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FULL NEST ALGEBRAS

ALAN HOPENWASSER AND JUSTIN R. PETERS[1]

I. Introduction

Direct limits of finite dimensional operator algebras have become a source for many interesting examples of non-self-adjoint operator algebras. This note will study a special class of such algebras—direct limits of full upper triangular matrix algebras with nest preserving embeddings. These limit algebras are members of three important classes of operator algebras: they are simultaneously nest subalgebras, analytic subalgebras, and strongly maximal triangular subalgebras of the UHF C*-algebras which they generate.

In what follows, we shall primarily study directed systems with the following form:

$$T_{n_1} \xrightarrow{\nu_1} T_{n_2} \xrightarrow{\nu_2} T_{n_3} \xrightarrow{\nu_3} \cdots \longrightarrow A$$

where T_n is the algebra of upper triangular $n \times n$ matrices and each ν is a unital isometric homomorphism. We further require that each embedding ν satisfy the following properties:

(1) ν has an extension to a *-homomorphism of M_n.

(2) ν maps a matrix unit in T_{n_k} to a sum of matrix units in $T_{n_{k+1}}$.

(3) ν maps $\mathscr{L}at\, T_{n_k}$ into $\mathscr{L}at\, T_{n_{k+1}}$.

Properties 1 and 2 are standard assumptions (but see [P3, P4] for information on what may happen if these assumptions are not satisfied). Property 3 is satisfied by the refinement embedding, ρ: $T_n \to T_{nk}$, which is defined as follows: $\rho[a_{ij}] = [a_{ij}I_k]$, for all $[a_{ij}] \in T_n$. The embedding most often contrasted with the refinement embedding, the standard embedding, σ: $T_n \to T_{nk}$, defined by $\sigma(a) = a \oplus \cdots \oplus a$ (k factors), does not satisfy property 3.

DEFINITION 1.1. An embedding ν which satisfies properties 1, 2, and 3 will be called a *nest embedding*.

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Remark. This definition is more restrictive than the definition used in [P4], where property 2 is not assumed to be satisfied for nest embeddings. In the literature, embeddings which satisfy property 2 are sometimes called *regular*. The restrictive definition is more convenient for the purposes of this paper. Thus, in this paper, all embeddings are assumed to be regular.

It can be seen easily that an embedding $\nu: T_n \to T_{nk}$ is a nest embedding if and only if its restriction to the diagonal algebra, D_n, is equal to the restriction of ρ to D_n. It is also easy to see that any nest embedding, ν, has the following form: $\nu[a_{ij}] = [a_{ij}U_{ij}]$, where each U_{ij} is a $k \times k$ permutation unitary matrix and the U_{ij} satisfy the cocycle condition $U_{ij}U_{jh} = U_{ih}$. Note that $U_{ii} = I_k$, for each $i = 1, 2, \dots, n$ and that ν is completely determined by the $n - 1$ unitaries on the first superdiagonal.

DEFINITION 1.2. If the $n - 1$ unitaries which determine ν are all equal, then ν is said to be *homogeneous*.

Let $\pi \in S_k$ be a permutation on k elements. Define U_π to be the permutation unitary matrix whose i, j-entry is 1 if, and only if, $\pi(j) = i$. The homogeneous embedding determined by the unitary U_π will be denoted by ν_π. A homogeneous embedding is given by the formula $\nu_\pi[a_{ij}] = [a_{ij}U_\pi^{j-i}]$.

Clearly, there is a one-to-one correspondence between $(n - 1)$-tuples of permutations in S_k and nest embeddings mapping T_n into T_{nk}. In particular, the number of such embeddings is $(k!)^{n-1}$. The number of homogeneous embeddings is, of course $k!$.

DEFINITION 1.3. A subalgebra, A, of a UHF C*-algebra which is the direct limit of a system

$$T_{n_1} \xrightarrow{\nu_1} T_{n_2} \xrightarrow{\nu_2} T_{n_3} \xrightarrow{\nu_3} \cdots \longrightarrow A,$$

where the ν's are nest embeddings, will be called a *full nest subalgebra*.

Different direct limit systems may yield (isometrically) isomorphic algebras. A complete invariant, the fundamental relation, introduced by Power in [P1], is very useful for determining when two direct limits are indeed isomorphic. This invariant is, in fact, a complete invariant for triangular subalgebras of AF C*-algebras whose diagonal is a canonical masa. The fundamental relation can be described in groupoid language: every UHF algebra is a groupoid C*-algebra and there is a subsemigroupoid which is the support set of the triangular subalgebra. This support subsemigroupoid can be identified with the fundamental relation. We shall not need to use the language of groupoids in this paper; rather, we will work with a particular representation of the fundamental relation. This representation will be easy to compute and will effectively determine whether two algebras are isomorphic.

To describe the fundamental relation in this context, let $A = \varinjlim (T_{n_i}, \nu_i)$ be a direct limit of a system with nest embeddings and let $D = \varinjlim (D_{n_i})$ be the direct limit of the diagonal algebras of the T_{n_i}. D is the diagonal of the triangular algebra, A. The maximal ideal space of D can be identified with the Cartesian product $X = \prod[n_i]$, where $[n_i] = \{0, 1, 2, \ldots, n_i - 1\}$. X carries the product topology, and so is a Cantor space. The fundamental relation is a topologized relation on X. While this relation is properly defined in terms of the partial isometries in A which normalize D, it turns out that it is sufficient to use only the matrix units from the T_{n_i}. It is exactly this point which makes it feasible to compute the fundamental relation in many examples.

If e is a matrix unit in A, then e induces a partially defined homeomorphism of X, whose graph we may denote by E. The fundamental relation, FR, is the union of the graphs of all the matrix units in A. The topology on FR is determined by taking the collection of graphs of matrix units as a basis.

An important result of Power [P2] states that two triangular subalgebras of AF C*-algebras are isometrically isomorphic if, and only if, their fundamental relations are isomorphic as topological relations. For full nest algebras with the specific representation of the fundamental relation as described above (and the same sequences of multiplicities in their presentations), it turns out that the topological relation isomorphism is the identity map. Consequently, we are able to distinguish algebras simply by showing that their fundamental relations are distinct as sets. Indeed we have the following lemma:

LEMMA 1.4. *Let $A = \varinjlim (T_{n_i}, \nu_i)$ and $B = \varinjlim (T_{n_i}, \mu_i)$, where the ν_i and μ_i are nest embeddings. If $\Phi\colon A \to B$ is an isometric isomorphism, then $\Phi\colon A \cap A^* \to B \cap B^*$ is the identity map. If $\phi\colon FR(A) \to FR(B)$ is the topological isomorphism of fundamental relations induced by Φ, then ϕ is the identity map.*

Proof. Since Φ is an isometric isomorphism, Φ maps $\mathcal{L}at\,(A)$ onto $\mathcal{L}at\,(B)$ and preserves trace. Since $C^*(\mathcal{L}at\, A) = C^*(\mathcal{L}at\, B) = D$, the diagonal, it follows that Φ is the identity map on D. Thus Φ induces the identity map on the maximal ideal space of D; the Cartesian product of this map with itself (i.e., the identity) restricted to $FR(A)$ is the fundamental relation isomorphism ϕ. □

Remark. There is a natural map from the Cantor space, X, to the unit interval, $[0, 1]$. To each $x = (x_1, x_2, \ldots)$ simply associate the real number $\sum_{i=1}^{\infty} x_i/n_i$. While we shall not use this map explicitly, it is very helpful for intuitive purposes to keep it in mind, as it permits a visualization of the fundamental relation as a subset of the unit square.

