^* = \frac{P^*}{P_\phi}\tilde{\phi} \quad \text{on } \partial\Delta.$$

From the fact that $\tilde{\phi} \neq 0$, it follows easily that P^*/P_ϕ is the boundary value of some holomorphic function defined on Δ. This implies that $P^* = \text{Const} \cdot P_\phi$ and thus that ϕ^* and ϕ differ by a positive constant. That is a contradiction, for $\phi^*(\xi_0) = \lim_{n\to\infty}\tilde{\phi}_n(\tau_n) = 0$ but $\tilde{\phi}(\xi_0) \neq 0$. □

LEMMA 4. *There exists a positive constant C_1, depending only on D and a, such that for every $\phi \in \mathcal{F}_a$ and for any $\tau_1, \tau_2 \in \Delta$, we have $|\phi'(\tau_1) - \phi'(\tau_2)| < C_1|\tau_1 - \tau_2|^{1/2}$.*

Proof. The argument is based on a careful examination of what is called the C^k-version of the Schwartz reflection principle.

Let $S = \{(p, T_p^{(1,0)}\partial D)\colon p \in \partial D\}$ and $\mathbf{P}_{n-1}$ the complex projective space of hyperplanes in $\mathbf{C}^n$. Then $S \subset \mathbf{C}^n \times \mathbf{P}_{n-1}$ is a compact totally real submanifold. Let

$$B(R_1, R_2) = \{z \in \mathbf{C}^n\colon R_1 < |z| < R_2\},$$

and denote by $\pi\colon \mathbf{C}^n \times B(R_1, R_2) \to \mathbf{C}^n \times \mathbf{P}_{n-1}$ the natural projection, where R_1, R_2 are as in Lemma 3.

We first find two open coverings $\{E_i\}_{i=1}^m$ and $\{\tilde{E}_i\}_{i=1}^m$ of S such that the following assertions hold for each i:

(1.c) $E_i \subset\subset \tilde{E}_i \subset\subset \mathbf{C}^n \times \mathbf{P}_{n-1}$.

(1.d) There exists a C^{k-1}-diffeomorphism $\Psi_i\colon \bar{\tilde{E}}_i \to \bar{V}_i \subset\subset \mathbf{C}^{2n-1}$ so that

$$\Psi_i\big(\tilde{E}_i \cap (\partial D \times S)\big) \subset \mathbf{R}^{2n-1} \subset \mathbf{C}^{2n-1}$$

and $D^\alpha(\bar{\partial}\Psi_i) = 0$ on $\tilde{E}_i \cap (\partial D \times S)$ for all multi-indices α with $|\alpha| \leq 1$.

(1.e) Let

$$O_i = \{z \in \mathbf{C}^n\colon \exists w \in B(R_1, R_2) \text{ so that } \pi(z, w) \in \tilde{E}_i\}.$$

Then $\max_{z \in O_i} \operatorname{dist}(z, \partial D) \ll 1$.

For every $\phi \in \mathcal{F}_a$, from Lemma 3, we can define a holomorphic mapping

$$\hat{\phi} \in \operatorname{Hol}(\Delta, \mathbf{C}^n \times B(R_1, R_2))$$

by

$$\phi(\tau) \triangleq \left(\phi(\tau), \tilde{\phi}(\tau)\right).$$

Let

$$b_0 \triangleq \min_i \left\{\operatorname{dist}\left(\partial\left(\pi^{-1}(\tilde{E}_i)\right), \partial\left(\pi^{-1}(E_i)\right)\right)\right.$$

and let

$$U_i = \left(\pi \circ \hat{\phi}\right)^{-1}(E_i).$$

From (1.c), we see that $b_0 > 0$. Furthermore the following properties are easy to verify:

(1.f) Let

$$\tilde{U}_i = \left\{\tau \in \tilde{\Delta}: \operatorname{dist}(\tau, U_i) < \left(\frac{b_0}{C_0}\right)^2\right\}$$

where C_0 is chosen as in Lemma 3. Then $\pi(\hat{\phi}(\tilde{U}_i)) \subset \tilde{E}_i$ (whenever $U_i \neq \varnothing$).

(1.g) There exists a constant $b_1 > 0$, independent of the choice of ϕ, so that for any $\tau_1, \tau_2 \in \overline{\Delta}$, if $1 - |\tau_1| < b_1$ and $|\tau_1 - \tau_2| < b_1$, then we may find some U_i, defined as above, which contains τ_1 and τ_2.

(1.h) There exists a constant $b_2 > 0$, independent of the choice of ϕ, so that for every $\tau \in \Delta$ with $|\tau| < b_2$, we have $\tau \notin \bigcup_i \tilde{U}_i$.

In fact, (1.f) follows easily from (1.a) and the definition of b_0, (1.h) follows from (1.a) and (1.e), while (1.g) is a simple application of the Lebesgue number lemma and (1.a).

