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THE HENSTOCK AND MCSHANE INTEGRALS OF VECTOR-VALUED FUNCTIONS

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Introduction

A familiar formula from undergraduate analysis is the 'Riemann sum' $\Sigma_{i=1}^{n} f(t_i)(b_i - b_{i-1})$ of a function f with respect to a tagged partition $0 = b_0 \leq t_1 \leq b_1 \leq \cdots \leq t_n \leq b_n = 1$ of $[0, 1]$. One of the standard definitions of the Riemann integral describes it as the limit of such sums as $\max_{1 \leq i \leq n}(b_i - b_{i-1}) \to 0$. It is a remarkable fact that the same formula may be used to define a vastly more powerful integral, if we take a different limiting process. Instead of requiring all partitions with $\max_i(b_i - b_{i-1}) \leq \delta_0$ to give good approximations to the integral, we can restrict our attention to those in which $b_i - b_{i-1} \leq \delta(t_i)$ for each i, where δ is a strictly positive function on $[0, 1]$. (See 1(c) below.) This refinement yields the 'Henstock' or 'Riemann-complete' integral; it agrees with the Lebesgue integral on non-negative functions but extends it on others (see 4(e) below). An ingenious modification of the construction, due to E.J. McShane, allows the t_i to lie outside the corresponding intervals (see 1(b)); this brings us back a step, to the Lebesgue integral precisely.

A common feature of the Riemann, McShane and Henstock integrals is that the use of Riemann sums gives us obvious formulations of integrals for vector-valued functions defined on $[0, 1]$. For the McShane and Henstock integrals I spell these out in 1(*b-c*) below. The Henstock integral obviously extends the McShane integral. In this paper I seek to elucidate the nature of this extension; in particular, to give criteria to distinguish McShane integrable functions among the Henstock integrable functions. In the real-valued case this is simple enough; a Lebesgue integrable function is just a Henstock integrable function with (Henstock) integrable absolute value; equivalently, a Henstock integrable function which is Henstock integrable over every measurable set. It turns out that the latter criterion is valid in the vector-valued case (Corollary 9 below). I give priority however to a more economically expressible result in terms of the Pettis integral: a vector-valued function is McShane integrable iff it is both Henstock integrable and Pettis integrable (Theorem 8). The Pettis integral being the widest of the standard integrals of vector-valued functions (see [7]), this suggests that the difference between the

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Henstock and McShane integrals for vector-valued functions is largely accounted for by the difference between the Henstock and Lebesgue integrals for real-valued functions.

1. Definitions

I recall the following definitions. Let X be a Banach space, with dual X^*.

(a) A function ϕ: $[0,1] \to X$ is *Pettis integrable* if for every Lebesgue measurable set $E \subseteq [0,1]$ there is a $w_E \in X$ such that $\int_E f(\phi(x))\mu(dx)$ exists and is equal to $f(w_E)$ for every $f \in X^*$; in this case $w_{[0,1]}$ is the *Pettis integral* of ϕ, and the map $E \mapsto w_E$ is the *indefinite Pettis integral* of ϕ.

(b) A *McShane partition* of $[0,1]$ is a finite sequence $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ such that $\langle[a_i, b_i]\rangle_{i\le n}$ is a non-overlapping family of intervals covering $[0,1]$ and $t_i \in [0,1]$ for each i. A *gauge* on $[0,1]$ is a function δ: $[0,1] \to]0,\infty[$. A McShane partition $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ is *subordinate* to a gauge δ if $t_i - \delta(t_i) \le a_i \le b_i \le t_i + \delta(t_i)$ for every $i \le n$.

Following [3], I say that a function ϕ: $[0,1] \to X$ is *McShane integrable*, with *McShane integral* w, if for every $\varepsilon > 0$ there is a gauge δ: $[0,1] \to]0,\infty[$ such that

$$\left\| w - \sum_{i\le n} (b_i - a_i)\phi(t_i) \right\| \le \varepsilon$$

for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ of $[0,1]$ subordinate to δ.

(c) A *Henstock partition* of $[0,1]$ is a McShane partition $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ of $[0,1]$ such that $t_i \in [a_i, b_i]$ for every $i \le n$. A function ϕ: $[0,1] \to X$ is *Henstock integrable*, with *Henstock integral* w, if for every $\varepsilon > 0$ there is a gauge δ: $[0,1] \to]0,\infty[$ such that $\|w - \sum_{i\le n}(b_i - a_i)\phi(t_i)\| \le \varepsilon$ for every Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ of $[0,1]$ subordinate to δ.

