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HANKEL OPERATORS ON WEIGHTED BERGMAN SPACES ON STRONGLY PSEUDOCONVEX DOMAINS

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Introduction

Let Ω be a C^∞-bounded strongly pseudoconvex domain, $\Omega = \{z \in \mathbf{C}^n: \rho(z) < 0\}$, $n > 1$. For $\nu > -1$, let $dm_\nu = |\rho(z)|^\nu dm$, where dm is the Lebesgue volume form. Let L^2_ν be the L^2-space $L^2(\Omega, dm_\nu)$. We consider the weighted Bergman space $A^{2,\nu}(\Omega)$, the closed subspace of L^2_ν consisting of the holomorphic functions. The orthogonal projection of L^2_ν onto $A^{2,\nu}$ will be denoted by P. Together with P we will consider a non-orthogonal projection $\tilde{P}$ of L^2_ν onto $A^{2,\nu}$, given by an explicit integral kernel $G(z,w)$. Such a kernel, and projection, have been introduced by Kerzman and Stein in [16], and studied by Ligocka in [14] and [15], and by Coupet in [6].

In this paper we consider the Hankel operator, and the so called *non-orthogonal* Hankel operator, denoted by H_f and $\tilde{H}_f$ respectively, and defined by

$$H_f g(z) = (I - P)(\bar{f}g)(z),$$

and

$$\tilde{H}_f g(z) = (I - \tilde{P})(\bar{f}g)(z).$$

The Hankel operators on Bergman spaces are considered to be classical by now. In [1] Axler proved that if f is holomorphic, then the Hankel operator H_f on the unweighted Bergman space $A^2(D)$ on the unit disc D, is bounded (respectively compact) if and only if f is a Bloch function (resp. a little Bloch function). About the same time, in [3] Arazy, Fisher, and Peetre proved the same characterization about boundedness and compactness for H_f in the case of the weighted Bergman spaces on the unit disc for f an analytic symbol. Moreover Arazy, Fisher, and Peetre proved that H_f belongs to the Schatten ideal $\mathscr{S}_p$ if and only if f is in a certain Besov space. These pioneering results have been extended in various directions. In [21] Zhu studied the Hankel operators H_f and $H_{\bar{f}}$ on the unweighted Bergman space

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$A^2(B)$ on the unit ball. He proved the same characterization as the previous cases for generic symbol f, but assuming that both H_f and $H_{\bar{f}}$ are respectively bounded, compact, in the Schatten class $\mathscr{S}_p$. For analytic symbols, the same results were also proved in the weighted case by Feldman and Rochberg in [8], by Arazy, Fisher, Janson, and Peetre in [2], and by Wallstèn in [20]. More recently Leucking [10] first in the case of the unit disc, and then Li in the case of smoothly bounded strongly pseudoconvex domain (see [13]), have been able to characterize the bounded and compact Hankel operators on the unweighted Bergman space for generic symbols.

In this paper, following an idea of Janson's (see [9]), we relate the properties of the Hankel operator H_f to the ones of the non-orthogonal Hankel operator $\tilde{H}_f$. We prove that H_f and $\tilde{H}_f$ have the same properties. Precisely we prove that, if f is holomorphic, H_f is bounded if and only if $\tilde{H}_f$ is bounded, and if and only if f is a Bloch function. Moreover we prove that H_f is compact if and only if $\tilde{H}_f$ is compact, and if and only if f is a little Bloch function. Next we turn to Schatten ideal properties of the operators H_f and $\tilde{H}_f$. Consider the Besov space B_p defined as

$$B_p = \Bigg\{ f \text{ holomorphic:} \\ \int_\Omega \left(|\rho(z)|^m \sum_{|\alpha|=m} \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| \right)^p |\rho(z)|^{-(n+1)} \, dm(z) < \infty \Bigg\}$$

where m is any integer such that $mp > n$. Let G be the explicit kernel mentioned before. Then we prove that the following four conditions are equivalent for f holomorphic in Ω, and $2n < p < \infty$:

(i) $f \in B_p$,
(ii) $H_f \in \mathscr{S}_p$,
(iii) $\tilde{H}_f \in \mathscr{S}_p$,
(iv) $\int_\Omega \int_\Omega |G(z,w)|^2 |f(z) - f(w)|^p \, dm_\nu(z) \, dm_\nu(w) < \infty$.

We also prove that if one of the conditions (ii) through (iv) holds for $0 \leq p < 2n$ then f is constant.

These results extend to the strongly pseudoconvex case results in the aforementioned papers. Some of these results also appear in [12] and [13].

We conclude this introduction by noticing the fact that by construction we consider only the case $n > 1$. For these operators defined on general planar domains, the reader can consult [4].

The paper is organized as follows. The first Section contains the definitions and the statement of the main results. In Section 2 we prove some basic facts about the non-orthogonal projection $\tilde{P}$ and relative kernel $G(z,w)$. The last two sections are devoted to the proofs of the main results.

1. Statement of the main results

Let Ω be a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, $n > 1$. Let ρ be a C^∞ pluri-subharmonic defining function for Ω, defined in a neighborhood of $\overline{\Omega}$:

$$\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\}.$$

Let dm be the Lebesgue volume form in $\mathbf{C}^n$. For $\nu > -1$ we let

$$dm_\nu(z) = |\rho(z)|^\nu \, dm(z),$$

and

$$L^2_\nu = L^2(\Omega, dm_\nu).$$

We consider the weighted Bergman spaces $A^{2,\nu}(\Omega)$, the closed subspaces consisting of the holomorphic functions. The orthogonal projection of L^2_ν onto $A^{2,\nu}$ will be denoted by P. It is well-known that $A^{2,\nu}$ admits a reproducing kernel $K(z,w)$, called the (weighted) Bergman kernel, and defined by the condition

$$Pf(z) = \int_\Omega K(z,w) f(w) \, dm_\nu(w),$$

for $z \in \Omega$ and $f \in L^2_\nu$. Notice that we adopt the convention of writing P and K without explicitly indicating the weight $|\rho|^\nu$. We do so because no confusion will arise.

Together with the orthogonal projection we will consider a non-orthogonal projection $\tilde{P}$, that we define in 2.3 that follows,

$$\tilde{P} : L^2_\nu \to A^{2,\nu}.$$

Such a projection is given by a kernel $G(z,w)$, holomorphic in $z \in \Omega$ and $C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$, where Δ is the diagonal of $b\Omega \times b\Omega$, and $b\Omega$ is the boundary of Ω. Moreover, for all $f \in A^{2,\nu}$

$$f(z) = \int_\Omega G(z,w) f(w) \, dm_\nu(w).$$

Such a kernel (and projection) has been introduced by Ligocka (see [14]), following a construction developed Kerzmen and Stein in [16]. The same projection also appears in [6]. We define the *non-orthogonal Hankel operator* $\tilde{H}_f$ *with symbol* $f \in L^2_\nu$ defined on $A^{2,\nu}$ as

$$\begin{aligned} \tilde{H}_f g(z) &= (I - \tilde{P})(\bar{f}g)(z) \\ &= \int_\Omega \overline{(f(z) - f(w))} G(z,w) g(w)\, dm_\nu(w). \end{aligned}$$

Now, let $f \in C^1(\Omega)$. Consider the modulus of the *covariant derivative* of f at z, i.e.,

$$|\tilde{D}f(z)| = \sup_{\xi \in \mathbf{C}^n,\, |\xi|_{B,z} = 1} |\nabla f(z) \cdot \xi|,$$

where $|\xi|_{B,z}$ is the norm of the vector ξ at the point z in the Bergman metric, and ∇f means the gradient of f. Here and in the rest of the paper we will write $f \in \mathscr{H}(\Omega)$ to indicate the holomorphic functions on Ω. For $f \in \mathscr{H}(\Omega)$ we say that f is a *Bloch function*, and we write $f \in \mathscr{B}$, if

$$\sup_{z \in \Omega} |\tilde{D}f(z)| < \infty.$$

We say that $f \in \mathscr{H}(\Omega)$ belongs to the *little Bloch space*, and we write $f \in \mathscr{B}_0$, if

$$\lim_{z \to b\Omega} |\tilde{D}f(z)| = 0.$$

The Bloch and little Bloch spaces on strongly pseudoconvex domains have been introduced and studied in [17].