II. Characterization theorems

Full nest subalgebras of UHF C*-algebras have been defined in terms of specific presentations of the algebras. This section addresses the issue of how one can recognize a full nest subalgebra of a UHF algebra amongst the canonical subalgebras of the UHF algebra.

Clearly, each full nest subalgebra, A, of a UHF algebra, B, is both a nest subalgebra and a strongly maximal triangular subalgebra. The diagonal of A is the canonical masa, $D = \varinjlim D_{n_i}$, where D_{n_i} is the diagonal of T_{n_i} and the nest, $\mathscr{N} = \mathscr{L}at\, A$ is the direct limit of the canonical nests for the algebras T_{n_i}. It is also evident that the nest, $\mathscr{N}$, is multiplicity free (in the sense that its commutant in A is D). See [PW; §2] for a discussion of this concept in the UHF C*-algebra context. In fact, a stronger property is true: $\mathscr{N}$ generates D as a C*-algebra. We remark in passing that $C^*(\mathscr{N}) = D$ by itself implies that $A = \mathscr{A}lg\, \mathscr{N}$ is a strongly maximal triangular algebra. [PW; Cor. 3.16]

Full nest subalgebras satisfy a second intrinsic property:

$$\operatorname{tr} \mathscr{N} = \{\operatorname{tr} p \colon p \in \mathscr{N}\} = K_0(B) \cap [0,1].$$

This is evident if one keeps in mind the fact that $K_0(B)$ is the dense subgroup of the rational numbers which consists of all fractions, a/b, where a/b is in lowest terms and b divides the supernatural number associated with B. The term "full" in the expression "full nest subalgebra" refers to this property of $\operatorname{tr} \mathscr{N}$.

These two properties "almost" characterize those nest subalgebras of a UHF algebra which are full nest subalgebras. In fact, there are nest subalgebras which satisfy these two properties and yet cannot be written as a direct limit of full upper triangular matrix algebras with nest embeddings. A complete characterization theorem requires a third property, which will be discussed below. On the other hand, it is true that nest subalgebras satisfying the two properties stated above can be written as direct limits of nest subalgebras of finite dimensional C*-algebras with nest embeddings. The conclusion is weaker than the conclusion of the complete characterization theorem proven later in that the finite dimensional C*-algebras need not be factors.

In the following theorem, it is more convenient to view UHF algebras as unions of an ascending chain of finite dimensional C*-algebras than as an abstract direct limit.

THEOREM 2.1. *Let A be a UHF C*-algebra with canonical trace,* tr, *and let $\mathscr{N}$ be a nest in A which satisfies the two properties*:

(1) *$C^*(\mathscr{N}) = D$ is a canonical masa in A.*

(2) $\operatorname{tr}(\mathscr{N}) = \{\operatorname{tr} p \colon p \in \mathscr{N}\} = K_0(A) \cap [0,1]$.

Then, there is an increasing sequence, $\{A_n\}_{n=1}^{\infty}$, of finite dimensional C^*-algebras such that:

(a) $A = \overline{\bigcup_n A_n}$.

(b) $A_n \cap \mathcal{N}$ *is a maximal nest in* A_n; *hence* $A_n \cap \mathcal{A}lg\ \mathcal{N}$ *is maximal triangular.*

(c) *The inclusion* $A_n \hookrightarrow A_{n+1}$ *is nest preserving when restricted to* $A_n \cap \mathcal{N}$.

Proof. Let B_n be an increasing sequence of finite dimensional factors contained in A such that $\overline{\bigcup B_n} = A$ and each $B_n \cap D$ is a masa in B_n. While the fact that A is UHF guarantees the existence of a sequence of factors, B_n, with dense union in A, it is not immediate that the B_n can be chosen so that D is canonical with respect to the B_n. However, a result of Renault [R; Cor. 1.16, Chapter 3] states that any two canonical masas are conjugate by an automorphism of the UHF algebra. (This result is actually stated in the AF context. A proof free of the language of groupoids is available in the monograph by Power [P5]). With the aid of this result, we can replace the original chain of factors by one which also satisfies the requirement that $B_n \cap D$ is a masa in B_n.

Our goal is to construct (inductively) the sequence of finite dimensional algebras, A_n, so that this sequence "interweaves" with a subchain of the B_n. First, observe that any minimal projection, f, in $B_1 \cap D$ is, in fact, in $C^*(\mathcal{N})$, by condition (1). It then follows from [PW; Lemma 3.1] that f can be written in the form

$$f = (p_2 - p_1) + (p_4 - p_3) + \cdots + (p_n - p_{n-1}),$$

where $p_1 < \cdots < p_n$ are uniquely determined projections in $\mathcal{N}$. Now, for every projection p in $\mathcal{N}$, there are integers $i(p)$ and $n(p)$ whose greatest common divisor is 1 such that $\operatorname{tr} p = i(p)/n(p)$. Note that $\operatorname{tr} p$ (and $1/n(p)$) lie in $K_0(A) \cap [0,1]$. Let j be the least common multiple of the set of numbers $n(p)$ obtained where f runs through all the minimal projections in $B_1 \cap D$ and p runs through all the projections which appear in the decomposition of each such f. Condition (2) now assures that for each integer i from 0 through j there is a projection, q_i, in $\mathcal{N}$ such that $\operatorname{tr} q_i = i/j$. The projections which appear in the decompositions of the minimal projections of $B_1 \cap D$ will all be amongst the q_i.

Now let D_1 be the abelian C*-algebra generated by the q_i and define $A_1 = C^*(D_1, B_1)$. Since each projection, q_i, must lie in some B_n, there is an integer, k_2, such that $A_1 \subset B_{k_2}$. Note that $A_1 \cap D$ is a masa in A_1 whose minimal projections are intervals from $\mathcal{N}$. From this, it follows immediately that $\mathcal{A}lg\ (\mathcal{N}) \cap A_1$ is maximal triangular.

The same argument will now yield a finite dimensional C*-algebra, A_2, and an integer k_3 such that $B_{k_2} \subset A_2 \subset B_{k_3}$ and $\mathcal{A}lg\ (\mathcal{N}) \cap A_2$ is maximal triangular in A_2. Continuing by induction, we can produce an increasing sequence, $\{A_n\}$, of finite dimensional subalgebras of B and an increasing sequence, k_n, of integers such that $B_{k_n} \subset A_n \subset B_{k_{n+1}}$ and $\mathscr{A}lg\,(\mathscr{N}) \cap A_n$ is maximal triangular in A_n. Condition (a) is trivially satisfied, since the A_n's interweave the B_{k_n}'s. It is not true that $\mathscr{L}at\ (A_n \cap \mathscr{A}lg\ (\mathscr{N})) \subseteq \mathscr{L}at\ (A_{n+1} \cap \mathscr{A}lg\ (\mathscr{N}))$, but it is easy to check that

$$\mathscr{N} \cap \mathscr{L}at\,(A_n \cap \mathscr{A}lg\,(\mathscr{N})) \subseteq \mathscr{N} \cap \mathscr{L}at\,(A_{n+1} \cap \mathscr{A}lg\,(\mathscr{N})),$$

which is the content of condition (c). □

Remark. Neither of the conditions (1) and (2) in the theorem implies the other. To see that condition (1) can be satisfied while (2) isn't, let $\mathscr{N}$ be the nest given in [PW; Example 3.17]. In this case, $C^*(\mathscr{N})$ is a canonical masa, but $1/2 \notin \operatorname{tr} \mathscr{N}$, and

$$K_0(A) = \bigcup_{n=1}^{\infty} \frac{1}{2^n}\mathbf{Z}.$$

To see that (2) can be satisfied without (1), let $\mathscr{N}$ be the uniform multiplicity 2 nest of [PW; Example 2.2.9]. $C^*(\mathscr{N})$ is not a masa, but

$$\operatorname{tr} \mathscr{N} = \left\{ \frac{k}{2^n} : 0 \le k \le 2^n,\ n = 1, 2, \ldots \right\} = K_0(A) \cap [0, 1].$$