2. For the general theory of the Pettis integral, see [10]; for the McShane integral, see [3] and [2]; for the Henstock integral see [6], [8]. The most important fact to note here is that if $X = \mathbf{R}$ then the Pettis and McShane integrals coincide with the ordinary Lebesgue integral, but the Henstock integral is a proper extension of the Lebesgue integral ([8], S8.2 and 3.2). Moreover, every Henstock integrable function is Lebesgue measurable. I believe that this result is due to R.O. Davies. A proof of a more general result is in [1], Theorem 2.12; all the necessary ideas are in [2], Proposition 2L (see Proposition 10 below).

We need a couple of elementary lemmas concerning Henstock partitions. It will be convenient to use the phrase *partial McShane partition* to mean a finite sequence $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ such that the $[a_i, b_i]$ are non-overlapping closed subintervals of $[0,1]$ and $t_i \in [0,1]$ for each i; and to say that it is a *partial Henstock partition* if $t_i \in [a_i, b_i]$ for each i, and that it is *subordinate* to a gauge δ if $t_i - \delta(t_i) \leq a_i \leq b_i \leq t_i + \delta(t_i)$ for each i.

3. LEMMA. *Let δ: $[0,1] \to]0,\infty[$ be a gauge and $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ any partial Henstock partition subordinate to δ. Then it may be extended to a Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i \leq m}$ of $[0,1]$ subordinate to δ.*

Proof. Use the technique of [8], S1.8.

4. PROPOSITION. *Let X be a Banach space and ϕ: $[0,1] \to X$, ψ: $[0,1] \to X$ Henstock integrable functions with Henstock integrals v, w.*

(a) *$\phi + \psi$: $[0,1] \to X$ is Henstock integrable, with Henstock integral $v + w$.*

(b) *For any $\alpha \in \mathbf{R}$, $\alpha\phi$: $[0,1] \to X$ is Henstock integrable, with Henstock integral αv.*

(c) *If $0 \leq a \leq b \leq 1$ then $\phi_{[a,b]} = \phi \times \chi([a,b])$, defined by writing $\phi_{[a,b]}(t) = \phi(t)$ if $t \in [a,b]$, 0 otherwise, is Henstock integrable.*

(d) *If $f \in X^*$ then $f\phi$: $[0,1] \to \mathbf{R}$ is Henstock integrable, with Henstock integral $f(v)$.*

(e) *Let θ: $[0,1] \to X$ be another function. If for every $a \in]0,1]$ we have a Henstock integral $F(a)$ of $\theta \times \chi([a,1])$, and if $\lim_{a \downarrow 0} F(a) = w$ exists in X, then θ is Henstock integrable, with Henstock integral w.*

Proof. Part (d) is immediate from the definitions. For the other parts use the methods of 2.1, 2.3 and S2.8 in [8].

5. LEMMA. *Let δ: $[0,1] \to]0,\infty[$ be a gauge. Suppose that $A \subseteq [0,1]$ is any set and that K is a compact subset of $[0,1] \cap \bigcup_{t \in A}]t - \delta(t), t + \delta(t)[$. Then there is a partial Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i < n}$, subordinate to δ, such that $t_i \in A$ for each i and $K \subseteq \bigcup_{i<n} [a_i, b_i]$.*

Proof. (a) Suppose first that A is finite. For this case I induce on $\#(A)$, as follows. For $\#(A) = 0$ the result is trivial. For the inductive step to $\#(A) = k > 0$, take $t^* \in A$ for which $t^* - \delta(t^*)$ is minimal. Then $]t - \delta(t), t + \delta(t)[\subseteq]t^* - \delta(t^*), t^* + \delta(t^*)[$ whenever $t \in A$ and $t \leq t^*$. Set

$$A' = \{t : t \in A, t > t^*\},$$

$$K' = \{t : t \in K, t \geq t^* + \delta(t^*)\}.$$

Then $K' \subseteq \bigcup_{t \in A'}]t - \delta(t), t + \delta(t)[$, so by the inductive hypothesis we have a partial Henstock partition $\langle([a'_i, b_i], t_i)\rangle_{i<m}$, subordinate to δ, with $t_i \in A'$