We let

$$\tilde{U}_i^* = \left\{\tau \in \mathbf{C}^1: \bar{\tau}^{-1} \in \overline{\tilde{U}_i} \cap \overline{\Delta}\right\}, \quad V_i^* = \left\{z \in \mathbf{C}^{2n-1}: \bar{z} \in V_i\right\},$$

and $\tilde{\Omega}_i = \tilde{U}_i \cup \tilde{U}_i^*$. Define $g_i\colon \tilde{\Omega}_i \to V_i \cup V_i^*$ by $\Psi_i \circ \pi \circ \hat{\phi}(\tau)$ when $\tau \in \tilde{U}_i$, and by $\overline{\Psi_i \circ \pi \circ \hat{\phi}(\bar{\tau}^{-1})}$ when $\tau \in \tilde{U}_i^*$. Consider $f_i \triangleq \partial g_i/\bar{\partial}\tau$. By the argument on Page 438 of [Lm1], we can conclude, from (1.a), (1.d), and the Hardy-Littlewood theorem, that f_i is uniformly bounded and uniformly Hölder-$\frac{1}{2}$ continuous on $\tilde{\Omega}_i$ with respect to $\mathscr{F}_a$ (i.e, there is a constant C, independent of the choice of ϕ, so that for every $\phi \in \mathscr{F}_a$ and $\tau_1, \tau_2 \in \Delta$, the corresponding f_i satisfies $|f_i(\tau_1) - f_i(\tau_2)| < C|\tau_1 - \tau_2|^{1/2}$).

Let

$$\psi_i(\tau) \triangleq \frac{1}{2\pi\sqrt{-1}} \int_{\tilde{\Omega}_i} \frac{f_i(\xi)}{\xi - \tau}\, d\xi \wedge \overline{d\xi}.$$

We then have the following facts:

(1.i) $\partial\psi_i/\partial\bar{\tau} = f_i$.

(1.j) ψ_i is uniformly bounded on $\tilde{\Omega}_i$ with respect to $\mathscr{F}_a$ (by (1.h) and the uniform boundedness of f_i).

(1.k) $\partial\psi_i/\partial\tau$ is uniformly Hölder-$\frac{1}{2}$ continuous on Ω_i (by (1.a) and Proposition 2.6.40 of [Ab1]). Here $U_i^* = \{\tau \in \mathbf{C}^1 \colon \bar{\tau}^{-1} \in U_i\}$ and $\Omega_i = U_i \cup U_i^*$.

Since $\psi_i - g_i$ is holomorphic and uniformly bounded on $\tilde{\Omega}_i$, it follows, from (1.f) and the Cauchy estimates, that $(\psi_i - g_i)'$ is uniformly bounded on U_i. Hence by (1.k), $\partial g_i/\partial\tau$ is uniformly Hölder-$\frac{1}{2}$ continuous on U_i. So by (1.a), (1.d), (1.g), and the Cauchy estimates, we can now find a constant C_1, depending only on D and a, so that for every $\phi \in \mathscr{F}_a$ and for any $\tau_1, \tau_2 \in \Delta$, we have $|\phi'(\tau_1) - \phi'(\tau_2)| < C_1|\tau_1 - \tau_2|^{1/2}$. This completes the proof. □

Remark. Let $\mathscr{F}$ be the set of all complex geodesics ϕ satisfying $\delta(\phi(0)) = \max_\tau\{\delta(\phi(\tau))\}$, where $\delta(z)$ is defined as the distance from z to ∂D. By making use of the uniform Hölder-$\frac{1}{4}$ continuity of $\mathscr{F}$ [CHL], the above argument might also be modified to prove the following:

PROPOSITION 1. *Let $D \subset\subset \mathbf{C}^n$ be a C^k-strongly convex domain ($k > 2$). If $k = \omega$, then there exists an open neighborhood U of $\overline{\Delta}$ so that all elements in $\mathscr{F}$ can be extended holomorphically to U; if $k < \omega$, then for any $j \le k - 2$ there exists a constant C_j so that for every $\phi \in \mathscr{F}$ and $\tau_1, \tau_2 \in \Delta$, it holds that $|\phi^{(j)}(\tau_1) - \phi^{(j)}(\tau_2)| < C_j|\tau_1 - \tau_2|^{1/4}$.*

Another key lemma which we need is the following version of the Fornaess embedding theorem:

LEMMA 5. *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and $p \in \partial D$ a strongly pseudoconvex point with at least C^2-smoothness. Suppose that either D has a stein neighborhood basis or D has a C^∞ boundary. Then there exist a neighborhood U of p, a bounded C^2-strongly convex domain Ω in $\mathbf{C}^n$, and a holomorphic mapping Φ from D to Ω such that*

(a) *Φ can be extended holomorphically to U with $\Phi^{-1}(\Phi(U \cap \overline{D})) = U \cap \overline{D}$;*

(b) *$\Phi(U \cap D) \subset \Omega$, $\Phi(U \cap \Omega^c) \subset \Omega^c$, and $\Phi(U \cap \partial\Omega) = \Phi(U) \cap \partial\Omega$.*

Proof. When D has a Stein neighborhood basis, the lemma is Proposition 1 of [Fn]. So it suffices for us to prove the lemma in case D is a smooth pseudoconvex domain with p being a strongly pseudoconvex point. The argument in this situation is also a slight modification of that in [Fn]. In fact, the only difference is in that we now have to make use of Kohn's global regularity result of the $\bar{\partial}$-equation [Ko] on smooth pseudoconvex domains to construct a nice bounded supporting function appeared in line 1 to line 5 of page 533 of [Fn] (this is the only place we need the global boundary smoothness of D). For the convenience of the reader, we present the following details:

First, let $\{w_1(z), \dots, w_n(z)\}$ be a local coordinates system on a neighborhood U of p so that $w(p) = 0$ and $U \cap D$ is defined by

$$\rho(w) = \operatorname{Re} w_1 + \sum_{j=1}^{n} |w_j|^2 + o(|w|^2).$$

Let $V \subset\subset U$ be a very small neighborhood of p (or $w = 0$). Choose χ to be a positive cutting function with $\operatorname{Supp}\chi \subset\subset V$ and $\chi(0) = \chi(w(p)) = 1$. For a positive number ε, define $D_\varepsilon = \{z \in \mathbf{C}^n$: either $z \in D$ or $z \in V$ with $\rho(w(z)) < \varepsilon\chi(w)\}$. By the above discussions and (3.4.2.2) of Theorem 3.4.2 in [Kr2], it is easy to check that when ε is small enough then $D_\varepsilon(\supset D)$ is also a smooth bounded pseudoconvex domain. Now when $|w| < \lambda_0$ with $\lambda_0 \ll 1$, we may assume that $\chi(w) < 2$ and $\rho(w) > \operatorname{Re} w_1 + 1/2\Sigma_{j=1}^{n}|w_j|^2$. Thus for $w \in D_\varepsilon \in \cap\{|w| < \lambda_0\}$, we have

$$\operatorname{Re} w_1 < \rho(w) - 1/2 \sum_{j=1}^{n} |w_j|^2 < \varepsilon\chi(w) - 1/2|w|^2 < 2\varepsilon - 1/2|w|^2,$$

where $|w|^2 = \Sigma_{j=1}^{n}|w_j|^2$. Hence, for $\lambda \ll 1$, if we let $\varepsilon = 1/4\lambda^2$, then the following claim holds (see Lemma 5.2.8 of [Kr2]):

Claim. Let ε, λ, and λ_0 as above. If $1 \gg \lambda_0 > \lambda$, $\lambda < |w| < \lambda_0$, and $w \in D_\varepsilon$, then $\operatorname{Re} w_1 < 0$.

Now, define a cutting function $\xi(t)$: $\mathbf{R}^1 \to [0,1]$ with $\xi(t) = 1$ for $|t| < \lambda'$ and 0 for $|t| > \lambda'_0$. Here $\lambda < \lambda' < \lambda'_0 < \lambda_0$. It thus follows that $\omega = \bar{\partial}_z(\xi(|w|)\log w_1$ is a well-defined C^∞ (0, 1)-form on $\overline{D_\varepsilon}$; for in case $\bar{\partial}_z(\xi|w|) \neq 0$, $\operatorname{Re} w_1 < 0$ and thus $\log w_1$ is well defined (see page 186–187 of [Kr2] for more details on this matter). Furthermore, it is easy to verify that $\bar{\partial}\omega \equiv 0$. Therefore, by a theorem of Kohn [Ko], there is a $g \in C^\infty(\overline{D_\varepsilon})$ so that $\bar{\partial}_z g = \omega$.

Define $f(z)$ with

$$f(z) = \exp(g + \xi(|w|)\log w_1) \text{ for } w \in D_\varepsilon \cap \{|w| < \lambda'_0\}$$

and

$$f(z) = \exp(g) \text{ for } w \in D_\varepsilon \cap \{|w| \geq \lambda'_0\}.$$

By the way these objects were constructed, we can conclude that

(i) $f(z) \in \mathrm{Hol}(D_\varepsilon) \cap C(\overline{D_\varepsilon})$ (see also page 186 of [Kr2]);
(ii) for w close to 0, $f(w(z)) = w_1 f^*(w)$ with $f^*(0) \neq 0$.

We now shrink λ_0 and λ (thus also ε) so that:

(iii) $|f^*(w) - f^*(0)| < 1/2|f^*(0)|$ for $|w| < \lambda_0$ and $w \in D_\varepsilon$;
(iv) $\mathrm{Re}\, w_1 < 0$ for $w \subset (\overline{D} - \{p\}) \cap \{w\colon |w| < \lambda_0\}$.

Therefore, we can also define the smooth (0,1)-closed form $\omega^* = \bar{\partial}(\xi(|w|)\log f \cdot f^{-3}$ on the closure of D_ε. Consider the similar equation $\bar{\partial}_z g^* = \omega^*$. By Kohn's theorem, we obtain again a solution g^* which is continuous on $\overline{D_\varepsilon}$ (actually smooth, but for our purpose here, all we need is the existence of a bounded solution). Now, define

$$\eta^*(z) = \exp\left(g^* f^3 + \xi(|w|)\log f\right) \quad \text{for } w \in D_\varepsilon \cap \{|w| < \lambda'_0\}$$

and

$$\eta^*(z) = \exp\left(g^* f^3\right) \quad \text{for } w \in D_\varepsilon \cap \{w\colon |w| \geq \lambda'_0\}.$$