Now we are ready to state our two first main theorems.

THEOREM 1.1. *Let $f \in \mathscr{H}(\Omega)$. Then the following are equivalent.*
(i) *$f \in \mathscr{B}$.*
(ii) *H_f is bounded.*
(iii) *$\tilde{H}_f$ is bounded.*

THEOREM 1.2. *Let $f \in \mathscr{H}(\Omega)$. Then the following are equivalent.*
(i) *$f \in \mathscr{B}_0$.*
(ii) *H_f is compact.*
(iii) *$\tilde{H}_f$ is compact.*

The idea of relating the study of Hankel operators to the one-orthogonal Hankel operators comes from [9], where similar results are obtained.

In order to state our next results we need to introduce some more space of functions and operators.

Let $t \in \mathbf{R}$ and $0 < p < \infty$. We define the (*diagonal*) *analytic Besov spaces* $B_p^{t,p}$ as

$$B_p^{t,p} = \left\{ f \in \mathscr{H}(\Omega) \colon \int_\Omega \left(|\rho(z)|^{m-t} \sum_{|\alpha|=m} \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| \right)^p \frac{dm(z)}{|\rho(z)|} < \infty \right\}, \quad (1)$$

where m is a non-negative integer such that $m > t$. We can make $B_p^{t,p}$ into a Banach space by fixing any compact set $E \subset\subset \Omega$ and set

$$\|f\|_{B_p^{t,p}}^p = \int_\Omega \left(|\rho(z)|^{m-t} \sum_{|\alpha|=m} |\partial^\alpha f(x)| \right)^p \frac{dm(z)}{|\rho(z)|} + \sum_{|\alpha|<m} \int_E |\partial^\alpha f(z)|^p \, dm(z).$$

Since we will deal particularly with the space $B_p^{n/p,p}$, we write

$$B_p = B_p^{n/p,p}.$$

Finally, let H be any Hilbert space. Let k be a positive integer. For any compact operator T on H define the k-singular number of T as

$$s_k(T) = \{\inf \|T - R\| \colon \operatorname{rank}(R) \le k\}.$$

We define the Schatten p-class $\mathscr{S}_p$ to be the linear space of compact operators on H for which

$$\sum_1^\infty s_k(T)^p < \infty.$$

THEOREM 1.3. *Let Ω be a C^∞-bounded strongly pseudoconvex domain. Let $2n < p < \infty$. Then the following are equivalent for $f \in \mathscr{H}(\Omega)$.*

(i) $f \in B_p$.

(ii) $\int_\Omega |\tilde{D} f(z)|^p \, d\lambda(z) < \infty$.

(iii) $\int_\Omega \int_\Omega |f(z) - f(w)|^p |G(z,w)|^2 \, dm_\nu(z) \, dm_\nu(w) < \infty$.

Moreover, if $0 < p \le 2n$ and either condition (ii) or (iii) holds, then f is a constant.

Here, and in the rest of the paper, we let $d\lambda \equiv |\rho(z)|^{-(n+1)} dm(z)$, and G is the kernel given by the non-orthogonal projection introduced in 2.3.

THEOREM 1.4. *Let $f \in \mathscr{H}(\Omega)$. Then the following are equivalent.*

(i) $f \in B_p$.

(ii) $H_f \in \mathscr{S}_p$.

(iii) $\tilde{H}_f \in \mathscr{S}_p$.

Moreover, if $0 < p \leq 2n$ and either condition (ii) or (iii) holds, then f is a constant.

2. Basic facts

In this section we construct the non-orthogonal reproducing kernel and describe some basic properties of it. In doing this we follow the construction in [14], whose ideas go back to [16]. We will compare this kernel with the Bergman kernel, of which we describe the asymptotic expansion due to Fefferman (see [7]).

Let Ω be a smoothly bounded strongly pseudoconvex domain,

$$\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\},$$

where ρ is such that the Levi form L_ρ satisfies

$$L_\rho(w)\xi \geq c_1|\xi|^2, \ \xi \in \mathbf{C}^n,$$

for $\rho(w) < \delta_0$, $\delta_0 > 0$, and c_1 depending only on Ω. Now set

$$F(z,w) = -\left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)(z-w) + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(z_j - w_j)(z_k - w_k)\right).$$

By strongly pseudoconvexity and Taylor formula it follows that there exist ε_0, $c_0 > 0$ such that if $\rho(w) > \delta_0$, $|z-w| < \varepsilon_0$, then

$$2\,\mathrm{Re}\, F(z,w) \geq -\rho(z) + \rho(w) + c_0|z-w|^2.$$

Now set

$$\Psi(z,w) = (F(z,w) - \rho(w))\chi(|z-w|) + (1 - \chi(|z-w|))|z-w|^2, \tag{2}$$

where χ is a C^∞ cut-off function of the real variable t, $\chi(t) \equiv 1$ for

$|t| < \varepsilon_0/2$, $\chi(t) = 0$ for $|t| \geq 3/4\varepsilon_0$. Thus, for $\rho(w) < \delta_0$, $|z - w| < \varepsilon_0/2$,

$$|\Psi(z,w)| \approx |\mathrm{Re}\,\Psi| + |\mathrm{Im}\,\Psi|$$
$$\approx |\rho(z)| + |\rho(w)| + |z - w|^2 + |\mathrm{Im}\,\Psi|.$$

Here and in the rest of the paper we adopt the following convention. The notation $\psi \lesssim \phi$ means that $\psi(\xi) \leq c \cdot \phi(\xi)$ for all ξ, and for a constant c depending only on the parameters involved, not on ξ. In the same manner, $\psi \approx \phi$ means $\psi \lesssim \phi$ and $\phi \lesssim \psi$.

2.1 *The weighted Bergman kernel.* For $\nu < -1$ we consider L^2_ν and the weighted Bergman space $A^{2,\nu}$. Let P be the orthogonal projection

$$P: L^2_\nu \to A^{2,\nu},$$

and $K = K(z,w)$ be the (weighted) Bergman kernel. In [7], Fefferman proved that, when $\nu = 0$, for $|\rho(w)| < \delta_0$, $|z - w| < \varepsilon_0/2$,

$$K(z,w) = c_\Omega |\nabla\rho(w)|^2 \det L_\rho(w) \Psi(z,w)^{-(n+1)} + E(z,w),$$

where $E \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$, Δ the diagonal of $b\Omega \times b\Omega$, and

$$|E(z,w)| \lesssim |\Psi(z,w)|^{-(n+1)+1/2} \cdot |\log|\Psi(z,w)||.$$

When $\nu = m$ is a positive integer, we can embed Ω into $\mathbf{C}^{n+m}$ and obtain Fefferman's result for the reproducing kernel of $A^{2,\nu}$. Put

$$\Omega^m = \left\{(z,\xi) \in \mathbf{C}^n \times \mathbf{C}^m \colon \rho_1(z,\xi) \equiv \rho(z) + |\xi|^2 < 0\right\}.$$

The following result is implicit in [15].