for every i and $K' \subseteq \bigcup_{i<m}[a'_i, b_i]$. Set

$$a_i = \max(a'_i, t^*) \text{ for } i < m,$$
$$t_m = t^*,\ a_m = \max(0, t^* - \delta(t^*)),$$
$$b_m = \min(\{t^* + \delta(t^*), 1\} \cup \{a_i: i < m\});$$

then $\langle([a_i, b_i], t_i)\rangle_{i\le m}$ is a partial Henstock partition, subordinate to δ, with $t_i \in A$ for every $i \le m$ and $K \subseteq \bigcup_{i\le m}[a_i, b_i]$.

(b) The general case now follows, because K is compact, so that there must be a finite $A' \subseteq A$ such that $K \subseteq \bigcup_{t\in A'}]t - \delta(t), t + \delta(t)[$.

6. LEMMA. *Let g: $[0,1] \to \mathbf{R}$ be a function. Let δ: $[0,1] \to]0,\infty[$ and ε, $\eta > 0$ be such that $\sum_{i<n}(b_i - a_i)g(t_i) \le \eta$ for every partial Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i<n}$ subordinate to δ. Then*

$$\mu\left([0,1] \cap \bigcup_{g(t)\ge\varepsilon}]t - \delta(t), t + \delta(t)[\right) \le \eta/\varepsilon.$$

Proof. Let K be any compact subset of $[0,1] \cap \bigcup_{g(t)\ge\varepsilon}]t - \delta(t), t + \delta(t)$. Then there is a partial Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i<n}$, subordinate to δ, with $g(t_i) \ge \varepsilon$ for every i and $K \subseteq \bigcup_{i<n}[a_i, b_i]$. Now

$$\varepsilon\mu K \le \sum_{i<n}(b_i - a_i)g(t_i) \le \eta,$$

so $\mu K \le \eta/\varepsilon$. As K is arbitrary,

$$\mu\left([0,1] \cap \bigcup_{g(t)\ge\varepsilon}]t - \delta(t), t + \delta(t)[\right) \le \eta/\varepsilon.$$

7. LEMMA. *Let X be a Banach space and ϕ: $[0,1] \to X$ a Henstock integrable function, with Henstock integral w. Suppose that $\varepsilon > 0$, δ: $[0,1] \to]0,\infty[$ are such that*

$$\left\| w - \sum_{i\le n}(b_i - a_i)\phi(t_i) \right\| \le \varepsilon$$

whenever $\langle([a_i, b_i], t_i)\rangle_{i\le n}$ is a Henstock partition of $[0,1]$ subordinate to δ. Let $\langle([a_i, b_i], t_i)\rangle_{i<n}$ be a partial Henstock partition of $[0,1]$ subordinate to δ, and set $H = \bigcup_{i<n}[a_i, b_i]$. Then the Henstock integral $\int_H\phi$ of $\phi \times \chi(H)$ exists, and $\|\int_H\phi - \sum_{i<n}(b_i - a_i)\phi(t_i)\| \le \varepsilon$.

Proof. As in [8], 3.1.

8. THEOREM. *Let X be a Banach space and ϕ: $[0,1] \to X$ a function. Then ϕ is McShane integrable iff it is Henstock integrable and Pettis integrable.*

Proof. (a) If ϕ is McShane integrable, then it is certainly Henstock integrable, because the Henstock integral involves a smaller class of partitions. Also ϕ is Pettis integrable by Theorem 2C of [2].

(b) For the rest of this proof, therefore, I assume that ϕ is Henstock integrable and Pettis integrable, and seek to show that it is McShane integrable. For measurable sets $E \subseteq [0,1]$ write $\int_E \phi$ for the Pettis integral of ϕ over E. I seek to show that $\int \phi = \int_{[0,1]} \phi$ is the McShane integral of ϕ. Note that from 4(d) above we see that the Henstock integral of ϕ must be $\int \phi$.

(c) Let $\varepsilon > 0$. Write

$$C = \{g\phi \colon g \in X^*, \|g\| \leq 1\}.$$

By 4-1-5 and 4-1-6 of [10], C is totally bounded for the seminorm $\| \quad \|_1$.

