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THE MULTIPLIER OPERATORS ON THE PRODUCT SPACES

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Introduction

Let $H^p(R^{n_1} \times R^{n_2})$ be the Hardy space defined on the product spaces (for more details, see [1]) and let a function $a(x_1, x_2)$ denote a rectangle p atom on $H^p(R^{n_1} \times R^{n_2})$ if (i) the $a(x_1, x_2)$ is supported on a rectangle $R = I \times J$ (I and J are cubes on R^{n_1} and R^{n_2} respectively), (ii) $\|a\|_2 \leq |R|^{1/2-1/p}$ and (iii) one picks and fixes two sufficiently large positive integers k and l (depending on p) such that

$$\int_I x_1^\alpha a(x_1, x_2)\, dx_1 = 0 \qquad \text{for all } x_2 \in J \text{ and } |\alpha| \leq k$$

$$\int_J x_2^\beta a(x_1, x_2)\, dx_2 = 0 \qquad \text{for all } x_1 \in I \text{ and } |\beta| \leq l.$$

In the paper [3], R. Fefferman gave a very powerful theorem (see Theorem 1) for studying the boundedness on the $H^p(R^{n_1} \times R^{n_2})$ spaces of a linear operator. In his theorem, it mentioned that to consider the boundedness on H^p of a linear operator one only needs to look at the boundedness of the linear operator acting on the rectangle p atoms. This is true despite the counterexample of L. Carleson which shows that the space $H^p(R^{n_1} \times R^{n_2})$ cannot be decomposed into rectangle atoms.

We will use $\wedge$ to denote the Fourier Transform and $\wedge_1$ to denote the Fourier Transform acting on the first variable. Throughout this paper, C represents a constant, although different in different places. T_m denotes the multiplier operator associated with the multiplier m, i.e.,

$$\widehat{T_m f}(\xi, \eta) = m(\xi, \eta)\hat{f}(\xi, \eta).$$

THEOREM 1 (R. Fefferman [3]). *Suppose that T is a bounded linear operator on $L^2(R^{n_1} \times R^{n_2})$. Suppose further that if a is an $H^p(R^{n_1} \times R^{n_2})$*

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rectangle p atom $(0 < p \leq 1)$ *supported on* R, *we have*

$$\int_{{}^c\tilde{R}_r} |T(a)|^p(x_1, x_2)\, dx_1\, dx_2 \leq Cr^{-\sigma}$$

for all $r \geq 2$ *and some fixed* $\sigma > 0$, *where* ${}^c\tilde{R}_r$ *denotes the complement of the r fold enlargement of* R. *Then* T *is a bounded operator from* $H^p(R^{n_1} \times R^{n_2})$ *to* $L^p(R^{n_1} \times R^{n_2})$.

The purpose of this paper is to study several multiplier operators on product spaces by establishing four general theorems, Theorem A, B, C, D.
Suppose C_1, C_2 are the arbitrary two real positive numbers and

$$E_1 = \{(x, y) \mid |x| \geq C_1, |y| \geq C_2\}, \quad E_2 = \{(x, y) \mid |x| \geq C_1, |y| \leq C_2\},$$
$$E_3 = \{(x, y) \mid |x| \leq C_1, |y| \geq C_2\}, \quad E_4 = \{(x, y) \mid |x| \leq C_1, |y| \leq C_2\}.$$

Let $Q(a_1, a_2, m)$ denote the following statement.

Statement. Let $a_1, a_2, p, 0 < p \leq 1$ be real numbers and let

$$b_i = a_i\left(\left[n_i\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1\right), \quad i = 1, 2.$$

Suppose m is a bounded function defined on $R^{n_1} \times R^{n_2}$ satisfying

(1) $$\int_{s_1 < |\xi| \leq 2s_1} \int_{s_2 < |\eta| \leq 2s_2} \left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right|^2 d\xi\, d\eta \leq C s_1^{-2b_1 + 2(a_1 - 1)|\alpha| + n_1} s_2^{-2b_2 + 2(a_2 - 1)|\beta| + n_2}$$