In the theorem above, the finite dimensional subalgebras, A_n, are each isomorphic to direct sums of full matrix algebras, i.e., algebras of the form $M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_k}$. An attempt to add matrix units to A_n so as to enlarge it to M_m, where $m = m_1 + \cdots + m_k$ will reveal the need for a decomposition property defined as follows:

DEFINITION 2.2. Let $\mathscr{N}$ be a nest in a UHF algebra, A, with canonical masa, D, satisfying $C^*(\mathscr{N}) = D$ and $\operatorname{tr} \mathscr{N} = K_0(A) \cap [0, 1]$. Say that $\mathscr{N}$ has the *decomposition property* relative to A if, for any finite set, $\mathscr{F}$, of partial isometries in the normalizer of D, there is a decomposition of the identity, $1 = \sum_{i=1}^{N} e_i$, such that:

(1) *Each e_i is an interval from the nest*, $1 \le i \le N$.

(2) $\operatorname{tr} e_i = \operatorname{tr} e_j$ *for all* $1 \le i, j \le N$.

(3) *if $v \in \mathscr{F}$ and $e_i \le vv^*$, then $v^* e_i v$ is an interval.*

(4) *For any positive integer n with $1/n \in K_0(A)$, N may be chosen to be a multiple of n.*

The decomposition property together with properties (1) and (2) in the theorem gives a characterization of the nest subalgebras of a UHF algebra which are direct limits of full upper triangular matrix algebras with nest embeddings.

It is easy to see that if A is the direct limit of a system

$$T_{n_1} \xrightarrow{\nu_1} T_{n_2} \xrightarrow{\nu_2} T_{n_3} \xrightarrow{\nu_3} \cdots \longrightarrow A$$

with nest embeddings, then the associated nest, $\mathcal{N}$, satisfies the decomposition property. This results from the fact that any D-normalizing partial isometry is the product of a partial isometry in D and a sum of matrix units in one of the T_n's. [PPW; Theorem 3.6] or [P1; Lemma 6.3]. So we may assume that the elements of $\mathcal{F}$ all lie in some T_n, for large n. The set of minimal diagonal projections in T_n will now satisfy properties (1), (2), and (3). Property (4) is met simply by choosing n sufficiently large.

Before presenting a proof of the characterization theorem, we give an example which shows that it is possible to satisfy all the hypotheses of the characterization theorem except for the decomposition property. The example will actually fail to satisfy the following weaker decomposition property: if e is an interval from $\mathcal{N}$ and v is a D-normalizing partial isometry with $e \le vv^*$, then there are intervals, e_i, of equal trace such that $e = \Sigma e_i$ and $v^* e_i v$ is an interval for each i.

Example 2.3. Let A be the realization of the 2^∞-UHF C*-algebra gotten by taking $A_n = M_{2^n}$ and $\nu_n \colon A_n \hookrightarrow A_{n+1}$ the refinement embedding. Also, let $e_{ij}^{(n)}$ be the usual set of matrix units for A_n. For convenience, diagonal matrix units will be denoted as $e_i^{(n)}$ instead of $e_{ii}^{(n)}$. The properties of this example are easiest to apprehend in the representation of A acting on the Hilbert space $L^2(0,1)$ usually associated with a UHF algebra. In this example, $e_{ij}^{(n)}$ is the partial isometry associated with translation of the jth dyadic interval of length $1/2^n$ to the ith dyadic interval.

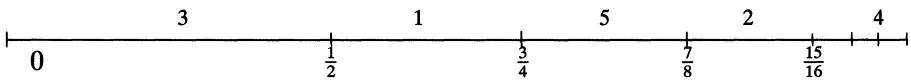
Let $\mathcal{L}$ denote the usual nest associated with the limit of the upper triangular matrices in the system for A, viz:

$$\mathcal{L} = \{0\} \bigcup \left\{ \sum_{i=1}^{j} e_i^{(n)} \colon 1 \le j \le 2^n,\ n = 1, 2, \ldots \right\}$$

In the representation, the subintervals of $[0,1]$ associated with nest projections are just the intervals of the form $[0,t]$, where t is a dyadic fraction. Next, define an auxiliary nest $\mathcal{M} = \{1\} \cup \{p_n\}_{n \ge 0}$ as follows:

$$\begin{aligned} p_0 &= 0 \\ p_1 &= e_3^{(2)} \\ p_{2n} - p_{2n-1} &= e_{2^{2n+2}-1}^{(2n+2)}, \qquad n \ge 1, \\ p_{2n+1} - p_{2n} &= e_{2^{2n-1}-1}^{(2n-1)}, \qquad n \ge 1. \end{aligned}$$

This subnest has been defined by specifying all the minimal intervals from it. The diagram below indicates the first five such intervals as they appear in the representation.



Finally, we define the nest, $\mathcal{N}$, which gives the desired example, as follows:

$$\mathcal{N} = \{1\} \bigcup_{n \geq 0} \{p_n + (p_{n+1} - p_n)\mathcal{L}\}.$$

It is routine to show directly that $\mathcal{N}$ is a maximal, multiplicity free nest containing $\mathcal{M}$. (Alternatively, use [PW, Lemma 2.20], which covers a more general situation.) Consequently, $\mathcal{A}lg\ (\mathcal{N})$ is a triangular nest subalgebra of A.

Since $C^*(\mathcal{L}) = D$, it is sufficient to show that $\mathcal{L} \subset C^*(\mathcal{N})$ in order to verify that $\mathcal{N}$ satisfies condition (1). It is clear that each of the intervals used in the construction of the auxiliary nest $\mathcal{M}$ lies in $C^*(\mathcal{N})$. From the construction itself, it now follows that $\mathcal{L} \subset C^*(\mathcal{N})$. It is also obvious from the construction that $\mathrm{tr}(\mathcal{N})$ consists of the dyadic rationals in $[0, 1]$, so condition (2) is also satisfied.

All that remains is to show that $\mathcal{N}$ fails to satisfy the decomposition property relative to $\mathcal{A}lg\ (\mathcal{N})$. In fact, even the weak decomposition property stated above fails. To see this, let $v = e_{12}^{(1)}$ and $f = e_2^{(2)}$. Then v is a D-normalizing partial isometry, f is an interval from $\mathcal{N}$, and $f \leq vv^*$. In the representation, v is the partial isometry which corresponds to the translation of $[\frac{1}{2}, 1]$ onto $[0, \frac{1}{2}]$ and f is the projection corresponding to the interval $[\frac{1}{4}, \frac{1}{2}]$.