LEMMA 2.2. *Let m be a positive integer, and let $K(z,w)$ be the weighted Bergman kernel for $A^{2,m}$, the subspace of $L^2(\Omega, |\rho|^m\, dm)$. Then*

$$K(z,w) = c|\nabla\rho(w)| \det L_\rho(w)(\Psi(z,w))^{-(n+1+m)} + E(z,w),$$

where $E \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$, and E satisfies the estimate

$$|E(z,w)| \lesssim |\Psi(z,w)|^{-(n+1+m)+1/2} \cdot |\log|\Psi(z,w)||.$$

Proof. It is clear that Ω^m is a C^∞-bounded strongly pseudoconvex domain. Therefore, by Fefferman's theorem,

$$\begin{aligned} &K_{\Omega^m}((z,\xi),(w,\eta)) \\ &\quad = c|\nabla\rho_1(w,\eta)|\det L_{\rho_1}(w,\eta)(\Psi((z,\xi),(w,\eta)))^{-(n+1+m)} \\ &\qquad + E((z,\xi)(w,\eta)), \end{aligned}$$

where Ψ is defined as in (2), with the obvious changes. We claim that

$$K_{\Omega^m}((z,0),(w,0)) \equiv K(z,w)$$

is the reproducing kernel for $A^{2,m}(\Omega)$. Indeed, $K(z,w)$ is holomorphic in z, and $K(z,w) = \overline{K(w,z)}$. Moreover, for each fixed z,

$$\begin{aligned} \infty &> \int_{\Omega^m} |K_{\Omega^m}((z,0),(w,0))|^2\, dm(w,\eta) \\ &= \int_\Omega |K(z,w)|^2 \int_{|\eta|^2<|\rho(w)|} dm(\eta)\, dm(w) \\ &= \int_\Omega |K(z,w)|^2 |\rho(w)|^m\, dm(w). \end{aligned}$$

So, $K(z,\cdot\,) \in L^2_m(\Omega)$. Finally, for $f \in L^2_m(\Omega)$ define $\tilde{f} \in L^2(\Omega^m)$ by setting $\tilde{f}(z,\xi) = f(z)$. We have that

$$\begin{aligned} f(z) &= \int_{\Omega^m} K_{\Omega^m}((z,0),(w,\eta))\tilde{f}(w)\, dm(w,\eta) \\ &= \int_\Omega f(w) \int_{|\eta|^2<|\rho(w)|} K_{\Omega^m}((z,0),(w,\eta))\, dm(\eta)\, dm(w) \\ &= \int_\Omega f(w) K(z,w) |\rho(w)|^m\, dm(w). \end{aligned}$$

Thus, $K(z,w)$ is the reproducing kernel for $A^{2,m}(\Omega)$ and

$$\begin{aligned} K(z,w) &= K_{\Omega^m}((z,0),(w,0)) \\ &= c|\nabla\rho(w)|\det L_\rho(w)\Psi(z,w)^{n+1+m} + E(z,w), \end{aligned}$$

and the lemma follows.

The non-orthogonal reproducing kernel. Let $\nu > -1$. Ligocka [15] proved the following.

THEOREM 2.3. *There exists a kernel $G(z,w)$ such that:*
(i) $G \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$, *$G$ is holomorphic in z.*
(ii) *G reproduces the holomorphic functions in $A^{2,\nu}$; i.e., for $f \in A^{2,\nu}$,*

$$f(z) = \int_\Omega G(z,w) f(w)\, dm_\nu(w).$$

(iii) $|G(z,w)| \approx |\Psi(z,w)|^{-(n+1+\nu)}$ *for* $|\rho(w)| < \delta_0$ *and* $|z - w| < \varepsilon_0$.
(iv) $G(z,w) - \overline{G(w,z)} = O(|z-w|^3)$.
Moreover, let

$$\tilde{P}\colon L^2_\nu \to A^{2,\nu}$$

be the integral operator defined by G. Let P and K be respectively the weighted Bergman projection and kernel. Then:
(v) $P = \tilde{P}(I - A)^{-1}$ *and* $P = (I + A)^{-1}\tilde{P}^*$ *where A is a smoothing operator of order $\mu/2$, where $\mu = |\nu - [\nu]|$.*
Here $[x]$ is the integral part of $x \in \mathbf{R}$.

2.4 We adopt the following convention. By the notation $G(z,w)$ we mean the kernel described above for $\nu > -1$, when ν is not an integer. When ν is an integer, $G \equiv K$, the weighted Bergman kernel.

Remark 2.5. The kernel described in 2.3 has the advantage of being explicit, that is the behaviour of $G(z,w)$ along the diagonal of Δ of the boundary is well described, as 2.7 will show. When ν is an integer, Fefferman's theorem [7] and 2.2 give complete information about the behaviour of the weighted Bergman kernel near Δ.

Standard coordinate systems. Near any boundary point $\zeta \in b\Omega$ we introduce a coordinate system that we call *standard*, and that allows us to make precise estimates for the integral kernels.

LEMMA 2.6. *Let Ω be a C^∞-bounded strongly pseudoconvex domain. There exist positive constants ε_0', δ_0', c_Ω, and M such that for any $\zeta \in \Omega$, $|\rho(\zeta)| < \varepsilon_0'$, on $B(\zeta, \delta_0')$ is defined a C^∞-diffeomorphism $t(z,\zeta)$ for which the following hold. The coordinates*

$$t = t(z,\zeta) = (t_1, t_2, t') \in \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^{2n-2}$$

satisfy:

(1) $t_1(z,\zeta) = -\rho(z)$, $t(\zeta,\zeta) = (-\rho(\zeta), 0, \ldots, 0)$.

(2) $t_2(z,\zeta) = \operatorname{Im}\Psi(z,\zeta)$.

(3) $|\mathrm{Jac}_{\mathbf{R}}\, t(\cdot,\zeta)| \le M$.

(4) $|\det \mathrm{Jac}_{\mathbf{R}}\, t(\cdot,\zeta)| \ge 1/M$.

Proof. By (2) we can find $\delta_0, \varepsilon_0 > 0$ so that $\Psi(z,w)$ is well defined. Then, using the same notation as before,

$$d_z(\operatorname{Im}\Psi(z,\zeta))|_{z=\zeta} = d_z(\operatorname{Im}F(z,\zeta))|_{z=\zeta} = d_z\left(\frac{1}{2i}\left(F(z,\zeta) - \overline{F(\zeta,z)}\right)\right)\Big|_{z=\zeta}$$
$$= \frac{1}{2i}\left(-\partial\rho(z) + \overline{\partial\rho(\zeta)}\right).$$

Therefore, at $z = \zeta$,

$$d_z(\operatorname{Im}\Psi) \wedge d_z(-\rho) = i\overline{\partial\rho}(\zeta) \wedge \partial\rho(\zeta) \neq 0.$$

Hence we can find smooth functions t_j, $3 \le j \le 2n$ with $t_j = 0$ for $z = \zeta$ and

$$d_z(-\rho) \wedge d_z(\operatorname{Im}\Psi) \wedge dt_3 \wedge \cdots \wedge dt_{2n} \neq 0$$

at $z = \zeta$. Now we use the inverse function theorem. The construction so obtained holds in a neighborhood of ζ. Since $\overline{\Omega}$ is compact, a finite subcollection of such neighborhoods covers $\overline{\Omega}$. Call these neighborhoods $U(\zeta)$, for some ζ near the boundary. Hence we can determine $\varepsilon_0', \delta_0', M$ so that the conclusions hold. □

Now we use this coordinate system to prove the next result. Put

$$D = \left\{t = (t_1, t_2, t') \in \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^{2n-2} : 0 < t_1 < 1,\ |t_2| < 1,\ |t'| < 1\right\}. \tag{3}$$

LEMMA 2.7. *Let Ω be a C^∞-bounded strongly pseudoconvex domain. Let $a \in \mathbf{R}$, $\nu > -1$, and let $\Psi(z,w)$ be the function defined in* (2). *Then*

$$\int_\Omega \frac{|\rho(w)|^\nu}{|\Psi(z,w)|^{n+1+\nu+a}}\, dm(w) \approx \begin{cases} 1 & \text{if } a < 0 \\ \log|\rho(z)|^{-1} & \text{if } a = 0\,. \\ |\rho(z)|^{-a} & \text{if } a > 0 \end{cases}$$

Proof. This is standard. Otherwise, it suffices to pass to standard coordinates, and use elementary estimates.