(2) $$\sup_{\eta \in R^{n_2}} \int_{s_1 < |\xi| \leq 2s_1} \left|\partial_\xi^\alpha m(\xi, \eta)\right|^2 d\xi \leq C s_1^{-2b_1 + 2(a_1 - 1)|\alpha| + n_1}$$

and

(3) $$\sup_{\xi \in R^{n_1}} \int_{s_2 < |\eta| \leq 2s_2} \left|\partial_\eta^\beta m(\xi, \eta)\right|^2 d\eta \leq C s_2^{-2b_2 + 2(a_2 - 1)|\beta| + n_2}$$

where

$$|\alpha| \leq \left[n_1\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1 \quad \text{and} \quad |\beta| \leq \left[n_2\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1.$$

THEOREM A. *Let $a_1 \geq 0$, $a_2 \geq 0$. Suppose m is supported on E_1 and the statement $Q(a_1, a_2, m)$. Then T_m maps $H^p(R^{n_1} \times R^{n_2})$ boundedly to $L^p(R^{n_1} \times R^{n_2})$, i.e.,*

$$\|T_m f\|_{L^p} \leq C\|f\|_{H^p}.$$

THEOREM B. *Let $a_1 \geq 0$, $a_2 \leq 0$. Suppose m is supported on E_2 and the statement $Q(a_1, a_2, m)$. Then T_m maps $H^p(R^{n_1} \times R^{n_2})$ boundedly to $L^p(R^{n_1} \times R^{n_2})$.*

THEOREM C. *Let $a_1 \leq 0$, $a_2 \geq 0$. Suppose m is supported on E_3 and the statement $Q(a_1, a_2, m)$. Then T_m maps $H^p(R^{n_1} \times R^{n_2})$ boundedly to $L^p(R^{n_1} \times R^{n_2})$.*

THEOREM D. *Let $a_1 \leq 0$, $a_2 \leq 0$. Suppose m is supported on E_4 and the statement $Q(a_1, a_2, m)$. Then T_m maps $H^p(R^{n_1} \times R^{n_2})$ boundedly to $L^p(R^{n_1} \times R^{n_2})$.*

Now we use those theorems to get the following theorems.

THEOREM 2. *Suppose $0 < p \leq 1$. Let*

$$k = \left[n_1\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1, \qquad l = \left[n_2\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1.$$

Suppose $m \in C^k(R^{n_1}) \times C^l(R^{n_2})$ and

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \leq C|\xi|^{-|\alpha|}|\eta|^{-|\beta|}$$

where $|\alpha| \leq k$, $|\beta| \leq l$. Then T_m maps $H^q(R^{n_1} \times R^{n_2})$ boundedly to $L^q(R^{n_1} \times R^{n_2})$ for $p \leq q \leq 2$.

Remark. R. Fefferman and K.C. Lin [2] have obtained the result for $p = 1$ in Theorem 2 under a weaker hypothesis,

$$\int_{s_1 < |\xi| \leq 2s_1} \int_{s_2 < |\eta| \leq 2s_2} \left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right|^2 d\xi\, d\eta \leq C s_1^{-2|\alpha| + n_1} s_2^{-2|\beta| + n_2}.$$

THEOREM 3. *Suppose* $0 < p \le 1$ *and* m *is defined on* $R^{n_1} \times R^{n_2}$ *satisfying*

(4)
$$\left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \le C(1 + |\xi|)^{-([n_1(1/p - 1/2)] + 1)}(1 + |\eta|)^{-([n_2(1/p-1/2)]+1)}$$

for

$$|\alpha| \le \left[n_1\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1, \qquad |\beta| \le \left[n_2\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1.$$

Then

$$\|T_m f\|_{L^q((R^{n_1}\times R^{n_2})} \le C\|f\|_{H^q(R^{n_1}\times R^{n_2})}$$

for $p \le q \le 2$.