Since f is a subprojection of one of the intervals used to build $\mathcal{M}$, any decomposition of f into a sum of intervals from $\mathcal{N}$ is also a decomposition into a sum of intervals from $\mathcal{L}$. If, then, $f = \Sigma f_i$, where the f_i are intervals of equal trace, then each f_i is a matrix unit from the original system. (Indeed, $\mathrm{tr}\, f_i = q/2^p$, for some integers p and q, and hence $\frac{1}{4} = \mathrm{tr}\, f = mq/2^p$, where m is the number of terms in the decomposition sum for f. It follows that $mq = 2^{p-2}$, and so q is a power of 2. Consequently, each f_i is a matrix unit.) Moreover, in this decomposition, one summand will be $e_{2^k-1}^{(k)}$, for some k. In the representation, $e_{2^k-1}^{(k)}$ corresponds to the interval $[1/2 - 1/2^k, 1/2]$.

To see that the decomposition property fails, we need only observe that for all $k \geq 2$, $v^* e_{2^k-1}^{(k)} v = e_{2^k}^{(k)}$ is not an interval from $\mathcal{N}$. This is clear from the representation, since $e_{2^k}^{(k)}$ corresponds to $[1 - 1/2^k, 1]$. For algebraic purists, some routine calculations yield

$$e^{(2n)}_{2^{2n}} = (1 - p_{2n+1}) + (p_{2n} - p_{2n-1}),$$

and

$$e^{(2n+1)}_{2^{2n+1}} = (1 - p_{2n+3}) + (p_{2n+2} - p_{2n+1}) + (p_{2n} - p_{2n-1})$$

again showing that $e^{(k)}_{2^k}$ is not an interval from $\mathscr{N}$.

This example shows that not only is the decomposition property necessarily satisfied by any full nest subalgebra, but it is independent of the other two necessary properties. The next theorem shows that these three properties are sufficient to characterize full nest subalgebras of UHF C*-algebras.

THEOREM 2.4. *Let A be a UHF C*-algebra with canonical trace*, tr, *and let $\mathscr{N}$ be a nest in A which satisfies the three properties*:

(1) *$C^*(\mathscr{N}) = D$ is a canonical masa in A.*

(2) $\operatorname{tr}(\mathscr{N}) = \{\operatorname{tr} p\colon p \in \mathscr{N}\} = K_0(A) \cap [0,1]$.

(3) *$\mathscr{N}$ satisfies the decomposition property relative to A.*

Then there is an increasing sequence, $\{A_n\}_{n=1}^{\infty}$, of finite dimensional factors in A such that:

(a) *$D_n \overset{\text{def}}{=} A_n \cap D$ is a masa in A_n and $A = \overline{\bigcup_n A_n}$.*

(b) *$\mathscr{N}_n \overset{\text{def}}{=} \mathscr{N} \cap A_n$ is a maximal nest in A_n and the inclusion $A_n \hookrightarrow A_{n+1}$ maps $\mathscr{N}_n$ into $\mathscr{N}_{n+1}$. (Hence, the inclusion $A_n \hookrightarrow A_{n+1}$ is a nest embedding.)*

(c) *$T_n \overset{\text{def}}{=} \mathscr{A}lg\,(\mathscr{N}) \cap A_n = \mathscr{A}lg\,(\mathscr{N}_n) \cap A_n$ is a maximal triangular subalgebra of A_n and $\mathscr{A}lg\,(\mathscr{N}) = \overline{\bigcup T_n}$.*

That is, $\mathscr{A}lg\,(\mathscr{N})$ is a full nest subalgebra of A.

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be an increasing sequence of factors with $\overline{\bigcup B_n} = A$ such that $D \cap B_n$ is a masa in B_n, $n = 1, 2, \ldots$. As in the proof of the previous theorem, the fact that any two canonical masas are conjugate is used here. Let $[n]$ denote the dimension of the factor, B_n, (so that $B_n \cong M_{[n]}$). Let $\{f^{(n)}_{ij}\}_{1 \le i,\, j \le [n]}$ be a system of matrix units for B_n compatible with D.

Property (1) implies that each projection in $D \cap B_1$ is a sum of intervals from $\mathscr{N}$; this, together with property (2) implies that there is a subalgebra, E_1, of D, such that the minimal projections in E_1 are intervals from $\mathscr{N}$, all of the same trace, say $1/n$, and $D \cap B_1 \subset E_1$.

Apply the decomposition property with $\mathscr{F} = \{f^{(1)}_{ij}\}_{1 \le i,\, j \le [1]}$, the matrix units of B_1, and N such that $n \mid N$: $1 = \sum_{i=1}^{N} e^{(1)}_i$, $\operatorname{tr} e^{(1)}_i = 1/N$, $1 \le i \le N$. Let D_1 be the C*-algebra generated by $\{e^{(1)}_i\}_{1 \le i \le N}$. Since each $f^{(1)}_i$ is a sum of minimal projections in E_1 and each minimal projection in E_1 is the sum of the minimal projections in D_1 which it dominates, it follows that for each i,

$1 \le i \le [1]$, there is a subset I_i of $\{1, 2, \ldots, N\}$ with $f_i^{(1)} = \sum_{t \in I_i} e_t^{(1)}$. The sets $I_1, \ldots, I_{[1]}$ form a partition of $\{1, \ldots, N\}$ with $\operatorname{card}(I_i) = \operatorname{card}(I_j)$, for all $1 \le i, j \le [1]$. Furthermore, the decomposition property implies that, for each fixed i, the collection

$$\left\{ f_{ji}^{(1)} e_t^{(1)} f_{ij}^{(1)} : t \in I_i, j = 1, \ldots, [1] \right\}$$

is a collection of disjoint intervals from $\mathcal{N}$ whose (equal) traces sum to 1, and hence is a decomposition of 1 into intervals of trace $1/N$. However, there is only one decomposition of 1 into disjoint intervals of trace $1/N$. Thus, for each $t \in I_i$, $j \in \{1, \ldots, [1]\}$, there is a $k = k(j, t)$ with $f_{ji}^{(1)} e_t^{(1)} f_{ij}^{(1)} = e_k^{(1)}$. In other words, each nonzero product $e_t^{(1)} f_{ij}^{(1)}$ is a partial isometry whose initial and final projections are minimal projections in D_1. If $e_t^{(1)} f_{ij}^{(1)}$ has initial projection $e_s^{(1)}$, set $e_{ts}^{(1)} = e_t^{(1)} f_{ij}^{(1)}$. The $\{e_{ts}^{(1)}\}$ form a system of matrix units for $C^*(D_1, B_1)$.

Now the projections $e_i^{(1)}$ are sums of matrix units in B_k, for some sufficiently large k. Replacing $\{B_n\}$ by a suitable subsequence, we may assume that each matrix unit $e_i^{(1)}$ can be expressed as a sum of matrix units in B_2. We claim that each matrix unit $e_{ts}^{(1)}$ of $C^*(D_1, B_1)$ is a sum of matrix units of B_2. Writing $e_t^{(1)} = \sum_{u \in J_t} f_u^{(2)}$, for some index set J_t, it follows that

$$e_{ts}^{(1)} = e_t^{(1)} f_{ij}^{(1)} = \sum_{u \in J_t} f_u^{(2)} f_{ij}^{(1)}.$$

Since each matrix unit of B_1 is a sum of matrix units of B_2, the claim is verified.