LEMMA 2.8. *Let $\nu > -1$, and let $K(z,\zeta)$ be the weighted Bergman kernel. Then*

$$\|K(\cdot,\zeta)\| \lesssim |\rho(\zeta)|^{-(n+1+\nu)/2}.$$

Proof. If ν is an integer the result follows from [7]. Let ν be non-integer. With the notation of 2.8

$$P = \tilde{P}(I-A)^{-1} = (I+A)^{-1}\tilde{P}^*.$$

Put $K_\zeta = K(\cdot,\zeta)$. Then

$$\begin{aligned} \|K_\zeta\|^2 = \|PK_\zeta\|^2 &= \left\|(I+A)^{-1}\tilde{P}^*K_\zeta\right\|^2 \\ &\lesssim \|\tilde{P}^*K_\zeta\|^2. \end{aligned}$$

Now, let $G^*(z,w) = \overline{G(w,z)}$. It follows that

$$\begin{aligned} \tilde{P}^*K_\zeta(z) &= \int_\Omega K_\zeta(w)G^*(z,w)\,dm_\nu(w) \\ &= \overline{\int_\Omega G(w,z)K(\zeta,w)\,dm_\nu(w)} \\ &= G^*(z,\zeta), \end{aligned}$$

where we have used the fact that G is holomorphic in the first variable. Therefore,

$$\|K_\zeta\| \lesssim \|G^*(\cdot,\zeta)\|,$$

and the result follows from 2.7. □

3. Boundedness and compactness

In this section we prove Theorems 1.1 and 1.2. Recall that we fixed $\nu < -1$ and we put $G(z,w)$ to be the reproducing kernel introduced in 2.3, with the convention 2.4. We begin with a lemma that is a generalization of Lemma 5 in [1].

LEMMA 3.1. *Let Ω be a strongly pseudoconvex domain, $\nu < -1$, and let $\Psi(z,w)$ be the function defined in* (2). *Moreover, let $\delta_0 > 0$ be fixed. For any*

$0 < 3a/2 < \nu + 1$ *there exists a constant* $C > 0$ *such that for all* z *with* $-\delta_0 < \rho(z) < 0$, *and for all* $f \in C^1(\Omega)$,

$$\int_{U(z)} \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |\rho(w)|^{-a}\, dm_\nu(w) \leq C|\rho(z)|^{-a} \sup_{\zeta \in U(z)} |\tilde{D}f(\zeta)|,$$

where $U(z)$ *is the neighborhood of* z *on which the standard coordinates are defined* (*see* 2.6).

Proof. Let $k(z,w)$ denote the Kobayashi distance between z and $w \in \Omega$ (see [17] for the definition). Then, for all $f \in C^1(\Omega)$ we have that

$$|f(z) - f(w)| \leq \sup_{\zeta \in U(z)} \tilde{D}f(\zeta) \cdot k(z,w) \text{ for } w \in U(z).$$

By [11] Theorem 4 it follows that, for $0 < \varepsilon < 1$,

$$k(z,w) \lesssim \frac{|\rho(z)|^{-\varepsilon} |\rho(w)|^{-\varepsilon}}{|\Psi(z,w)|^{-2\varepsilon}}.$$

Therefore, taking $\varepsilon = a/2$, we have

$$\begin{aligned} &\int_{U(z)} \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |\rho(w)|^{-a}\, dm_\nu(w) \\ &\lesssim |\rho(z)|^{-a/2} \sup_{\zeta \in U(z)} |\tilde{D}f(\zeta)| \int_{U(z)} \frac{|\rho(w)|^{\nu-3a/2}}{|\Psi(z,w)|^{n+1+\nu-a}}\, dm(w) \\ &\lesssim |\rho(z)|^{-a} \sup_{\zeta \in U(z)} |\tilde{D}f(\zeta)|, \end{aligned}$$

applying 2.7 again. □

Proof of 1.1. Recall that for $f \in \mathscr{H}(\Omega)$, $g \in A^{2,\nu}$

$$\begin{aligned} \tilde{H}_f(g) &= (I - \tilde{P})(\bar{f}g) \\ &= (I - P)(\bar{f}g) + PA(\bar{f}g). \end{aligned}$$

(i) ⇒ (ii). Let $f \in \mathscr{B}$,

$$|\tilde{H}_f(g)(z)|^2 \leq \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |g(w)| dm_\nu(w).$$

Let $0 < 3a/2 < \nu + 1$. Then by the Schwarz inequality we have

$$\left|\tilde{H}_f g(z)\right|^2 \le \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |\rho(w)|^{\nu - a}\, dm_\nu(w)$$
$$\times \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |g(w)|^2 |\rho(w)|^{\nu+a}\, dm_\nu(w). \quad (4)$$

Fix a finite partition of unity on a neighborhood of $\overline{\Omega}$ such that on each open set the standard coordinates are defined. Now using 3.1 it follows that the right hand side of (4) is less or equal to a constant times

$$|\rho(z)|^{-a} \|f\|_{\mathscr{B}} \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{N+1+\nu}} |g(w)|^2 |\rho(w)|^{\nu+a}\, dm_\nu(w).$$

Therefore, by Fubini's theorem and 3.1 again,

$$\|\tilde{H}_f g\|^2_{A^{2,\nu}} \lesssim \|f\|_{\mathscr{B}} \int_\Omega \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |\rho(z)|^{\nu-a} |g(w)|^2 |\rho(w)|^{\nu+a}\, dm_\nu\, dm_\nu$$
$$\lesssim \|f\|^2_{\mathscr{B}} \int_\Omega |g(w)|^2 |\rho(w)|^{\nu}\, dm_\nu(w)$$
$$= \|f\|^2_{\mathscr{B}} \|g\|^2_{A^{2,\nu}}.$$

(ii) $\Rightarrow$ (iii). recall that $\tilde{H}_f = H_f + PA(\bar{f}\,\cdot\,)$. Since H_f and $PA(\bar{f}\,\cdot\,)$ have orthogonal ranges, $\tilde{H}_f$ bounded implies that both H_f and $PA(\bar{f}\,\cdot\,)$ are bounded.