THEOREM 4. *Suppose* $0 < p \le 1$ *and* m *is defined on* $R^{n_1} \times R^{n_2}$ *satisfying*

$$|m(\xi, \eta)| \le (1 + |\xi|)^{-([n_1(1/p-1/2)]+1)}(1 + |\eta|)^{-([n_2(1/p-1/2)]+1)}$$

and the inverse Fourier transform of m *has compact support. Then*

$$\|T_m f\|_{L^q(R^{n_1}\times R^{n_2})} \le C\|f\|_{H^q(R^{n_1}\times R^{n_2})}$$

for $p \le q \le 2$.

Proofs of the theorems. Without loss of generality, one assumes $C_1 = C_2 = 1$ in the definitions of E_i, $i = 1, 2, 3, 4$. The idea of the proof of Theorem A is basically from [4]. Let a be a smooth rectangle atom with vanishing moments and $\operatorname{supp} a \subset I \times J \equiv R$, $\|a\|_2 \le |I|^{1/2-1/p}|J|^{1/2-1/p}$ where I and J are cubes on R^{n_1} and R^{n_2}, respectively. Let us take a smooth function on R^1 and its Fourier transform $\hat{\phi}(t)$ has compact support $\{1/2 < |t| < 2\}$ such that $\sum_{j\in Z}\hat{\phi}(2^{-j}|t|) = 1$ for all $t \neq 0$. Let

$$m_{i,j}(\xi, \eta) = m(\xi, \eta)\hat{\phi}(2^{-i}|\xi|)\hat{\phi}(2^{-j}|\eta|)$$

and

$$\widehat{T_{ij}f}(\xi, \eta) = m_{i,j}(\xi, \eta)\hat{f}(\xi, \eta) \equiv (K_{ij} * f)^\wedge(\xi, \eta)$$

It is clear $Tf = \sum_{ij} T_{ij} f$.

Let us decompose ${}^c\tilde{R}_r$, the complement of $\tilde{R}_r$, into three pieces

$$ {}^c\tilde{R}_r^1 = \left\{(\xi,\eta) | \xi \in {}^c\tilde{I}_r, \eta \in \tilde{J}_2\right\} $$
$$ {}^c\tilde{R}_r^2 = \left\{(\xi,\eta) | \xi \in \tilde{I}_2, \eta \in {}^c\tilde{J}_r\right\} $$

and

$$ {}^c\tilde{R}_r^3 = {}^c\tilde{R}_r \setminus \left({}^c\tilde{R}_r^1 \cup {}^c\tilde{R}_r^2\right). $$

LEMMA A. *Let $a_1 \geq 0$, $a_2 \geq 0$. Suppose $m(\xi,\eta)$ satisfies* (1), (2), (3) *in Theorem A and $m(\xi,\eta)$ is supported on E_1. Then*

(5)
$$ \begin{aligned} &\int_{{}^c\tilde{R}_r^3} |T_{ij}a|^p \, dx\, dy \\ &\leq C\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right) \\ &\cdot |I|^{(\frac{1}{2}-\frac{1}{p}+\frac{\lambda_1}{n_1})p+(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})+\frac{p}{2}} \\ &\cdot |J|^{(\frac{1}{2}-\frac{1}{p}+\frac{\lambda_2}{n_2})p+(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})+\frac{p}{2}} \\ &\cdot 2^{i((a_1-1)k+\lambda_1-b_1+\frac{n_1}{2})p} 2^{j((a_2-1)l+\lambda_2-b_2+\frac{n_2}{2})p}, \end{aligned} $$

(6)
$$ \int_{{}^c\tilde{R}_r^2} \left|\sum_i T_{ij}a\right|^p dx\, dy \leq Cr^{-lp+n_2(\frac{2-p}{2})} 2^{j((a_2-1)l+\lambda_2-b_2+\frac{n_2}{2})p} |J|^{-\frac{l}{n_2}p+\frac{\lambda_2}{n_2}p+\frac{p}{2}} $$

and

(7)
$$ \int_{{}^c\tilde{R}_r^1} \left|\sum_j T_{ij}a\right|^p dx\, dy \leq Cr^{-kp+n_1(\frac{2-p}{2})} 2^{i((a_1-1)k+\lambda_1-b_1+\frac{n_1}{2})p} |I|^{-\frac{k}{n_1}p+\frac{\lambda_1}{n_1}p+\frac{p}{2}} $$

where

$$ k = \left[n_1\left(\frac{1}{p}-\frac{1}{2}\right)\right]+1, \qquad l = \left[n_2\left(\frac{1}{p}-\frac{1}{2}\right)\right]+1, $$