Next, we claim that $C^*(D_1, B_1)$ can be embedded in a factor by adjoining additional matrix units which are sums of matrix units in B_2. Now, $C^*(D_1, B_1)$ is isomorphic to a direct sum of full matrix algebras, $M_{k_1} \oplus \cdots \oplus M_{k_r}$. By relabeling, if necessary, we may assume that the indexing on the minimal projections $\{e_i^{(1)}\}$ of $C^*(D_1, B_1)$ is compatible with the direct sum decomposition. Define a new matrix unit $e_{k_1, k_1+1}^{(1)}$ to be any sum of orthogonal matrix units in B_2 with initial projection $e_{k_1+1}^{(1)}$ and final projection $e_{k_1}^{(1)}$. Similarly, define $e_{k_s, k_s+1}^{(2)}$, $2 \le s < r$. The C*-algebra generated by these matrix units, together with the matrix units $e_{st}^{(1)}$ defined earlier, is a factor. Denote this factor by A_1.

The embeddings, $B_1 \hookrightarrow A_1 \hookrightarrow B_2$, are *-embeddings such that each matrix unit in the domain of each embedding is a sum of matrix units in the range. Since the minimal projections in the diagonal, D_1, of A_1 are intervals from the nest, it follows that $\mathcal{A}lg\ (\mathcal{N}) \cap A_1$ is maximal triangular in A_1. Now repeat the construction with B_2 in the role of B_1 to obtain $B_2 \hookrightarrow A_2 \hookrightarrow B_3$. Since the minimal diagonal projections of A_2 are intervals, and since under the embedding $A_1 \hookrightarrow B_2 \hookrightarrow A_2$ each matrix unit in A_1 is a sum of matrix units in A_2, and, in particular, each interval in D_1 is a sum of intervals in D_2, the embedding $A_1 \hookrightarrow A_2$ is nest preserving.

Continue in this fashion to get factors and embeddings $A_{n-1} \hookrightarrow B_n \hookrightarrow A_n$. Since $\overline{\cup A_n} \supseteq \overline{\cup B_n}$, we have $A = \overline{\cup A_n}$. Since $\mathscr{A}lg\ (\mathscr{N}) \cap A_n = T_n$ is maximal triangular in the factor A_n, and since $\mathscr{N}_n = \{0\} \cup \{e_1^{(n)}, e_1^{(n)} + e_2^{(n)}, \dots\}$ is a maximal nest in A_n, $T_n = A_n \cap \mathscr{A}lg\ (\mathscr{N}_n)$. □

Remark. The characterization theorem presented above is intrinsic within the context of subalgebras of a UHF algebra. Not all direct limits of finite dimensional operator algebras are realizable as subalgebras of AF C*-algebras; indeed, the natural category in which to consider such direct limits is the category of Banach algebras. In fact, the direct limits of finite dimensional operator algebras considered in the literature are almost invariably abstract operator algebras, at the least. In any event, this suggests the following problem: determine which Banach algebras are isometrically isomorphic to a full nest algebra and which abstract operator algebras are completely isometrically isomorphic to a full nest algebra.

A special class of full nest algebras was classified in [HP]. The general classification problem for full nest subalgebras of UHF algebras is probably very difficult. A (presumably) more tractable classification problem arises from the consideration of a smaller class of algebras—those which arise from homogeneous embeddings.

A direct limit of homogeneous nest embeddings satisfies the homogeneity property which appears in the following definition.

DEFINITION 2.5. A nest subalgebra, A, of a UHF C*-algebra is said to be *homogeneous* if, for any two interval projections, p and q, with the same trace, the algebras pAp and qAq are isometrically isomorphic.

The following open question is provocative: if A is a homogeneous nest subalgebra of a UHF C*-algebra which also satisfies properties (1), (2) and (3) of the characterization theorem, i.e. is a full nest subalgebra, then can it be written as a direct limit with homogeneous nest embeddings?

III. Stationary systems

We next consider full nest algebras which arise from systems in which every embedding is the nest embedding induced by a fixed permutation, π, in S_b:

$$T_b \xrightarrow{\nu_\pi} T_{b^2} \xrightarrow{\nu_\pi} T_{b^3} \xrightarrow{\nu_\pi} \cdots \longrightarrow A_\pi$$

The fundamental relation of such a system will be denoted by $FR(\pi)$, except for the fundamental relation of the refinement algebra, which will be denoted by $FR(0)$.

We now describe $FR(\pi)$ explicitly. The diagonal D of A_π is the direct limit of the finite dimensional diagonals D_{b^n}, each of what can be identified with the tensor product $D_b \otimes D_b \otimes \cdots \otimes D_b$, ($n$ factors). The minimal projections of D_{b^n} can therefore be indexed by $[b] \times \cdots \times [b]$ with the lexicographic order, which we denote by $\preccurlyeq$. Ordered pairs (i, j) with $i, j \in [b] \times \cdots \times [b]$ and $i \preccurlyeq j$ serve as the indices for the matrix units of T_{b^n}. The maximal ideal space, X, of D is identified with $\prod_1^\infty [b]$. The minimal projection, e_i, in D_{b^n}, with $i = (i_1, \dots, i_n)$ corresponds to the subset $\{x \in X: x_1 = i_1, \dots, x_n = i_n\}$.

Now suppose that e_{ij} is a matrix unit in T_{b^k}, where $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_k)$ and $i \preccurlyeq j$. Let E_{ij} denote the graph of the partial homeomorphism induced by the matrix unit e_{ij}. We shall describe the points (r, c) in E_{ij}. Here, $r, c \in X$ and are intended to connote "row" and "column." For (r, c) to be in E_{ij} we must have $r_1 = i_1, \dots, r_k = i_k$ and $c_1 = j_1, \dots, c_k = j_k$. Beyond this, for $n > k$ each coordinate r_n is determined by π, the corresponding coordinate c_n, and an exponent for π which depends on all the preceeding coordinates of r and c. To describe the exponent, let $x, y \in \prod_1^n [b]$ and define a "diagonal number" d_n by

$$d_n(x, y) = b^{n-1}(y_1 - x_1) + b^{n-2}(y_2 - x_2) + \cdots + b(y_{n-1} - x_{n-1}) + (y_n - x_n).$$

We have in mind that $x \preccurlyeq y$, so that $d_n(x, y) \geq 0$; the entry 1 in e_{ij} lies on the d_nth superdiagonal of the matrix. For each $n = k, k+1, \dots$, let $t_n = d_n((r_1, \dots, r_n), (c_1, \dots, c_n))$. The remaining condition on r and c is that $r_{n+1} = \pi^{t_n}(c_{n+1})$, for $n = k, k+1, \dots$.

It should be noted that the value of the exponent, t_n, is significant only modulo the order of π. Consequently, when the order of π divides a power of the base, b, the formula for t_n simplifies. If, say, the order of π divides b^t, then only terms involving powers of b less than t are necessary in the sum for d_n. In particular, when the order of π divides b itself, then $t_n = c_n - r_n$.