(iii) $\Rightarrow$ (i). This follows from [6], Theorem 5. □

Proof of 1.2. (i) $\Rightarrow$ (ii). Let $f \in \mathscr{B}_0$ and $g \in A^{2,\nu}$. Let $\varepsilon > 0$ be fixed, and let $\delta > 0$. Then

$$\tilde{H}_f g(z) = \int_\Omega \overline{(f(z) - f(w))} G(z,w) g(w)\, dm_\nu(w)$$
$$= \left(\int_{\rho(w) \le -\delta} + \int_{-\delta < \rho(w) < 0} \right) \overline{(f(z) - f(w))} G(z,w) g(w)\, dm_\nu(w)$$
$$\equiv T_1 g(z) + T_2 g(z).$$

We claim that T_1 is compact and that $\|T_2\| < \varepsilon$. From this it follows that $\tilde{H}_f$

is compact. Let $\{g_j\} \in A^{2,\nu}$ be such that $g_j \to 0$ weakly, and hence uniformly on compact subsets. Then

$$\begin{aligned}\|T_1 g_j\|^2 &\leq C_\delta \int_\Omega \left(\int_\Omega |f(z) - f(w)| |g_j(w)| dm_\nu(w) \right)^2 dm_\nu(z) \\ &\leq C_\delta \int_\Omega \left(\int_\Omega |\log|\rho(w)|| + |\log|\rho(z)|| |g_j(w)| dm_\nu(w) \right)^2 dm_\nu(z) \\ &\leq C_\delta \varepsilon,\end{aligned}$$

if $j \geq j_0(\varepsilon)$. Then T_1 is compact. Next, as in the proof of 3.1, it follows that if $0 < 3a/2 < \nu + 1$,

$$\begin{aligned}|T_2 g(z)|^2 \lesssim |\rho(z)|^{-a} &\sup_{|\rho(z)|<\delta} |\tilde{D}f(z)| \\ &\times \int_{-\delta<\rho<0} \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |g(z)|^2 |\rho(w)|^{\nu+a} dm_\nu(w).\end{aligned}$$

Therefore,

$$\begin{aligned}\|T_2 g\|^2_{A^{2,\nu}} &\lesssim \sup_{|\rho(z)|<\delta} |\tilde{D}f(z)| \int_{-\delta<\rho<0} |g(z)|^2 |\rho(w)|^{\nu+a} \\ &\qquad \times \int_\Omega \frac{|f(z) - f(w)|}{|\Psi(z,w)|^{n+1+\nu}} |\rho(z)|^{\nu-a} dm_\nu(z)\, dm_\nu(w) \\ &\lesssim \|f\|_{\mathscr{B}} \|g\|^2_{A^{2,\nu}} \sup_{|\rho(\zeta)|<\delta} |\tilde{D}f(\zeta)|.\end{aligned}$$

Since $f \in \mathscr{B}_0$, $\sup_{|\rho(\zeta)|<\delta} |\tilde{D}f(\zeta)|$ can be as small as we like by taking δ small enough. Therefore $\|T_2\| < \varepsilon$ for $\delta < \delta(\varepsilon)$ and the claim is established.

(ii) $\Rightarrow$ (iii). This is as in the proof of 1.1.

(iii) $\Rightarrow$ (i). This follows from [6], Theorem 7. □

Remark 3.2. The assumption f holomorphic has been used only in proving the implication "H_f bounded (compact) implies $f \in \mathscr{B}(\mathscr{B}_0)$". Consider the linear space $\mathscr{C} \equiv \mathrm{I}^1(\Omega) \cap L^2(\Omega)$ with the norm

$$\|f\|_{\mathscr{C}} = \sup_{z\in\Omega} |\tilde{D}f(z)| + \|f\|_{L^2(\Omega)}.$$

Notice that if f is holomorphic then $\|f\|_{\mathscr{C}} \approx \|f\|_{\mathscr{B}}$. Moreover, consider the subspace $\mathscr{E}_0$ of $\mathscr{E}$ of the functions for which

$$\lim_{\rho(\zeta)\to 0^-} |\tilde{D}f(\zeta)| = 0.$$

Then we have the following

Corollary 3.3. *Let $f \in \mathscr{E}$ (respectively $\mathscr{E}_0$). Then the Hankel and non-orthogonal Hankel operators H_f and $\tilde{H}_f$ are bounded (resp. compact) on $A^{2,\nu}$.*

It would be interesting to prove the final implication, that is "H_f, $\tilde{H}_f$ bounded (resp. compact), implies $f \in \mathscr{E}$ (resp. $\mathscr{E}_0$)". So far, we have not been able to prove the statement. Related results are contained in [10], [12], and [13].

4. Besov spaces and Schatten ideal classes

In this section we prove Theorems 1.3 and 1.4. We begin with 1.3, the proof of which requires us to show few lemmas.

Lemma 4.1. *Let $\nu > -1$, $\beta > 0$. For $\alpha, t > 0$ set*

$$h_\alpha(t) = \frac{t^\nu}{(\alpha + t)^\beta},$$

and

$$H(t) = \int_0^t h_\alpha(\tau)\, d\tau.$$

Then for all $M > 0$ there exists a positive constant $C = C(M, \nu, \beta)$, such that

$$H(t) \leq C t h_\alpha(t)$$

for all $\alpha > 0$, and $0 < t < M\alpha$.

Proof. First of all we dispose of the case $\beta \leq \nu$. In this case an integration by parts give that

$$H(t) \leq \frac{1}{\nu + 1} t h_\alpha(t) + \frac{\beta}{\nu + 1} H(t).$$

Thus,

$$H(t) \leq \frac{1}{\nu + 1 - \beta} t h_\alpha(t) \quad \text{for all } \alpha, t > 0.$$

Suppose now that $\beta > \nu$. For any positive integer m, integrating by parts m-times gives that

$$H(t) = \sum_{j=1}^{m} c_j(\nu,\beta)\frac{t^{\nu+j}}{(\alpha+t)^{\beta+j-1}} + c(\nu,\beta)\int_0^t \frac{\tau^{\nu+m}}{(\alpha+t)^{\beta+m}}\,d\tau. \qquad (5)$$

where $c_1 = 1/(\nu+1)$, and for $j \geq 2$,

$$c_j = \beta(\beta+1)\cdots(\beta+j-2)((\nu+1)\cdots(\nu+j))^{-1}.$$

By applying the mean value theorem we see that

$$\int_0^t \frac{\tau^{\nu+m}}{(\alpha+\tau)^{\beta+m}}\,d\tau \leq t\frac{t^{\nu+m}}{(\alpha+t)^{\beta+m}} \qquad (6)$$

for

$$t < \frac{\nu+m}{\beta-\nu}\alpha.$$

Having fixed M, we can choose m such that

$$M < \frac{\nu+m}{\beta-\nu}.$$

Plugging (5) into (6) we find that

$$H(t) \leq C(M,\nu,\beta)th_\alpha(t) \text{ for } 0 < t < \frac{\nu+m}{\beta-\nu}\alpha.$$

This finishes the proof. □

Proposition 4.2. *Let $\nu > -1$ and $1 < p < \infty$. Then there exists a constant $C > 0$ such that for all $f \in C^1(\Omega)$,*

$$\int_\Omega\int_\Omega \frac{|f(z)-f(w)|^p}{|\Psi(z,w)|^{2(n+1+\nu)}}\,dm_\nu(z)\,dm_\nu(w)$$
$$\leq C\left(\int_\Omega |\tilde{D}f(z)|^p|\rho(z)|^{-(n+1)}\,dm(z) + \|f\|_{L^p_\nu}\right).$$