$0 < p \leq 1$, *λ_1, λ_2 are arbitrarily nonnegative integers and $\lambda_1 \leq k$, $\lambda_2 \leq l$.*

Proof. Since m is supported on E_1, without loss of generality, we assume $m_{i,j}(\xi,\eta) = 0$ if $i < 0$ or $j < 0$. After a translation, it suffices to assume the origin (0, 0) is the center of the rectangle $I \times J$. Write

$$T_{ij}a(x, y)$$

$$= \int K_{ij}(x - x', y - y') a(x', y')\, dx'\, dy'$$

(8)

$$= \int \left(K_{ij}(x - x', y - y') - \sum_{|\alpha| \le \lambda_1 - 1} \frac{1}{\alpha!} \partial_x^\alpha K_{ij}(x, y - y')(-x')^\alpha \right) \times a(x', y')\, dx'\, dy'$$

$$= \lambda_1 \sum_{|\tilde{\alpha}| = \lambda_1} \frac{1}{\tilde{\alpha}!} \int_{I \times J} \int_0^1 (1 - t)^{\lambda_1 - 1} \partial_x^{\tilde{\alpha}} K_{ij}(x - tx', y - y')(-x')^{\tilde{\alpha}} a(x', y')\, dt\, dx'\, dy'$$

(9)

$$= \lambda_1 \sum_{|\tilde{\alpha}| = \lambda_1} \int_0^1 \int_{I \times J} (1 - t)^{\lambda_1 - 1} \left(\partial_x^{\tilde{\alpha}} K_{ij}(x - tx', y - y') - \sum_{|\beta| \le \lambda_2 - 1} \frac{1}{\beta!} \partial_y^\beta \partial_x^{\tilde{\alpha}} K_{ij}(x - tx', y)(-y')^\beta \right) (-x')^{\tilde{\alpha}} a(x', y')\, dy'\, dx'\, dt$$

(10)

$$= \lambda_1 \lambda_2 \sum_{|\tilde{\alpha}| = \lambda_1} \frac{1}{\tilde{\alpha}!} \sum_{|\tilde{\beta}| = \lambda_2} \frac{1}{\tilde{\beta}!} \left\{ \int_{I \times J} \int_0^1 \int_0^1 (1 - t)^{\lambda_1 - 1} (1 - s)^{\lambda_2 - 1} \cdot (-x')^{\tilde{\alpha}} (-y')^{\tilde{\beta}} \partial_y^{\tilde{\beta}} \partial_x^{\tilde{\alpha}} K_{ij}(x - tx', y - sy') \times a(x', y')\, ds\, dt\, dx'\, dy' \right\},$$

where λ_1 and λ_2 are integers and $0 \le \lambda_1 \le k$, $0 \le \lambda_2 \le l$. Here we should remark that if one sets $\lambda_1 = 0$ or $\lambda_2 = 0$ then it means one does not subtract the Taylor polynomial on the equation (8) or (9). For example, if $\lambda_1 = 0$ and $\lambda_2 \ne 0$ then

$$T_{ij}a(x, y) = \int K_{ij}(x - x', y - y') a(x', y')\, dx'\, dy'$$

$$= \lambda_2 \sum_{|\tilde{\beta}| = \lambda_2} \frac{1}{\tilde{\beta}!} \int_{I \times J} \int_0^1 (1 - s)^{\lambda_2 - 1} (-y')^{\tilde{\beta}} \times \partial_y^{\tilde{\beta}} K_{ij}(x - x', y - sy')\, ds\, dx'\, dy'.$$