With this description, the reader will find it a straightforward matter to work out the fundamental relation, $FR(\pi)$, for specific choices of π, especially if she keeps in mind the canonical projection of $FR(\pi)$ into the unit square. In the simplest case of all, when π is the identity permutation, then $r_n = c_n$ for all $n > k$ and the well-known description of $FR(0)$, the fundamental relation of the refinement algebra, results.

In the next proposition, which gives the first step in the determination of a necessary and sufficient condition on π so that A_π is the refinement algebra, the containment is not merely set-theoretic. Rather, it is containment as a topologized relation.

PROPOSITION 3.1. *$FR(0) \subseteq FR(\pi)$ if and only if the order of π divides b^n, for some $n \geq 1$.*

Proof. $\Rightarrow$. Let $p = \text{order}(\pi)$ and suppose that p does not divide b^n, for all $n \in \mathbf{N}$. Let $\phi\colon \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ be the canonical quotient map. Since $\phi(b^n) \neq 0$, for all $n \geq 1$, there is an integer $d \in \{1, 2, \ldots, p-1\}$ such that $\phi(b^n) = d$ for infinitely many values of $n \geq 1$. Let $a \in [b]$ be such that $\pi^d(a) \neq a$.

Now let

$$c = (1, a, a, a, \ldots) \quad \text{and} \quad r = (0, a, a, a, \ldots).$$

Clearly, $(r, c) \in FR(0)$. We claim that $(r, c) \notin FR(\pi)$.

If $(r, c) \in FR(\pi)$, then there exists a positive integer, m, such that for all $n \geq m$, we have $r_{n+1} = \pi^{t_n}(c_{n+1})$, where

$$\begin{aligned} t_n &= b^{n-1}(c_1 - r_1) + b^{n-2}(c_2 - r_2) + \cdots + b(c_{n-1} - r_{n-1}) + c_n - r_n \\ &= b^{n-1}. \end{aligned}$$

Thus, for infinitely many choices for n, $t_n \equiv d \bmod p$. In particular, n can be chosen so that $a = r_{n+1} = \pi^d(c_{n+1}) = \pi^d(a)$, a contradiction.

$\Leftarrow$. Assume that $p = \text{order}(\pi)$ divides b^m. Let $i = (i_1, \ldots, i_k) \preccurlyeq (j_1, \ldots, j_k) = j$ be given. The graph of the matrix unit e_{ij} in the refinement fundamental relation is

$$E_{ij}^0 = \{(r, c)\colon r_n = i_n, c_n = j_n \quad \text{for } n = 1, 2, \ldots, k \quad \text{and } r_n = c_n \text{ for } n > k\}$$

Since $FR(0) = \bigcup_{i,j} E_{ij}^0$, it suffices to show that $E_{ij}^0 \subseteq FR(\pi)$.

For each finite sequence $s = (s_{k+1}, \ldots, s_{k+m})$ with all $s_n \in [b]$, let

$$F_{ij}^s = \{(r, c) \in E_{ij}^0\colon r_n = c_n = s_n \quad \text{for } k+1 \leq n \leq k+m\}.$$

Then $E_{ij}^0 = \bigcup_s F_{ij}^s$, a finite union. Note that each F_{ij}^s is the graph of a matrix unit in the refinement fundamental relation. It now suffices to prove that $F_{ij}^s \subseteq FR(\pi)$, for each s.

We shall show that F_{ij}^s is, in fact, the graph in $FR(\pi)$ of the matrix unit with row index $(i_1, \ldots, i_k, s_{k+1}, \ldots, s_{k+m})$ and column index $(j_1, \ldots, j_k, s_{k+1}, \ldots, s_{k+m})$. Let $(r, c) \in F_{ij}^s$. We must show that for $n \geq k + m$, $r_{n+1} = \pi^{t_n}(c_{n+1})$, where t_n is given by

$$t_n = b^{n-1}(c_1 - r_1) + b^{n-2}(c_2 - r_2) + \cdots + (c_n - r_n).$$

But if $(r, c) \in F_{ij}^s$ and t_n is defined by the formula above, then all terms except for the first k are equal to 0; thus t_n is actually given by

$$t_n = b^{n-1}(j_1 - i_1) + \cdots + b^{n-k}(j_k - i_k).$$

Since $n \geq k + m$, every exponent in the sum is greater than or equal to m. Thus b^m divides t_n. But by hypothesis, $p | b^m$, so $p | t_n$ and $\pi^{t_n} = \text{id}$. Since $r_n = c_n$, for all $n \geq k + m$; this verifies that $r_{n+1} = \pi^{t_n}(c_{n+1})$, as required.

This completes the proof that $FR(0)$ is a subset of $FR(\pi)$. To see that the containment is topological, observe that each F_{ij}^t is a compact and open subset of $FR(\pi)$, since it is the graph of a matrix unit. It follows that E_{ij}^0 is compact and open as a subset of $FR(\pi)$ and hence that the inclusion $FR(0) \hookrightarrow FR(\pi)$ is a continuous and open mapping. □

Remark 1. While the containment in Proposition 3.1 is topological containment, the proof of the implication ($\Rightarrow$) only used set theoretic containment. Thus, if $FR(0)$ is contained in $FR(\pi)$ as a set, it follows automatically that the inclusion map is continuous and open. In the more general context of triangular subalgebras of AF C*-algebras, it is possible that two non-isomorphic algebras have fundamental relations that are equal as sets (but, of course, different as topological relations). An example of this phenomenon is given in [HP]. This raises the following question: if the fundamental relations of two full nest algebras are identical as sets, must the two algebras be isomorphic?

Remark 2. When $FR(0)$ is a proper subset of $FR(\pi)$, the algebra A_π contains a proper subalgebra which is isometrically isomorphic to the refinement algebra. At first glance this might seem paradoxical, since both algebras are maximal triangular algebras. But the C*-algebra generated by the copy of the refinement algebra is a proper subalgebra of the C*-algebra generated by A_π (even though both of these C*-algebras are isomorphic to the UHF algebra $M(b^\infty)$), so no problem arises.

In the discussion which follows, fix a base, b, and a permutation, π; let $p = \text{order}(\pi)$, and let ϕ denote the canonical map $\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$. If $(r, c) \in FR(\pi)$, then for large n we have

$$r_{n+1} = \pi^{t_n}(c_{n+1}), \quad \text{where}$$
$$t_n = b^{n-1}(c_1 - r_1) + b^{n-2}(c_2 - r_2) + \cdots + b(c_{n-1} - r_{n-1}) + (c_n - r_n).$$

From this it follows that

$$\begin{aligned} t_{n+1} &= b^n(c_1 - r_1) + b^{n-1}(c_2 - r_2) + \cdots + b^2(c_{n-1} - r_{n-1}) \\ &\quad + b(c_n - r_n) + (c_{n+1} - r_{n+1}) \\ &= c_{n+1} - \pi^{t_n}(c_{n+1}) + bt_n. \end{aligned}$$

This equation, together with the fact that $\pi^t = \pi^{\phi(t)}$, for any t, suggests the following definition of a sequence of "exponent" sets for π:

$$\begin{aligned} E_1 &= \{0, 1, 2, \dots, p-1\} \\ E_2 &= \{\phi(x - \pi^n(x) + bn) : x \in [b], n \in E_1\} \\ E_3 &= \{\phi(x - \pi^n(x) + bn) : x \in [b], n \in E_2\} \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

It is easy to check that $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, and that, once two successive E_i are equal, all subsequent ones are also equal. Also obvious: $0 \in E_i$, for all i. Since E_1 contains p elements, stabilization must occur no later than at E_p.