Proof. In [16] it was proved that

$$\Psi(z,w) = \overline{\Psi(w,z)} + O(|z-w|^3) \text{ for } z,w \in \Omega.$$

Because of this symmetry it suffices to estimate the integral over the subset $\mathscr{D}$ of $\Omega \times \Omega$,

$$\mathscr{D} = \{(z,w) \in \Omega \times \Omega : |\rho(z)| < |\rho(w)|\}.$$

Moreover, it is also clear that it suffices to estimate integrals of the kind

$$\int_{U(\zeta)} \int_{U(z)\cap\{|\rho(z)|<|\rho(w)|\}} \frac{|f(z) - f(w)|^p}{|\Psi(z,w)|^{2(n+1+\nu)}} \, dm_\nu(w) \, dm_\nu(z),$$

where ζ is any point on $b\Omega$. We apply the change of coordinates described in 2.6. Put

$$E_t = \{(s_1, s_2, s') \in \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^{2n-2} : t_1 < s_1 < 1, |t_2 - s_2| < 1, |t' - s'| < 1\}.$$

We find that the above double integral is less than or equal to a constant times

$$\int_D \int_{E_t} \frac{|f(t) - f(s)|^p}{\left(t_1 + s_1 + |s_2 - t_2| + |s' - t'|^2\right)^{2(n+1+\nu)}} s_1^\nu t_1^\nu \, ds \, dt$$
$$= \int_D \int_{t_1}^1 \int_{|s_2|<1} \int_{|s'|<1} \frac{|f(t_1, t_2, t') - f(s_1, s_2 + t_2, s' + t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)}} s_1^\nu t_1^\nu \, ds \, dt.$$

Now we break the integral into three different ones, called I, II, and III respectively, by majorizing the numerator of the integrand as follows:

$$\begin{aligned} &|f(t_1, t_2, t') - f(s_1, s_2 + t_2, s' + t')|^p \\ &\lesssim |f(t_1, t_2, t') - f(s_1, t_2, t')|^p \\ &\quad + |f(s_1, s_2 + t_2, t') - f(s_1, t_2, t')|^p \\ &\quad + |f(s_1, s_2 + t_2, s' + t') - f(s_1, s_2 + t_2, t')|^p. \end{aligned}$$

We estimate the three different terms I, II, and III in a sequence of claims.

Claim 1.

$$I \lesssim \int_D \left| t_1 \frac{\partial}{\partial t_1} f(t_1, t_2, t') \right|^p t_1^{-(n+1)} \, dt.$$

Proof of Claim 1. We need to estimate the double integral

$$\int_D \int_{D\cap\{t_1<s_1\}} \frac{|f(t_1,t_2,t') - f(s_1,t_2,t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)}} s_1^\nu t_1^\nu \, ds\, dt.$$

Now set $\beta = 2(n+1+\nu)$, $\alpha = s_1 + |s_2| + |s'|^2$. Also, put

$$h_\alpha(t_1) = \frac{t_1^\nu}{(\alpha + t_1)^\beta},$$

and

$$H(t_1) = \int_0^{t_1} h_\alpha(\tau_1)\, d\tau.$$

By 4.1 we know that $H(t_1) \lesssim t_1 h_\alpha(t_1)$ for $t_1 < \alpha$, in particular for $t_1 < s_1 \le \alpha$. Now, we proceed with an integration in the t_1 variable

$$\begin{aligned}\int_0^{s_1} &|f(t_1,t_2,t') - f(s_1,t_2,t')|^p h_\alpha(t_1)\, dt_1 \\ &= \left[|f(t_1,t_2,t') - f(s_1,t_2,t')|^p H(t_1)\right]_0^{s_1} \\ &\quad - p\int_0^{s_1} |f(t_1,t_2,t') - f(s_1,t_2,t')|^{p-1} \frac{\partial}{\partial t_1} |f(t_1,t_2,t')| H(t_1)\, dt_1.\end{aligned} \tag{7}$$

Now we use the estimate

$$\left|\frac{\partial}{\partial x} |\phi(x)|\right| \le \left|\frac{\partial}{\partial x}\phi\right|.$$

This inequality holds for all $\phi \in C^1$ and for all x for which $\phi(x) \ne 0$. When we pass to an integral we see that we can simply extend the above inequality to all x. Since the first term on the right hand side of (7) is zero, we see that the left hand side is majorized by a constant times

$$\begin{aligned}\int_0^{s_1} &|f(t_1,t_2,t') - f(s_1,t_2,t')|^{p-1} \left| t_1 \frac{\partial}{\partial t_1} f(t_1,t_2,t')\right| h_\alpha(t_1)\, dt_1 \\ &\le \left\{\int_0^{s_1} |f(t_1,t_2,t') - f(s_1,t_2,t')|^p h_\alpha(t_1)\, dt_1\right\}^{1/p'} \\ &\quad \times \left\{\int_0^{s_1} \left| t_1 \frac{\partial}{\partial t_1} f(t_1,t_2,t')\right|^p h_\alpha(t_1)\, dt_1\right\}^{1/p},\end{aligned}$$

where we have applied Hölder's inequality with conjugate exponents p and p'. Hence,

$$\int_0^{s_1} |f(t_1, t_2, t') - f(s_1, t_2, t')|^p h_\alpha(t_1)\, dt_1 \lesssim \int_0^{s_1} \left| t_1 \frac{\partial}{\partial t_1} f(t_1, t_2, t') \right|^p h_\alpha(t_1)\, dt_1.$$

Therefore, by 2.7,

$$I \lesssim \int_D \left| t_1 \frac{\partial}{\partial t_1} f(t_1, t_2, t') \right|^p t_1^\nu \int_D \frac{s_1^\nu}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)}}\, ds\, dt$$
$$\lesssim \int_D \left| t_1 \frac{\partial}{\partial t_1} f(t_1, t_2, t') \right|^p t_1^{-(n+1)}\, dt.$$

This establishes Claim 1.

Claim 2.

$$II \lesssim \int_D \left| t_1 \frac{\partial}{\partial t_2} f(t_1, t_2, t') \right|^p t_1^{-(n+1)}\, dt.$$

Proof of Claim 2. We argue essentially as in Claim 1. First we need an integration by parts. Notice that,

$$\int_0^1 \frac{t_1^\nu}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)}}\, dt_1$$
$$\lesssim \frac{1}{\left(1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)}} + \int_0^1 \frac{t_1^\nu\, dt_1}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)+1}}$$
$$\lesssim H(s) + \int_0^1 \frac{t_1^{\nu+k}}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)+k}}\, dt_1,$$

where $H \in C^\infty(\overline{D})$, and k is an integer. Then, if we choose $k > p$,

$$II \lesssim \int_D \int_0^1 \int_{|s_2|<1} \int_{|s'|<1} \frac{|f(s_1, s_2 + t_2, t') - f(s_1, t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)+k}} s_1^\nu\, ds t_1^{\nu+k}\, dt.$$

Now consider the integral

$$\int_{|s_2|<1} \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^\beta} \, ds_2,$$

where we have set $\beta = 2(n+1+\nu)+k$. By symmetry we can integrate over $\{0 < s_2 < 1\}$. Then set

$$II' = \int_0^1 \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^\beta} \, ds_2.$$

We integrate by parts in II'.

$$II' = -\frac{p}{\beta-1}\int_0^1 \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^{p-1}}{\left(t_1 + s_2 + |s_2| + |s'|^2\right)^{\beta-1}} \frac{\partial}{\partial s_2} |f(s_1, s_2, t')| ds_2$$
$$+ \left[\frac{1}{\beta-1} \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta-1}}\right]_0^1. \tag{8}$$

Notice that the second term on the right hand side of (8) can be easily estimated. Thus, using Hölder's inequality with p and p' conjugate exponents, it follows that

$$II' \lesssim \int_0^1 \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^{p-1}}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta-1}} \left|\frac{\partial}{\partial s_2} f(s_1, s_2, t')\right| ds_2$$
$$\leq \left\{\int_0^1 \frac{|f(s_1, s_2+t_2, t') - f(s_1, t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^\beta} \, ds_2\right\}^{1/p'}$$
$$\times \left\{\int_0^1 \frac{\left|\frac{\partial}{\partial s_2} f(s_1, s_2, t')\right|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta-p}} \, ds_2\right\}^{1/p}.$$