For each $k \in \mathbf{N}$ set $\eta_k = 2^{-k}\varepsilon^2/(2\varepsilon + 12(k+1)) > 0$. Choose $h_{k0}, \ldots, h_{k,r(k)} \in C$ such that

$$\forall h \in C \exists i \leq r(k), \qquad \int |h - h_{ki}| \leq \eta_k.$$

Let δ_k: $[0,1] \to]0,\infty[$ be a gauge such that

(i) for every Henstock partition $\langle([a_i, b_i], t_i)\rangle_{i<n}$ of $[0,1]$ subordinate to δ_k,

$$\left\| \int \phi - \sum_{i \leq n} (b_i - a_i)\phi(t_i) \right\| \leq \eta_k,$$

(ii) for every $j \leq r(k)$, every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ of $[0,1]$ subordinate to δ_k,

$$\left| \int h_{kj} - \sum_{i \leq n} (b_i - a_i) h_{kj}(t_i) \right| \leq \eta_k.$$

(d) For each $k \in \mathbf{N}$ write $A_k = \{t \colon k \leq \|\phi(t)\| < k+1\}$. Define a gauge δ by writing

$$\delta(t) = \delta_k(t) \text{ if } t \in A_k.$$

Let $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ be a McShane partition of $[0,1]$ subordinate to δ, and take any $h \in C$.

(e) Fix k for the moment. Set

$$I_k = \{i : i \le n,\, t_i \in A_k\},\; H_k = \bigcup_{i \in I_k} [t_i - \delta(t_i), t_i + \delta(t_i)].$$

I seek to estimate $|\int_{H_k} h - \sum_{i \in I_k}(b_i - a_i)h(t_i)|$.

Take $j \le r(k)$ such that $\int |h - h_{kj}| \le \eta_k$. Then

$$\left| \int_{H_k} h - \int_{H_k} h_{kj} \right| \le \eta_k,$$

$$\left| \int_{H_k} h_{kj} - \sum_{i \in I_k} (b_i - a_i) h_{kj}(t_i) \right| \le \eta_k$$

because $\langle([a_i, b_i], t_i)\rangle_{i \in I_k}$ is a partial McShane partition subordinate to δ_k ([3], Theorem 5).

Set

$$V = \bigcup \left\{]t - \delta_k(t), t + \delta_k(t)[: h(t) - h_{kj}(t) \ge \varepsilon \right\}.$$

If $\langle([c_i, d_i], u_i)\rangle_{i<m}$ is a partial Henstock partition subordinate to δ_k, and $H = \bigcup_{i<m}[c_i, d_i]$, then the Henstock integral of $\phi \times \chi(H)$ must be the Pettis integral $\int_H \phi$, so by Lemma 7 we have

$$\left\| \int_H \phi - \sum_{i<m} (d_i - c_i)\phi(u_i) \right\| \le \eta_k$$

and

$$\left| \int_H g - \sum_{i<m} (d_i - c_i) g(u_i) \right| \le \eta_k \quad \text{for every } g \in C;$$

consequently

$$\left| \int_H (h - h_{kj}) - \sum_{i<m} (d_i - c_i)(h - h_{kj})(u_i) \right| \le 2\eta_k$$

and

$$\sum_{i<m} (d_i - c_i)(h - h_{kj})(u_i) \le 3\eta_k.$$

By Lemma 6,

$$\mu([0,1] \cap V) \leq 3\eta_k/\varepsilon.$$

But of course

$$\bigcup \left\{[a_i, b_i]: i \in I_k,\, h(t_i) - h_{kj}(t_i) \geq \varepsilon\right\} \setminus V$$

is finite, so

$$\sum_{i \in I_k,\, h(t_i) - h_{kj}(t_i) \geq \varepsilon} b_i - a_i \leq 3\eta_k/\varepsilon.$$

Similarly,

$$\sum_{i \in I_k,\, h_{kj}(t_i) - h(t_i) \geq \varepsilon} b_i - a_i \leq 3\eta_k/\varepsilon.$$

So

$$\sum_{i \in I_k} (b_i - a_i)\left|h(t_i) - h_{kj}(t_i)\right| \leq \varepsilon \sum_{i \in I_k} (b_i - a_i) + 12\eta_k(k+1)/\varepsilon$$

because

$$\left|h(t_i) - h_{kj}(t_i)\right| \leq 2\|\phi(t_i)\| \leq 2(k+1)$$

for each $i \in I_k$.