LEMMA 3.2. *If $E_p = \{0\}$, then $FR(\pi) \subseteq FR(0)$. The containment is topological.*

Proof. Assume $E_p = \{0\}$ (and hence, $E_n = \{0\}$, for all $n \geq p$). Let $(r, c) \in FR(\pi)$, where $r = (r_1, r_2, \dots)$ and $c = (c_1, c_2, \dots)$. Then there exists an integer, k, such that

$$(r_1, \dots, r_k) \preccurlyeq (c_1, \dots, c_k) \quad \text{and} \quad r_{n+1} = \pi^{t_n}(c_{n+1}) \text{ for all } n \geq k.$$

From the discussion above, it is clear that $\phi(t_k) \in E_1$, $\phi(t_{k+1}) \in E_2$, $\phi(t_{k+2}) \in E_3$, etc. In particular, for $m \geq p - 1$, we have $\phi(t_{k+m}) \in E_p = \{0\}$. Thus, for $n \geq k + p - 1$, $\phi(t_n) = 0$ and hence $c_{n+1} = r_{n+1}$. This shows that $FR(\pi) \subseteq FR(0)$ as a set.

To show that the containment is topological, let $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_k)$ and E_{ij}^{π} be the graph of the matrix unit e_{ij} in $FR(\pi)$. If $(r, c) \in E_{ij}^{\pi}$, then

(1) $c_n = j_n$ and $r_n = i_n$, for $n = 1, \dots, k$,
(2) c_{n+1} is arbitrary for $n \geq k$,
(3) $r_{n+1} = \pi^{t_n}(c_{n+1})$ for $n \geq k$.

As shown above, $r_n = c_n$, for $n \geq k + p$.

Given

$$v = (c_{k+1}, \dots, c_{k+p-1}) \in \prod_1^{p-1} [b],$$

let

$$u = u(v) = \left(\pi^{t_k}(c_{k+1}), \dots, \pi^{t_{k+p-2}}(c_{k+p-1})\right).$$

Let jv and iu denote $(j_1, \ldots, j_k, c_{k+1}, \ldots, c_{k+p-1})$ and $(i_1, \ldots, i_k, r_{k+1}, \ldots, r_{k+p-1})$, respectively, where, as usual, $r_{n+1} = \pi^{t_n}(c_{n+1})$, for $k \leq n \leq k+p-2$. Then,

$$E^{\pi}_{ij} = \bigcup_{v \in \Pi_1^{p-1}[b]} E^{\pi}_{iu,jv}.$$

Now, by the argument above, $E^{\pi}_{iu,jv} = E^{0}_{iu,jv}$; hence $E^{\pi}_{iu,jv}$ is a compact and open subset of $FR(0)$. Since E^{π}_{ij} is a finite union of the $E^{\pi}_{iu,jv}$, E^{π}_{ij} is also a compact and open subset of $FR(0)$. Thus, the inclusion $FR(\pi) \hookrightarrow FR(0)$ is a continuous and open mapping. □

LEMMA 3.3. *Let π be a permutation in S_b with order p. Suppose that $E_p = \{0\}$. Then p divides a power of b.*

Proof. First suppose that π has a fixed point, say x, in $[b]$. Since $1 \in E_1$, it follows that $\phi(b) = \phi(x - \pi(x) + b)$ is an element of E_2. Similarly, $\phi(b^2) = \phi(x - \pi^b(x) + b^2) \in E_3$ and, by induction, $\phi(b^m) \in E_{m+1}$, for all m. But $E_{m+1} = \{0\}$, for large m, i.e., p divides all large powers of b.

Now suppose only that π^n has a fixed point, for some value of n at most $p-1$. If x is the fixed point, then $x = \pi^{an}(x)$, for any integer a. Arguing much as above, $n \in E_1$ and $bn = x - \pi^n(x) + bn$ implies that $\phi(bn) \in E_2$. Induction now yields $\phi(b^m n) \in E_{m+1}$ for all m, and hence p divides $b^m n$ for all large m.

Since we know that p divides large powers of b when π has a fixed point, we may now assume that π has no fixed points. It suffices to show that if q is a prime factor of p, then it also divides b. Write π as a product of disjoint cycles of lengths $a_1, \ldots, a_m$. Then $b = a_1 + \cdots + a_m$, each $a_i \geq 2$, and $p = \operatorname{lcm}(a_1, \ldots, a_m)$.

If $q | a_i$, for all i, then $q | b$. So assume that q does not divide a_i, for some i. Then $a_i \leq p-1$ and π^{a_i} has a fixed point. Consequently, by the second paragraph, p divides $b^m a_i$ for large m. Since $q | p$ and q does not divide a_i, we have q divides b, as desired. □

We now come to the main result of this section—the characterization of those permutations for which A_π is a refinement algebra. As it turns out, the condition is surprisingly simple and highly computable.

THEOREM 3.4. *Let $\pi \in S_b$ and let A_π be the limit algebra of the stationary system of nest embeddings associated with π. Let p denote* order(π). *A_π is a refinement algebra if, and only if, $E_p = \{0\}$, where E_p is the* pth *set in the sequence of exponent sets for π.*

Proof. First assume that $E_p = \{0\}$. By the two lemmas, $FR(\pi) \subseteq FR(0)$ and p divides a power of b. Proposition 3.1 now implies that $FR(0) \subseteq FR(\pi)$; thus $FR(\pi) = FR(0)$ and A_π is a refinement algebra.

For the converse, assume that $FR(\pi) = FR(0)$. From Proposition 3.1, we may conclude that p divides a power of b. We shall consider sequences, $(t_n)_{n \geq 1}$, which satisfy the property that for each $n > 1$, there is some $x \in [b]$ such that $t_n = \phi(x - \pi^{t_{n-1}}(x) + bt_{n-1})$. A sequence satisfying this property will be called an *exponent sequence*.

Observe that if $t_s = 0$ for some term of an exponent sequence, then $t_n = 0$ for all $n \geq s$. Also, from the definition of the exponent sets, it is clear that $t_n \in E_n$, for all n (assuming $t_1 \in E_1$).

We next claim that if t is an exponent sequence, then we do indeed have $t_n = 0$ for large n. Suppose, to the contrary, that all $t_n \neq 0$. For each $n > 1$, choose $x_{n-1} \in [b]$ so that

$$t_n = \phi\big(x_{n-1} - \pi^{t_{n-1}}(x_{n-1}) + bt_{n-1}\big).$$

Now let

$$c = (1, x_1, x_2, \dots, x_n, \dots)$$
$$r = \big(0, \pi^{t_1}(x_1), \pi^{t_2}(x_2), \dots, \pi^{t_n}(x_n), \dots\big).$$

Then $(r, c) \in FR(\pi)$. But, since $\pi^{t_n}(x_n) \neq x_n$, for infinitely many n, $(r, c) \notin FR(0)$. (If $\pi^{t_n}(x_n) = x_n$ for all large n, then t_n is congruent to a multiple of b^k for sufficiently large n; but $p | b^k$ so $t_n = 0$ for large n.)