Then we similarly see that $\eta^*(z) \in \mathrm{Hol}(D_\varepsilon) \cap C(\overline{D_\varepsilon})$. Moreover, it holds, for $w \simeq 0$ (or $z \simeq p$), that

$$\eta_1(w) \triangleq \eta^*(z) f^{*-1}(w(p)) = w_1 + O(|w|^3).$$

As in [Fn], we now change the coordinates $\{w_1, \cdots, w_n\}$ to the globally defined functions $\{\eta_1, \cdots, \eta_n\}$ on $\overline{D_\varepsilon}$ which also serve the local coordinates near p, where η_1 is as above and η_j is the linear term of the Taylor expansion of $w_j(z)$ at $z = p$ $(j > 1)$. Note that for $z \in D$, it still holds that

$$\rho(w(\eta)) = \mathrm{Re}\, \eta_1 + \sum_{j=1}^{n} |\eta_j|^2 + o\left(|\eta|^2\right) < 0.$$

We therefore see that $\mathrm{Re}\, \eta_1(z) < 0$ for $z(\simeq p) \in \overline{D} - p$. Since $\eta_1(z) \neq 0$ for $z \in \overline{D} - p$ (by the construction of η_1 and the property (iv)), and since η_1 is continuous on $\overline{D}$, we therefore conclude that there is a small positive ε_0 so that $|\eta_1(z) - \varepsilon_0| > \varepsilon_0$ for $z \in \overline{D} - p$. Also, notice that $\eta_1 \in \mathrm{Hol}(D_\varepsilon) \cap C(\overline{D_\varepsilon})$ and η_1 is holomorphic near p. Thus starting from such a supporting function, we can now obtain the Φ in our lemma by copying the argument of [Fn] from line 6, page 533 to line 11, page 536. □

Proof of Theorem 1. Seeking a contradiction, we suppose that there is a sequence of extremal mappings $\{\phi_k\}$ of D so that

$$\phi_k(0) \to p, \frac{|((\phi'_k(0))_{tn}|}{|((\phi'_k(0))_{nr}|} \to \infty,$$

but for each k, $\phi_k(\Delta) \cap V \neq \varnothing$ for some fixed neighborhood V of p.

Let Ω, U, Φ be as in Lemma 5 and let $\phi_k^* = \Phi \circ \phi_k$. It is then easy to see that

$$\phi_k^*(0) \to \Phi(p)(\triangleq q) \quad \text{and} \quad \frac{|(\phi_k^{*\prime}(0))_{tn}|}{|(\phi_k^{*\prime}(0))_{nr}|} \to \infty.$$

Construct another strongly convex domain Ω_0, which is contained in $\Phi(U)$, so that $\partial\Omega_0 \cap \partial\Phi(D)$ $(\subset \partial\Omega \cap \partial\Phi(D))$ is a piece of strongly convex hypersurface, and find a sequence of complex geodesics $\{\psi_k\}$ of Ω with $\psi_k(0) = \phi_k^*(0)$ and $\psi_k'(0) = \lambda_k \phi_k^{*\prime}(0)$ $(\lambda_k > 0)$ for each k. We claim that $\psi_k(\Delta) \to p$ as $k \to \infty$, thus that $\psi_k(\Delta) \subset \Omega_0$ for $k \gg 1$. In fact, let $\{\sigma_k\} \subset \mathrm{Aut}(\Delta)$ be such that $\psi_k^* \triangleq \psi_k \circ \sigma_k \in \mathscr{F}$ and $\psi_k^*(\tau_k) = \psi_k(0)$ for some $\{\tau_k\} \subset (0, 1)$. If $\psi_k(\Delta)$ does not reduce to q as $k \to \infty$, it then follows easily from the assumptions that, for infinitely many k, $\tau_k \to 1$ and $\{\psi_k^*\} \subset \mathscr{F}_a$ for some $a > 0$. By a normal family argument, we may assume, without loss of generality, that $\psi_k^* \to \psi \in \mathscr{F}_a$ (see proposition 4 of [CHL]). Hence, from Lemma 4 and the above hypotheses, we obtain $\psi'(1) \in T_q^{(1,0)}\partial\Omega$. This is a contradiction [Lm1].

Now, by Lemma 1, we see that ψ_k is also a complex geodesic of both Ω_0 and $\Phi(D)$ when $k \gg 1$. Hence, by making use of the monotonicity property of the Kobayashi metric and this fact, we have for $k \gg 1$ that

$$\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) \leq \kappa_D(\phi_k(0), \phi_k'(0)) = 1$$

and

$$\begin{aligned}\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) &= \kappa_{\Omega_0}(\phi_k^*(0), \phi_k^{*\prime}(0)) \\ &= \kappa_{\Phi^{-1}(\Omega_0)}(\phi(0), \phi'(0)) \geq \kappa_D(\phi(0), \phi'(0)) = 1.\end{aligned}$$

Thus, $\kappa_{\Phi(D)}(\phi_k^*(0), \phi_k^{*\prime}(0)) = 1$.