Thus,

$$II' \lesssim \int_0^1 \frac{\left|\frac{\partial}{\partial s_2} f(s_1, s_2, t')\right|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta-p}} \, ds_2.$$

Finally,

$$II \lesssim \int_D \int_0^1 \frac{\left| \frac{\partial}{\partial s_2} f(s_1, s_2 + t_2, t') \right|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)+k-p}} s_1^\nu \, ds \, t_1^{\nu+k} \, dt$$

$$\lesssim \int_D \int_0^1 \frac{\left| \frac{\partial}{\partial s_2} f(s_1, s_2 + t_2, t') \right|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{2(n+1+\nu)+k-p}} s_1^{\nu+k} \, ds \, t_1^\nu \, dt.$$

Next we switch the integration order having enlarged the region of integration of s_1 to $\{0 < s_1 < 1\}$. Applying 2.7 to the kernel at the denominator of the fraction in the last integral, over the region

$$\{0 < t_1 < 1\} \times \{|s_2| < 1\} \times \{|s'| < 1\}$$

we find that

$$II \lesssim \int_0^1 \int_0^2 \int_{|t'|<1} \left| \frac{\partial}{\partial s_2} f(s_1, s_2, t') \right|^p s_1^{\nu+k-(k-\nu-p+n+1)} \, dt' \, ds_2 \, ds_1$$

$$\lesssim \int_D \left| s_1 \frac{\partial}{\partial s_2} f(s_1, s_2, s') \right|^p s_1^{-(n+1)} \, ds.$$

This proves Claim 2.

Claim 3.

$$III \lesssim \int_D |s_1^{1/2} \nabla_{s'} |f(s_1, s_2, s')|^p s_1^{-(n+1)} \, ds.$$

Proof of Claim 3. All the ingredients appeared already in the proofs of Claim 1 and Claim 2. Integrating by parts in the t_1 variable we see that

$$III \lesssim \int_D \int_{D \cap \{t_1 < s_1\}} \frac{|f(s_1, s_2 + t_2, s' + t') - f(s_1, s_2 + t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta+k}} s_1^\nu \, ds_1 \, t_1^{\nu+k} \, dt,$$

where k is an integer larger than p, and $\beta = 2(n + 1 + \nu)$. Next we consider the integral

$$III' = \int_{|s'|<1} \frac{|f(s_1, s_2 + t_2, s' + t') - f(s_1, s_2 + t_2, t')|^p}{\left(t_1 + s_1 + |s_2| + |s'|^2\right)^{\beta+k}} \, ds'.$$

By passing into polar coordinates, setting $s' = ru$, $u \in S$, $0 < r < 1$, we find that

$$III' = \int_S \int_0^1 \frac{|f(s_1, s_2 + t_2, ru + t') - f(s_1, s_2 + t_2, t')|^p}{(t_1 + s_1 + |s_2| + r^2)^{\beta + k}} r^{2n-3}\, dr\, d\sigma(u).$$

We apply the same procedure as in Claim 2 to the inner integral. It follows

$$III' \lesssim \int_{|s'|<1} \frac{|\nabla_{s'} f(s_1, s_2 + t_2, s' + t')|^p}{(t_1 + s_1 + |s_2| + |s'|^2)^{\beta + k - p/2}}\, ds'.$$

Then, using 2.7 again

$$\begin{aligned} III &\lesssim \int_D \int_{D \cap \{t_1 < s_1\}} \frac{|\nabla_{s'} f(s_1, s_2 + t_2, s' + t')|^p}{(t_1 + s_1 + |s_2| + |s'|^2)^{2(n+1+\nu)+k-p/2}} s_1^{\nu+k}\, ds\, t_1^{\nu}\, dt \\ &\lesssim \int_0^1 \int_0^1 \int_{|s'|<2} |\nabla_{s'} f(s_1, s_2 + t_2, s' + t')|^p s_1^{k-(k-p/2)-(n+1)}\, ds, \end{aligned}$$

which proves Claim 3, and the proposition. □

LEMMA 4.3. *Let $\tilde{\Omega}$ be a C^2-bounded strongly pseudoconvex domain. Let $\nu > -1$ and $0 < r < \infty$. Let $\beta(z, r)$ denote the ball in the Bergman metric centered at $z \in \Omega$ with radius r. Then there exists a constant $C > 0$ such that for all $f \in \mathscr{H}(\Omega)$*

$$|\tilde{D} f(z)| \le \frac{C}{|\beta(z, r)|_\nu} \int_{\beta(z,r)} |f(\zeta) - f(z)|\, dm_\nu(\zeta),$$

where $|\beta|_\nu$ denotes the volume of the set β with respect to dm_ν.

Proof. Since $\tilde{\Omega}$ is strongly pseudoconvex we have that

$$|\tilde{D} f(z)| \approx |\rho(z)| |\nabla_N f(z)| + |\rho(z)|^{1/2} |\nabla_T f(z)|,$$

where ∇_N and ∇_T denote the derivatives in the complex normal and complex tangential directions respectively, see [17]. It is well known that $\beta(z, r)$ is comparable with the product of a disc and of a $2n - 2$ real dimensional ball,

$$\beta(z, r) \simeq D_{N(z)}(z, c_1 |\rho(z)|) \times B'_{T(z)}(z, c_2 \rho^{1/2}),$$

where

$$D_{N(z)} = \{\zeta \in \mathbf{C}^n : \zeta = z + c_1|\rho(z)|\eta N(z)\},$$

and $N(z)$ indicates the normal direction at $z \in \tilde{\Omega}$, $\eta \in \mathbf{C}$, $|\eta| < 1$ and c_1 is a constant that depends only on $\tilde{\Omega}$. Moreover,

$$B_{T(z)} = \left\{\zeta \in \mathbf{C}^n : \zeta = z + c_2|\rho(z)|^{1/2}\xi,\ \xi \cdot \overline{N(z)} = 0\right\},$$

and c_2 is another constant. Then

$$|\nabla_N f(z)| \le \frac{C}{|\rho(z)|^3}\int_{D_{N(z)}} |f(z + c_1|\rho(z)|\eta) - f(z)|\, dm(\eta).$$

(Here dm is the 2-dimensional Lebesgue measure.) Therefore, using the submean value theorem in the tangential directions we see that

$$|\nabla_N f(z)| \le \frac{C}{|\rho(z)|^{n+2}} \times \int_{D_{N(z)}}\int_{B'_{T(z)}} \left|f\left(z + c_1|\rho(z)|\eta + c_2|\rho(z)|^{1/2}\xi\right) - f(z)\right| dm(\xi)\, dm(\eta).$$

(Here $dm(\xi)$ is the $(2n-2)$-dimensional Lebesgue measure, thinking of ξ as vector in $\mathbf{C}^{n-1}$.) Since $|\beta(z,r)| \approx |D_{N(z)} \times B'_{T(z)}| \approx |\rho(z)|^{n+1}$, we have bounded one term of the desired estimate. In order to estimate the term $|\rho(z)|^{1/2}|\,|\nabla_T f(z)|$ we argue in the same fashion:

$$\begin{aligned}|\nabla_T f(z)| &\le \frac{C}{|\rho(z)|^{n-1/2}}\int_{B'_{T(z)}} \left|f\left(z + c_2|\rho(z)|^{1/2}\xi\right) - f(z)\right| dm(\xi)\\ &\le \frac{C}{|\rho(z)|^{n+3/2}} \times \int_{B'_{T(z)}}\int_{D_{N(z)}} \left|f\left(z + c_1|\rho(z)|\eta + c_2|\rho(z)|^{1/2}\xi\right) - f(z)\right| dm(\eta)\, dm(\xi)\end{aligned}$$

The estimate now follows. □

Proof of 1.3. Suppose $2n < p < \infty$ first. The implication (i) ⇒ (ii) is trivial. The proof of (ii) ⇒ (iii) is contained in 4.2 where f is assumed to be only $C^1(\Omega)$. The implication (iii) ⇒ (i) now follows from 4.3 and the implication (ii) ⇒ (i) valid for $f \in \mathscr{H}(\Omega)$. A proof of this fact can be found in [5].