Let us look at the integral in the parentheses of (10). It is dominated by

$$\int_{I\times J}\int_0^1\int_0^1 \left|(-x')^{\tilde{\alpha}}(-y')^{\tilde{\beta}}\partial_y^{\tilde{\beta}}\partial_x^{\tilde{\alpha}}K_{ij}(x-tx', y-sy')a(x',y')\right| ds\,dt\,dx'\,dy'$$
$$\leq \left(\int_{I\times J}\int_0^1\int_0^1 \left|(-x')^{\tilde{\alpha}}(-y')^{\tilde{\beta}}\partial_y^{\tilde{\beta}}\partial_x^{\tilde{\alpha}}K_{ij}(x-tx', y-sy')\right|^2 ds\,dt\,dx'\,dy'\right)^{1/2}$$
$$\times\left(\int_{I\times J}|a|^2\right)^{1/2}$$
$$\leq |I|^{1/2-1/p+\lambda_1/n_1}|J|^{1/2-1/p+\lambda_2/n_2}$$
$$\times\left(\int_{I\times J}\int_0^1\int_0^1 \left|\partial_y^{\tilde{\beta}}\partial_x^{\tilde{\alpha}}K_{ij}(x-tx', y-sy')\right|^2 ds\,dt\,dx'\,dy'\right)^{1/2}$$
$$\equiv |I|^{1/2-1/p+\lambda_1/n_1}|J|^{1/2-1/p+\lambda_2/n_2}L_{ij}(x,y).$$

Hence

$$\int_D |T_{ij}a|^p \leq C|I|^{(1/2-1/p+\lambda_1/n_1)p}|J|^{(1/2-1/p+\lambda_2/n_2)p}\int_D \left|L_{ij}(x,y)\right|^p dx\,dy \tag{11}$$

for any measurable set D. Next, one will compute the integral $\int_D |L_{i,j}(x,y)|^p\,dx\,dy$ with respect to $D = {}^c\tilde{R}_r^3$, ${}^c\tilde{R}_r^1$ and ${}^c\tilde{R}_r^2$, respectively.

First let us compute

$$\int_{{}^c\tilde{R}_r^3}\left|L_{ij}(x,y)\right|^p dx\,dy$$
$$= \int_{{}^c\tilde{R}_r^3}(A|x|)^{-kp}(B|y|)^{-lp}\left((A|x|)^k(B|y|)^l L_{ij}(x,y)\right)^p dx\,dy$$

where A and B will be given later. By Hölder's inequality, it is not bigger than

$$\left(\int_{{}^c\tilde{R}_r^3}(A|x|)^{-k(\frac{2p}{2-p})}(B|y|)^{-l(\frac{2p}{2-p})}\,dx\,dy\right)^{\frac{2-p}{2}}$$
$$\cdot\left(\int_{{}^c\tilde{R}_r^3}\left|(A|x|)^k(B|y|)^l L_{ij}(x,y)\right|^2 dx\,dy\right)^{p/2}$$
$$\leq CA^{-kp}B^{-lp}\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right)$$
$$\cdot\ |I|^{(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})}|J|^{(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})}\left(\int_{{}^c\tilde{R}_r^3}(A|x|)^{2k}(B|y|)^{2l}\right.$$
$$\left.\cdot\int_{I\times J}\int_0^1\int_0^1\left|\partial_y^{\tilde{\beta}}\,\partial_x^{\tilde{\alpha}}K_{ij}(x-tx', y-sy')\right|^2 ds\,dt\,dx'\,dy'\,dx\,dy\right)^{p/2}.$$