On the other hand, since $\kappa_{\Phi(D)}(\phi_k^*(0), \lambda_k\phi_k^{*\prime}(0)) = 1$, for ψ_k is a complex geodesic of $\Phi(D)$, we obtain $\lambda_k = 1$. So we can conclude that ϕ_k^* is a complex geodesic of Ω when $k \gg 1$. By the uniqueness property of the complex geodesics on strongly convex domains, we therefore have $\phi_k^* = \psi_k$

for $k \gg 1$. However, from the above argument this implies that $\phi_k^*(\Delta) \to q$ as $k \to \infty$. That is a contradiction and hence completes the proof for the second assertion of our theorem.

To conclude the proof, we let ε be small enough so that we can choose V in the theorem to be $\Phi^{-1}(\Omega_0)$. Suppose that ϕ is an extremal mapping of D with $\phi(\Delta) \subset V$. Then by Lemma 1, it is also an extremal mapping for V, thus a complex geodesic of V; for it is bilomorphic to the strongly convex domain Ω_0. By making use of Lemma 2, we see that $\Phi \circ \phi$ is a complex of Ω. Now, by the monotonicity property for the Kobayashi distance, we have, for any $\tau_1, \tau_2 \subset \Delta$, that

$$\begin{aligned} K_\Delta(\tau_1, \tau_2) &= K_V(\phi(\tau_1), \phi(\tau_2)) = K_\Omega(\Phi \circ \phi(\tau_1), \Phi \circ \phi(\tau_2)) \\ &\le K_{\Phi(D)}(\Phi \circ \phi(\tau_1), \Phi \circ \phi(\tau_2)) \le K_D(\phi(\tau_1), \phi(\tau_2)) \le K_\Delta(\tau_1, \tau_2). \end{aligned}$$

Therefore ϕ is a complex geodesic of D. □

Proof of Corollary 1. This follows easily from Theorem 1. □

Proof of Corollary 2. Let ϕ be as in Corollary 2. Then from the argument in Theorem 1 of [FR], we easily see that ϕ is continuous at 1.

By the well-known estimates of the Kobayashi metric near a strongly pseudoconvex point (see [Al], for example), we may find a neighborhood U of p and a constant C so that for every $z \in U \cap D$ and $X \in T_z^{(1,0)}D$, we have $\kappa_D(z, X) \ge C|X|_{nr}/\delta(z)$. Meanwhile, we recall that ϕ is also an infinitesimal complex geodesic (see [Ab1]), i.e.,

$$\kappa_D(\phi(\tau), \phi'(\tau)) = \kappa_\Delta(\tau, 1) = 1/(1 - |\tau|^2).$$

Hence, from the fact that $\delta(\phi(\tau)) \simeq 1 - |\tau|^2$ [Ab1], it follows easily that $|(\phi'(\tau))_{nr}| <$ Const. near 1. To finish the proof, it now suffices to show that $|(\phi'(\tau))_{tn}|$ is bounded near 1. Suppose this is not the case. Then there exists a sequence $\{\tau_k\}$ converging to 1, so that

$$|(\phi'(\tau_k))_{tn}| / |(\phi'(\tau_k))_{nr}|$$

goes to the infinity as $k \to \infty$. Let ϕ_k be a reparametrization of ϕ so that $\phi_k(0) = \phi(\tau_k)$ for each k. From Theorem 1, it then follows that $\phi_k(\Delta)(= \phi(\Delta)) \to p$. This is obviously a contradiction. □

We recall that a subset E of a bounded domain D is called a holomorphic retract if there is an $h \in \mathrm{Hol}(D, D)$ with $h^2 = h$ so that $h(D) = E$. An obvious observation is that for a holomorphic retract E, it holds that

$K_E(z_1, z_2) = K_D(z_1, z_2)$ for any $z_1, z_2 \in E$. Combining this fact with Corollary 2, we have the following:

COROLLARY 4. *Let D and p be as in Theorem 1. Suppose that E is a simply connected one dimensional holomorphic retract of D with $p \in \overline{E}$, and suppose that ϕ is a biholomorphic mapping from Δ to E with $\phi(\tau_k) \to p$ for some $\tau_k \to 1$. Then ϕ is Lipschtz-1 continuous near 1.*

Remark. From the proof, we can actually see that Theorem 1, Corollary 1, Corollary 2, and Corollary 3 hold for all bounded domains which possess the local embedding property in Lemma 5. In particular, we can replace D by the bounded domain of the form $D - K$, where D is as in Theorem 1 and K is a compact subset of D.

Remark 2. When D has a C^1-smooth plurisubharmonic defining function, then all extremal mappings satisfy the Euler-Lagrange equation (see §5 of [P]). This result with the reflection principle and Corollary 2 gives the following:

PROPOSITION 2. *Let $D \subset \mathbf{C}^n$ be a bounded pseudoconvex domain with a C^1 plurisubharmonic defining function. Let $p \in \partial D$ be a C^k-smooth strongly pseudoconvex point ($3 \leq k \leq \omega$). Suppose that ϕ is a complex geodesic of D. If there exists a sequence $\{\tau_m\}$, converging to 1, such that $\phi(\tau_m) \to p$, then ϕ can be extended Λ^{k-1} smoothly across $1 \in \partial\Delta$, where Λ^{k-1} is the standard Zygmund space of order $k - 1$.*