Putting these together,

$$\begin{aligned}\left|\int_{H_k} h - \sum_{i \in I_k} (b_i - a_i)h(t_i)\right| &\leq 2\eta_k + \varepsilon\mu H_k + 12\eta_k(k+1)/\varepsilon \\ &\leq 2^{-k}\varepsilon + \varepsilon\mu H_k.\end{aligned}$$

(f) Summing over k,

$$\left|\int h - \sum_{i \leq n} (b_i - a_i)h(t_i)\right| \leq \varepsilon \sum_{k \in \mathbf{N}} \left(2^{-k} + \mu H_k\right) = 3\varepsilon.$$

Thus

$$\left|f\left(\int \phi\right) - \sum_{i \leq n} (b_i - a_i) f(\phi(t_i))\right| \leq 3\varepsilon$$

for every f in the unit ball of X^*. But this means that

$$\left\| \int \phi - \sum_{i \le n} (b_i - a_i)\phi(t_i) \right\| \le 3\varepsilon.$$

This is true for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ of $[0, 1]$ subordinate to δ. As ε is arbitrary, ϕ is McShane integrable, as required.

9. Corollary. *Let X be a Banach space and ϕ: $[0,1] \to X$ a function. Then the following are equivalent*:

(i) *ϕ is McShane integrable*;

(ii) *$\phi \times \chi(E)$ is Henstock integrable for every measurable $E \subseteq [0,1]$*;

(iii) *ϕ is Henstock integrable and $\sum_{k \in \mathbf{N}} \int_{I_k} \phi$ exists in X for every sequence $\langle I_k \rangle_{k \in \mathbf{N}}$ of non-overlapping intervals in $[0,1]$, writing $\int_I \phi$ for the Henstock integral of $\phi \times \chi(I)$.*

Proof. (i) $\Rightarrow$ (ii) If ϕ is McShane integrable and $E \subseteq [0,1]$ is measurable, then $\phi \times \chi(E)$ is McShane integrable, by [2], Theorem $2E$, therefore Henstock integrable.

(ii) $\Rightarrow$ (i) Assume (ii). If $f \in X^*$ then $f\phi \times \chi(E)$ must be Henstock integrable for every measurable $E \subseteq [0,1]$, so $f\phi$ is Lebesgue integrable (because it is measurable, as remarked in §2 above); and $\int_E f\phi = f(\int_E \phi)$ for every E, f. Thus ϕ is Pettis integrable. By Theorem 8 it is McShane integrable.

(i) $\Rightarrow$ (iii) If ϕ is McShane integrable, then it is Pettis integrable, so that $\sum_{k \in \mathbf{N}} \int_{I_k} \phi$ exists for any sequence $\langle I_k \rangle_{k \in \mathbf{N}}$ of non-overlapping intervals, by Proposition 2B of [2].

(iii) $\Rightarrow$ (i) Assume (iii). If $f \in X^*$ then $h = f\phi$ is Henstock integrable and $\sum_{k \in \mathbf{N}} \int_{I_k} h$ exists for any sequence $\langle I_k \rangle_{k \in \mathbf{N}}$ of non-overlapping intervals in $[0,1]$. Consequently the indefinite Henstock integral $t \mapsto \int_0^t h$ of h as bounded variation and h is Lebesgue integrable ([8], 3.2).

This shows that ϕ is Dunford integrable. But now writing ν for the indefinite Dunford integral of ϕ ([2], 2A) we have $\nu I = \int_I \phi \in X$ for every interval $I \subseteq [0,1]$, and $\sum_{k \in \mathbf{N}} \nu I_k$ exists in X for every sequence $\langle I_k \rangle_{k \in \mathbf{N}}$ of non-overlapping intervals in $[0,1]$. So ϕ is Pettis integrable by Proposition 2B of [2].

Now Theorem 8 shows that ϕ is McShane integrable.

10. The Henstock integral is close to the McShane integral in a further respect. See [10] for the notion of 'properly measurable' function from a probability space to a Banach space.

PROPOSITION. *Let X be a Banach space such that the unit ball of X^* is w^*-separable. If ϕ: $[0, 1] \to X$ is a Henstock integrable function then it is properly measurable.*

Proof. As 2L of [2].

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