Since $(x, y) \in {}^c\tilde{R}_r^3$ and $(x', y') \in I \times J$, $0 \le s, t \le 1$, one has $|x| \approx |x - tx'|$, $|y| \approx |y - sy'|$. Therefore, the above inequality is equivalent to

$$CA^{-kp}B^{-lp}\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right)$$
$$\cdot |I|^{(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})}|J|^{(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})}$$
$$\cdot\left(\int_0^1\int_0^1\int_{I\times J}\int_{{}^c\tilde{R}_r^3}\Big|(A|x-tx'|)^k(B|y-sy'|)^l\right.$$
$$\left.\times\partial_y^{\tilde{\beta}}\partial_x^{\tilde{\alpha}}K_{ij}(x-tx', y-sy')\Big|^2 dx\,dy\,dx'\,dy'\,ds\,dt\right)^{p/2}.$$

After a change of variables, the last inequality is dominated by

$$A^{-kp}B^{-lp}\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right)$$
$$(12)\qquad \cdot |I|^{(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})+\frac{p}{2}}|J|^{(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})+\frac{p}{2}}$$
$$\cdot\left(\int_{R^{n_1}\times R^{n_2}}\Big||Ax|^k|By|^l\partial_y^{\tilde{\beta}}\partial_x^{\tilde{\alpha}}K_{ij}(x,y)\Big|^2 dx\,dy\right)^{p/2}.$$

Here one lets

$$A = 2^{-i(a_1-1)}, \qquad B = 2^{-j(a_2-1)}.$$

From the hypothesis (1), one concludes

$$(13)\qquad \left(\int_{|\xi|\approx 2^i}\int_{|\eta|\approx 2^j}\Big|(A\partial_\xi)^\sigma(B\partial_\eta)^\delta\big(\xi^{\tilde{\alpha}}\eta^{\tilde{\beta}}m_{i,j}(\xi,\eta)\big)\Big|^2 d\xi\,d\eta\right)^{1/2}$$
$$\le C2^{i(\lambda_1-b_1+n_1/2)}2^{j(\lambda_2-b_2+n_2/2)}$$

for every multi-indexes σ, δ. (Recall $|\tilde{\alpha}| = \lambda_1$ and $|\tilde{\beta}| = \lambda_2$.) Then, applying Plancherel's Theorem on the integral (12) and using formula (13), one has

$$\int_{{}^c\tilde{R}_r^3}|L_{ij}(x,y)|^p\,dx\,dy$$
$$\le C\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right)$$
$$\cdot |I|^{(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})+\frac{p}{2}}|J|^{(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})+\frac{p}{2}}$$
$$\cdot 2^{i((a_1-1)k+\lambda_1-b_1+\frac{n_1}{2})p}2^{j((a_2-1)l+\lambda_2-b_2+\frac{n_2}{2})p}.$$

From (11) and the above inequality, one has

$$\begin{aligned}
&\int_{{}^c\tilde{R}_r^3} |T_{ij}a|^p\,dx\,dy \\
&\quad\le C\left(r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})}\right) \\
&\qquad\cdot |I|^{(\frac{1}{2}-\frac{1}{p}+\frac{\lambda_1}{n_1})p+(-k(\frac{2p}{2-p})\frac{1}{n_1}+1)(\frac{2-p}{2})+\frac{p}{2}} \\
&\qquad\times |J|^{(\frac{1}{2}-\frac{1}{p}+\frac{\lambda_2}{n_2})p+(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})+\frac{p}{2}} \\
&\qquad\cdot 2^{i((a_1-1)k+\lambda_1-b_1+\frac{n_1}{2})p}\,2^{j((a_2-1)l+\lambda_2-b_2+\frac{n_2}{2})p}.
\end{aligned}$$

(5) is proved.

Since the proofs of (6) and (7) are similar, we show (6). Let $T_j a \equiv \sum_i T_{ij}a$. Hence, it is clear that

$$\widehat{T_j a}(\xi,\eta) = m(\xi,\eta)\hat{\phi}(2^{-j}|\eta|)\hat{f}(\xi,\eta) \equiv \hat{K}_j(\xi,\eta)\hat{f}(\xi,\eta).$$