Proof. Since the argument is standard (see [Lm1] for example), we omit the details and just sketch the idea: We can prove first the Hölder continuity of the dual mapping $\tilde{\phi}$ defined from the Euler-Lagrange equation of ϕ, by making use of the Riemann-Hilbert problem. We then can show that $\tilde{\phi}$ can at most vanish to some integer order m at 1. Thus, by applying the reflection principle to $[\phi, \tilde{\phi}/(1 - \tau)^m]$, the regularity result of ϕ follows. □

3. Proof of Theorem 2

In this section, we start by proving the local version of a result of Krantz [Kr2]. Then we use iteration theory and Lipschitz-1 continuity for holomorphic retracts (Corollary 3) to obtain Theorem 2. We leave the proof of a technical lemma to the last.

LEMMA 6. *Let D and p as in Theorem 1. Suppose that σ is a biholomorphism of D such that $\sigma(z) = z + o(|z - p|^k)$ as $z \to p$. Then $\sigma \equiv \mathrm{id}$ if either $k = 2$ or $k = 1$ and D is not biholomorphic to the ball.*

Remark. By Example 2 in Section 1, we see that exponent 2 in the theorem is necessary for the ball. In fact, even in the one dimensional case, we have the following example.

Example 5. Let

$$\sigma(\tau) = \frac{1 + (2i - 1)\tau}{2i + 1 - \tau} \quad \text{for } \tau \in \Delta.$$

Then $\sigma(1) = \sigma'(1) = 1$ and $\sigma \in \mathrm{Aut}(\Delta)$. However, by a direct computation, it can be seen that an automorphism of Δ which has contact of order 2 with the identity at some boundary point must be the identity.

Proof of Lemma 6. Let Ω, Ω_0, Φ, and U be as in the proof of Theorem 1, and let ϕ be a complex geodesic of Ω with $\phi(1) = \Phi(p)$ (see [Lm1]). As argued before, we see that when $\phi'(1)$ is close enough to the tangential direction, then $\phi(\Delta) \subset \Omega_0$ and $\phi_0 \triangleq \Phi^{-1}|_{\Omega_0} \circ \phi$ is a complex geodesic of D. Hence, $\sigma \circ \phi_0$ is also a complex geodesic of D. Now when $\sigma \circ \phi_0(\Delta)$ is close enough to p (we can do this by shrinking $\phi(\Delta)$ and by the continuity of σ at p), it follows from Lemma 1 that $\sigma \circ \phi_0$ is also a complex geodesic of $\Phi^{-1}(\Omega_0)$. So $\Phi \circ \sigma \circ \phi_0$ is a complex geodesic of Ω_0 and therefore a complex geodesic of Ω (by Lemma 2). We note that ϕ and $\Phi \circ \sigma \circ \phi_0$ coincide at 1 up to the first order. Thus, by the uniqueness property of complex geodesics on strongly convex domains, we can find a biholomorphism α of Δ so that $\alpha(1) = 1$, $\alpha'(1) = 1$, and $\Phi \circ \sigma \circ \phi_0 = \phi \circ \alpha$. If $k = 2$ or $k = 1$ and α is elliptic (i.e, the sequence $\{\alpha^n\}$ is a pre-compact family), we have that $\alpha(\tau) \equiv \tau$ and hence that σ fixes $\phi_0(\Delta)$. If α is non-elliptic, then by noting the fact that $\sigma(\phi_0(\Delta)) \subset \phi_0(\Delta)$, we have $\Phi \circ \sigma^m \circ \Phi^{-1} \circ \phi = \phi \circ \alpha^m \to \Phi(p)$. Thus p is a boundary accumulation point of the automorphism sequence $\{\sigma^m\}_{m=1}^{\infty}$ of D. By the Wong-Rosay theorem [Kr2], this implies that D is biholomorphic to the ball. So when D is not biholomorphic to the ball, by making use of the uniqueness theorem of holomorphic functions and the fact that the union of all such $\phi_0(\Delta)$'s occupies an open subset of D, we see that $\sigma \equiv \mathrm{id}$. □

LEMMA 7. *Let $D \subset \mathbf{C}^n$ $(n > 1)$ be a pseudoconvex domain and $p \in \partial D$ a C^2-strongly pseudoconvex point. Assume furthermore that either D has a Stein neighborhood basis or D has a C^∞ boundary. If $f \in \mathrm{Hol}(D, D)$ is such that $f(z) = z + o(|z - p|)$ as $z \to p$, then for any neighborhood V of p, there exists a point $z \in V \cap D$ such that $f^k(z) \in V$ for $k = 1, 2, 3, \cdots$.*

We assume this lemma for the moment and pass to Theorem 2.

Proof of Theorem 2. Let D, p, and f be as in the theorem. Then Case (1) is the local version of the Burns-Krantz theorem (see [BK] and [H]). Now if

$\{f^n\}$ does not converge compactly to p, then by iteration theory of holomorphic mappings (see [Bd] or [Ab1]) and Lemma 7, we have the following possibilities:

(i) $\{f^n\}$ converges compactly to some $z_0 \in D$;
(ii) Some subsequence of $\{f^n\}$ converges to a one dimensional holomorphic retract h of D so that $f \in \mathrm{Aut}(h(D))$;
(iii) f is an automorphism of D.