Finally, suppose $0 < p \le 2n$, and $f \in \mathscr{H}(\Omega)$. Moreover assume that (ii) or (iii) holds. Lemma 4.3 gives that (iii) $\Rightarrow$ (ii). Therefore it suffices to prove that the condition

$$\int_\Omega |\nabla_T f(z)|^p |\rho(z)|^{p/2-(n+1)}\, dm(z) < \infty,\ f \in \mathscr{H}(\Omega) \tag{9}$$

implies that f is constant. Since Ω is strongly pseudoconvex it follows that the functions that are holomorphic in a neighborhood of $\overline{\Omega}$ are dense in B_p (see [18] for instance). Hence we can assume that f in the integral in (9) is holomorphic across the boundary. This implies that $|\nabla_T f| \equiv 0$ near $b\Omega$. Thus, f is constant on the level sets $\{\rho(z) = -\varepsilon\}$, for $0 < \varepsilon < \varepsilon_0$, for some $\varepsilon_0 > 0$. Since f can be reproduced from its boundary values (on a slightly smaller domain), it follows that f is constant. This finishes the proof. □

Now we turn to the proof of 1.4. We need a proposition which is a version in the strongly pseudoconvex case as a result of Russo's, (see [19]), refined by Arazy, Fisher, Janson, and Peetre (see [2] Lemma 3.6 and Theorem 6). We begin with a lemma. In this lemma $L^{1\infty}(dm_\nu)$ denotes the weak-L^1 space with respect to the measure dm_ν, (recall also the notation introduced in 2.4).

LEMMA 4.4. *Let $\nu > -1$. Then*

$$\sup_{z\in\Omega} \|G(\cdot, z)\|_{L^{1\infty}(dm_\nu)} < \infty.$$

Proof. The statement is clear when $|\rho(z)| \ge \delta_0 > 0$. Then we want to show that

$$G(\cdot, z) \in L^{1\infty}(dm_\nu)$$

with norm uniformly bounded in $z \in \Omega$, $|\rho(z)| < \delta_0$. Let $\tau > 0$. Set $r = \tau^{-1/(n+1+\nu)}$. Using the special coordinates we see that

$$\begin{aligned} m_\nu\{w\colon |G(w,z)| > \tau\} &\lesssim \int_D \chi_{\{s_1+|s_2|+|s'|^2<r\}} s_1^\nu\, ds \\ &\le \int_0^r \int_{|s_r|<r} \int_{|s'|<r^{1/2}} ds'\, ds_2\, s_1^\nu\, ds_1 \\ &\lesssim \int_0^r s_1^{n+\nu}\, ds_1 \\ &\lesssim \tau^{-1}, \end{aligned}$$

which is the desired inequality. □

PROPOSITION 4.5 (*Russo-Arazy, Fisher, Janson, Peetre*). *Let $2 \le p < \infty$, and let H be any measurable function on $\Omega \times \Omega$. Suppose that*

$$\int_\Omega \int_\Omega |H(z,w)|^p |G(z,w)|^2 \, dm_\nu(z) \, dm_\nu(w) < \infty.$$

Then the kernel $H(z,w)G(z,w)$ defines an operator in $\mathscr{S}_p$ of L^2_ν.

Proof. Given 4.4 and Theorem 6 of [2], the proof is the same as the proof of Lemma 3.6 of [2]. □

Proof of 1.4. The implication (i) ⇒ (ii) follows from 1.3 and 4.5.

(ii) ⇒ (iii). Recall that if ν is an integer, H_f and $\tilde{H}_f$ are the same operator. For ν not an integer

$$P = \tilde{P}(I - A)^{-1},$$

that is

$$P = \tilde{P} + PA,$$

where A is a $\mu/2$-smoothing operator, $\mu = |\nu - [\nu]|$. Then

$$(I - \tilde{P}) = (I - P) + PA.$$

Notice that the operators H_f and $PA(\bar{f}\,\cdot\,)$ have orthogonal ranges. Thus, if $\tilde{H}_f \in \mathscr{S}_p$, both H_f and $PA(\bar{f}\,\cdot\,) \in \mathscr{S}_p$.

(iii) ⇒ (i). Suppose now $p > 0$ and $H_f \in \mathscr{S}_p$. Then $h_f \equiv P(\bar{f}\,\cdot\,) \in \mathscr{S}_p$ and therefore also the operator T,

$$T \equiv (I - P)(\bar{f}\,\cdot) - AP(\bar{f}\,\cdot).$$

Recall that $(I + A)P = \tilde{P}^*$. Hence,

$$\begin{aligned} Tg(z) &= (I - \tilde{P})(\bar{f}g)(z) \\ &= \int_\Omega \overline{(f(z) - f(w))} G^*(z,w) \, dm_\nu(w), \end{aligned}$$

where $G^*(z,w) = \overline{G(w,z)}$. Now recall that for all operators S on L^2_ν,

$$\|S\|_{\mathscr{S}_2} = \int_\Omega \|Sk_\zeta\| d\lambda(\zeta),$$

where $k_\zeta = K(\cdot,\zeta)/\|K(\cdot,\zeta)\|$. Recall that by 2.8

$$\|K(\cdot,\zeta)\| \lesssim \|\rho(\zeta)\|^{-(n+1+\nu)/2}.$$

Moreover notice that

$$\begin{aligned} TK(\cdot,\zeta)(z) &= \int_\Omega \overline{(f(z)-f(w))}G^*(z,w)K(w,\zeta)\,dm_\nu(w) \\ &= \overline{\int_\Omega (f(z)-f(w))G(w,z)K(\zeta,w)\,dm_\nu(w)} \\ &= \overline{(f(z)-f(\zeta))}G^*(z,\zeta), \end{aligned}$$

since $(f(z)-f)G(\cdot,z)$ is holomorphic. Thus,

$$\begin{aligned} &\int_\Omega \|Tk_\zeta\|^2\,d\lambda(\zeta) \\ &\quad= \int_\Omega \|K(\cdot,\zeta)\|^2 \int_\Omega |f(z)-f(\zeta)|^2 |G(z,\zeta)|^2\,dm_\nu(z)\,d\lambda(\zeta) \\ &\quad\gtrsim \int_\Omega |\rho(\zeta)|^{n+1+\nu} \int_\Omega |f(z)-f(\zeta)|^2 |G(z,\zeta)|^2\,dm_\nu(z)\,d\lambda(\zeta) \\ &\quad\approx \int_\Omega\int_\Omega |f(z)-f(\zeta)|^2 |G(z,\zeta)|^2\,dm_\nu(z)\,dm_\nu(\zeta). \end{aligned}$$

Finally, Theorem 1.3 finishes the proof in both cases, $2n < p < \infty$, and $0 < p \le 2n$. □

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