Not only must the terms of an exponent sequence eventually equal zero, but actually $t_p = 0$ for any exponent sequence. For if not, then $t_1, \dots, t_p$ are all elements of $\{1, 2, \dots, p-1\}$. Consequently, two of them must be equal, say t_r and t_s, with $r < s$. But then

$$(t_1, \dots, t_r, \dots, t_{s-1}, t_r, \dots, t_{s-1}, t_r, \dots)$$

is also an exponent sequence. This, however, contradicts the fact that the terms of every exponent sequence are eventually zero.

The conclusion of the theorem now follows from the simple observation that

$$E_p = \{s\colon \text{there is an exponent sequence with } s = t_p\} = \{0\}. \qquad \square$$

Remark. When order (π) divides b the calculation of the exponent sets for π and of $FR(\pi)$ is simplified. (The terms bn in the definition of the exponent sets may be deleted and the exponents, t_n, needed to determine $FR(\pi)$ are given by the simple formula $t_n = c_n - r_n$.) This will occur, for example, if order (π) is prime, i.e. if π is a product of disjoint cycles each of the same prime length.

Question. Does $FR(\pi) \subseteq FR(0)$ (rather than equality) imply that $E_p = \{0\}$? In other words, are these two conditions equivalent in the absence of the condition that p divides a power of b?

IV. Further discussion of stationary systems

The problem of classifying all the algebras, A_π, which arise as limits of stationary systems can be broken into two parts:

1. For a fixed base b, classify the algebras, A_π, which arise from the permutations in S_b.
2. Determine when permutations π in S_b and τ in S_c yield isomorphic algebras.

With regard to problem 2, one necessary condition is evident: the supernatural numbers b^∞ and c^∞ must be equal. (The enveloping C*-algebras, $M(b^\infty)$ and $M(c^\infty)$, are isomorphic when the full nest algebras are.) On the other hand, examples of permutations, π and τ, with isomorphic limit algebras are easy to come by. For example, take $\pi = (0\ 1)$ in S_3 and $\tau = (0\ 3)(1\ 4)(2\ 5)(6\ 7)$ in S_9 or $\pi = (0\ 2\ 1)$ in S_3 and $\tau = (0\ 8\ 4)\ (1\ 6\ 5)\ (2\ 7\ 3)$ in S_9. These examples are obtained by letting τ be the permutation in S_9 for which $\nu_\tau = \nu_\pi \circ \nu_\pi$. Examples obtained in a less trivial way might be more enlightening.

The algebras obtained from permutations in S_3 and S_4 shed some light on problem 1. (But some phenomena do not occur for such low values of b.) The six permutations in S_3 yield six non-isomorphic limit algebras, as can be checked by direct comparison of the fundamental relations. In S_4, four permutations yield the refinement algebra. In addition to the identity permutation, they are $(0\ 2)$, $(1\ 3)$ and $(0\ 2)\ (1\ 3)$. (This can be checked either by direct calculation or by applying the theorem in the preceeding section.) The remaining twenty permutations all yield distinct algebras.

Clearly, this can be verified through an inordinate number of direct comparisions, but several observations greatly reduce the tedium. For example, the eight permutations of order 3 in S_4 all have fundamental relations which do not contain $FR(0)$, while the remaining permutations, all of which have order 2 or 4, do have fundamental relations which contain $FR(0)$. So no permutation in one set can yield the same limit algebra as a permutation in the other set.

To separate the permutations of order 3, observe that each one has precisely one fixed point. If π has order 3 and d is a fixed point for π, then

$FR(\pi)$ contains every point, (r, c), where

$$r = (i_1, \dots, i_k, d, d, d, \dots)$$
$$c = (j_1, \dots, j_k, d, d, d, \dots)$$

and $(i_1, \dots, i_k) \preccurlyeq (j_1, \dots, j_k)$. On the other hand, when d is not a fixed point, $FR(\pi)$ cannot contain the point (r, c) if, in addition $\sum_{h=1}^{k}(j_h - i_h) \not\equiv 0 \bmod 3$. Thus the eight permutations of order 3 are divided into four pairs with no common algebra associated to permutations in different pairs. The two algebras in each pair can be separated directly.

Another example of an invariant which can be used to distinguish classes of permutations is the pair of points $r_{\min} = (0, 0, 0, \dots)$ and $c_{\max} = (3, 3, 3, \dots)$. These two points are the unique minimal and maximal points in the maximal ideal space, X, under the fundamental relation. It is easy to show that, for π of even order, $FR(\pi)$ contains the point $(r_{\min}, c_{\max})$ only for π equal to one of (0 3), (0 3) (1 2), (0 3 1 2) and (0 3 2 1). In a similar vein, if we let

$$G_c(\pi) = \{r: (r, c) \in FR(\pi)\} \quad \text{and} \quad G^r(\pi) = \{c: (r, c) \in FR(\pi)\},$$

then $G_{c_{\max}} = \{r\colon r \text{ has a tail of all 3's}\}$ if, and only if, π is one of (0 1), (1 2), (0 1 2), or (0 2 1) while $G^{r_{\min}} = \{c\colon c \text{ has a tail of all 0's}\}$ if, and only if, π is one of (1 2), (2 3), (1 2 3), or (1 3 2). With invariants such as these, the labor needed to separate the twenty non-refinement algebras associated with S_3 becomes manageable.

The classification of the permutations in S_3 and S_4 might suggest that distinct permutations which do not give the refinement algebra must give distinct algebras. But it is easy to find examples of the form $\pi = \mu\sigma$, where μ and σ are disjoint cycles, A_μ is a refinement algebra, and $A_\pi \cong A_\sigma$. This leaves open the possibility that the isomorphism class of A_π is determined by those cycles in the decomposition of π into disjoint cycles which do not correspond to refinement algebras. A couple of examples shows that the situation is more complicated.

First, let $b = 20$ (or any number of the form $4k$ for $k \geq 5$), $\pi = (2\ 6\ 8\ 10)$ and $\sigma = (2\ 14\ 8\ 18)$. The exponent set for either π or σ stabilizes at $\{0, 2\}$, so neither permutation gives a refinement algebra. It is not difficult to show that $FR(\pi) = FR(\sigma)$, so $A_\pi \cong A_\sigma$. (A point (r, c) lies in the common fundamental relation when an initial segment of r proceeds an initial segment of s and one of two conditions hold for the corresponding tails: either $r_n = c_n$ for all large n or the only possible values for coordinates in the tails are 2 and 8 and the patterns in r and c are complementary ($r_n \neq c_n$, all n).)

In the second example, take $b = 8$, $\mu = (0\ 2)$, $\sigma = (4\ 3\ 1)$, and $\pi = \mu\sigma = (0\ 2)(4\ 3\ 1)$. Then A_μ is a refinement algebra while $A_\pi \not\cong A_\sigma$. In fact, if

$$c = (1, 2, 4, 2, 4, 2, 4, 2, 4, 2, 4, \dots)$$
$$r = (0, 0, 3, 0, 1, 0, 3, 0, 1, 0, 3, \dots)$$

then $(r, c) \in FR(\pi)$ and $(r, c) \notin FR(\sigma)$.

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University of Alabama
Tuscaloosa, Alabama
Iowa State University
Ames, Iowa