In view of Lemma 7, (i) cannot happen, while by Lemma 6 (iii) can occur only when $k = 1$ and D is biholomorphic to the ball. Hence, all we actually have to study is the case (ii).

Since $h(D)$ is a simply connected hyperbolic Riemann Surface (this follows from the simple connectivity of D) and since $\{f^n\}$ is a precompact family, we may conclude that f fixes some point on $h(D)$ [Ab1]. From Lemma 7, it follows easily that $p \in \overline{h(D)}$. Hence, we may choose a biholomorphism ϕ from Δ to $h(D)$ and a sequence $\{\tau_k\}$, converging to 1, so that $\phi(\tau_k) \to p$ as $k \to \infty$. By Corollary 4, we see that $\phi(1) = p$ and ϕ is Lipschitz-1 continuous near 1. Since $\phi^{-1} \circ f \circ \phi$ $(\in \mathrm{Aut}(\Delta))$ fixes two points on $\overline{\Delta}$; one is in Δ and the other one is on $\partial\Delta$, we can easily conclude that f fixes $h(D)$.

Now if $k = 2$, then we let

$$\lambda(\tau) = \left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\right) \circ \phi(\tau),$$

which is the sum of the eigenvalues of the Jacobian of f at $\phi(\tau)$. We claim that $\mathrm{Re}\,\lambda(\tau) \equiv 2$ under these assumptions. In fact, by using the Cauchy estimates, the Lipschitz-1 continuity of ϕ at 1, and the fact that $\delta(\phi(\tau)) \simeq C(1 - |\tau|)$, we have the following estimates for $\tau \in (0, 1)$:

$$\left|\frac{\partial f_j}{\partial z_j} \circ \phi(\tau) - 1\right| \le C \frac{1}{\delta(\phi(\tau))} \sup_{|z - \phi(\tau)| \le \delta(\phi(\tau))} |f_j(z) - z_j|$$
$$= \frac{1}{|1 - \tau|} o\left((\delta(\phi(\tau)) + |\phi(\tau) - p|)^2\right) = o(1 - \tau),$$

as $\tau \to 1$. On the other hand, since $\mathrm{Re}(\lambda(\tau))$ is harmonic and is never larger than 2, it follows from the Hopf lemma that $\mathrm{Re}(\lambda(\tau)) \equiv 2$. This is a contradiction, for from the Cartan-Carathéodory-Kaup-Wu theorem it implies that $f(z) \equiv z$. The proof is now complete. □

Proof of Lemma 7. Let D, p, f be as in the lemma, and let $\vec{n}$ be the inward normal vector of D at p. Denote by L the inward $\pi/4$-cone at p, i.e, $L \triangleq \{z \in D$: the angle between $\overrightarrow{pz}$ and $\vec{n}$ is less than $\pi/4\}$. We then define the big and small horospheres for any $z_0 \in D$ and $R > 0$ as follows (we note that the definition is somewhat different from that in [Ab1], but is more suitable for our purpose here):

$$E(z_0, R) \triangleq \left\{ z \in D\colon \limsup_{w(\in L)\to p} \left(K_D(z,w) - K_D(z_0,w)\right) < 1/2 \log R \right\},$$

$$F(z_0, R) \triangleq \left\{ z \in D\colon \liminf_{w(\in L)\to p} \left(K_D(z,w) - K_D(z_0,w)\right) < 1/2 \log R \right\}.$$

Claim 1. Let $R > 0$ and $z_0 \in \vec{n}$ close to p. Then $z \in E_D(z_0, R)$ for $z \in \vec{n}$ close enough to p.

Proof of Claim 1. Let $z_0 \in \vec{n}$ be close to p and let z be in the segment $\overline{z_0 p}$. Denote by $B(z)$ the ball with center z and radius $\rho(z)(= |z - p|)$. We then see that $B(z) \subset D$ when $z \sim p$. By the estimate that $K_D(z_0, w) \geq C - \frac{1}{2}\log \delta(w)$ for $w(\in L) \sim p$ (see Claim 2 for more discussions on this matter) and some basic properties of the Kobayashi distance, we then have for $w(\in L) \sim p$,

$$\begin{aligned} K_D(z,w) - K_D(z_0,w) &\leq K_{B(z)}(z,w) - K_D(z_0,w) \\ &\leq \frac{1}{2}\log\left(\frac{1 + |z-w|/\delta(z)}{1 - |z-w|/\delta(z)}\right) - C + \frac{1}{2}\log\delta(w) \\ &\leq \frac{1}{2}\log\delta(z) + C + \frac{1}{2}\log\frac{\delta(w)}{\delta(z) - |z-w|}, \end{aligned}$$

where C denotes a constant, which may be different in different context.

We now use the special property for L which makes that

$$\frac{\delta(w)}{\delta(z) - |z-w|} \to 1$$

as $w(\in L) \to p$. We therefore obtain $K_D(z,w) - K_D(z_0,w) \leq \frac{1}{2}\log\delta(z) + C$. So for any $R > 0$, when $z(\in \vec{n})$ is close enough to p, we have $z \in E(z_0, R)$.

Claim 2. Let z_0 be as in Claim 1. For every small neighborhood V of p, there exists a $R > 0$ such that $F(z_0, R) \subset V$.

Proof of Claim 2. Let Ω_0, Ω, U and Φ be as in the proof of Theorem 1. Without loss of generality, we assume that $\Phi^{-1}(\Omega_0) \subset V$ and $d\Phi(p) = \mathrm{id}$. Let $z_0^* = \Phi(z_0)$, $w^* = \Phi(w)$, and $\vec{n}^*$ the inward normal vector of Ω at $q(= \Phi(p))$. Then $\Phi(\vec{n})$ is tangent to $\vec{n}^*$ at q.

By noting the fact that the Kobayashi distance of Ω between any two points can be realized by a complex geodesic, we then have

$$K_\Omega(z^*, w^*) \geq \inf_{u \in \partial\Omega_0 - \partial\Omega} K_\Omega(u, w^*)$$

for any $z^* \notin \Omega_0$ and $w^* \sim q$. Let $B^* \supset \Omega$ be a ball, which is tangent to Ω at q and has $\vec{n}$ as part of its diameter. Then, from a direct computation of the Kobayashi distance for B^*, we obtain

$$K_\Omega(u, w^*) \geq K_{B^*}(u, w^*) \geq -1/2 \log \delta^*(w^*) + C.$$

Here C is a constant independent of the choice of $u \in \partial\Omega_0 - \partial\Omega$, $\delta^*(w^*)$ denotes the distance from w^* to $\partial\Omega$, and $w^*(\in \Phi(K)) \sim q$. So, from the monotonicity property of the Kobayashi distance, it follows that

$$\begin{aligned} K_D(z, w) - K_D(z_0, w) &\geq K_\Omega(z^*, w^*) - K_{\Omega_0}(z^*, w^*) \\ &\geq -\frac{1}{2} \log \delta^*(w^*) + C - C' + \frac{1}{2} \log \delta^*(w^*) \\ &\geq C - C'. \end{aligned}$$

Thus, if we choose $\frac{1}{2} \log R = C - C'$, then $z \notin F(z_0, R)$ when $z \notin \Phi^{-1}(\Omega_0)$. This completes the argument for claim 2.

Claim 3. $f(E(z_0, R)) \subset F(z_0, R)$.

Proof of Claim 3. This is essentially a lemma in [H]. However, for completeness, we include a proof here.

Let $z_k = p + \vec{n}/k (\in \vec{n})$. Then for $k \gg 1$, $f(z_k)$ is in L and converges to p as $k \to \infty$. For any $z_0 \in D$, $R > 0$, and $z \in E(z_0, R)$, we have

$$\begin{aligned} &\inf \lim_{w(\in L) \to p} (K_D(f(z), w) - K_D(z_0, w)) \\ &\quad \leq \inf \lim_{k \to \infty} (K_D(f(z), f(z_k)) - K_D(z_0, f(z_0))) \\ &\quad \leq \inf \lim_{k \to \infty} (K_D(z, z_k) - K_D(z_0, f(z_0))) \\ &\quad \leq \inf \lim_{k \to \infty} (K_D(z, z_k) - K_D(z_k, z_0)) \\ &\qquad + \sup \lim (K_D(z_k, z_0) - K_D(z_0, f(z_k)) \\ &\quad \leq \frac{1}{2} \log R + \sup \lim_{k \to \infty} (K_D(z_k, z_0) - K(z_0, f(z_k)). \end{aligned}$$

So to complete the proof of Claim 3, we have only to show that

$$\overline{\lim}(K_D(z_0, z_k) - K_D(z_0, f(z_k))) \leq 0.$$

In fact, let γ_k: $[0, 1] \to D$ be the segment joining z_k and $f(z_k)$. Obviously, when $k \gg 1$, then γ_k stays in D. Denote by $B(a, r)$ the ball of center a and radius r. We then have for every $X \in T^{(1,0)}_{\gamma_k(t)}D$, that

$$\kappa_D(\gamma_k(t), X) \leq \kappa_{B(\gamma_k(t), 1/(2k))}(\gamma_k(t), X) \leq C|X|k.$$

Here C is a constant which is independent of k and t. Hence,

$$\begin{aligned} K_D(z_0, z_k) - K_D(z_0, f(z_k)) \leq K_D(z_k, f(z_k)) &\leq \int_{\gamma_k} \kappa_D(\gamma_k(t), \gamma_k'(t))\, dt \\ &\leq Ck|f(z_k) - z_k| \\ &\leq o(1) \quad \text{as } k \to \infty. \end{aligned}$$

This completes the argument for Claim 3.

Now, for any given V, a small neighborhood of p, by Claim 1 and Claim 2, we can find a point z_0 and $R > 0$, so that $V \supset F(z_0, R) \supset E(z_0, R) \neq \varnothing$. From Claim 3, it follows easily that

$$f^k(E(z_0, R)) \subset F(z_0, R)$$

for each k, since for any k, f^k also satisfies the condition in Lemma 7. Hence, every element in $E(z_0, R)$ does our job. □

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