Let us write

$$\begin{aligned}
\int_{{}^c\tilde{R}_r^2}\left|\sum_i T_{ij}a\right|^p dx\,dy &= \int_{{}^c\tilde{R}_r^2}|T_j a|^p\,dx\,dy \\
&\le \int_{{}^c\tilde{R}_r^2}(B|y|)^{-lp}\left((B|y|)^l|T_j a|\right)^p dx\,dy \\
&\le \left(\int_{{}^c\tilde{R}_r^2}(B|y|)^{-l(\frac{2p}{2-p})}\,dx\,dy\right)^{(2-p)/2} \\
&\quad\times\left(\int_{{}^c\tilde{R}_r^2}\left|(B|y|)^l T_j a\right|^2 dx\,dy\right)^{p/2} \\
&\le CB^{-lp}|I|^{\frac{2-p}{2}}\, r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})} \\
&\quad\times|J|^{(-l(\frac{2p}{2-p})\frac{1}{n_2}+1)(\frac{2-p}{2})} \\
&\quad\cdot\left(\int_{{}^c\tilde{R}_r^2}\left|(B|y|)^l T_j a\right|^2 dx\,dy\right)^{p/2},
\end{aligned}$$

where the last second inequality is obtained by applying Hölder inequality.

Following the same procedure as in the proof of (5), one has

$$\int_{{}^c\tilde{R}_r^2}\left|(B|y|)^l T_j a\right|^2 dx\,dy$$
$$\leq C\sum_{|\tilde{\beta}|=\lambda_2}\int_{{}^c\tilde{R}_r^2}(B|y|)^{2l}\left|\int_0^1\int_{I\times J}(-y')^{\tilde{\beta}}\partial_y^{\tilde{\beta}}K_j(x-x',y-sy')\right.$$
$$\left.\times a(x',y')\,dx'\,dy'\,ds\right|^2 dx\,dy.$$

Since

$${}^c\tilde{R}_r^2=\left\{|x|<2\sqrt{n_1}|I|^{1/n_1}\right\}\cap\left\{|y|>r|J|^{1/n_2}\right\},$$

by Minkowski's inequality, the last inequality is less than

$$C\sum_{|\tilde{\beta}|=\lambda_2}\left\{\int_{|y|>r|J|^{1/n_2}}(B|y|)^{2l}\left(\int_0^1\int_J|y'|^{\lambda_2}\left(\int_{|x|<2\sqrt{n_1}|I|^{1/n_1}}\left|\int_I\partial_y^{\tilde{\beta}}K_j(x-x',y-sy')\right.\right.\right.\right.$$
$$\left.\left.\left.\left.a(x',y')\,dx'\right|^2 dx\right)^{1/2} dy'\,ds\right)^2 dy\right\}.$$

By Plancherel's Theorem, the above integral on the parentheses is dominated by

$$\int_{|y|>r|J|^{1/n_2}}\left(\int_0^1\int_J|y'|^{\lambda_2}(B|y|)^l\right.$$
$$\left.\times\left(\int_{R^{n_1}}\left|\partial_y^{\tilde{\beta}}\widehat{K_j}^1(\xi,y-sy')\hat{a}^1(\xi,y')\right|^2 d\xi\right)^{1/2} dy'\,ds\right)^2 dy$$
$$\leq|J|^{2\frac{\lambda_2}{n_2}}\int_{|y|>r|J|^{1/n_2}}\left(\int_0^1\int_J(B|y|)^l\right.$$
$$\left.\times\left(\int_{R^{n_1}}\left|\partial_y^{\tilde{\beta}}\widehat{K_j}^1(\xi,y-sy')\hat{a}^1(\xi,y')\right|^2 d\xi\right)^{1/2} dy'\,ds\right)^2 dy$$
$$\leq|J|^{2\frac{\lambda_2}{n_2}+1}\int_{|y|>r|J|^{1/n_2}}\int_0^1\int_J\int_{R^{n_1}}\left|(B|y|)^l\partial_y^{\tilde{\beta}}\widehat{K_j}^1(\xi,y-sy')\right.$$
$$\left.\times\hat{a}^1(\xi,y')\right|^2 d\xi\,dy'\,ds\,dy$$
$$\leq C|J|^{2\frac{\lambda_2}{n_2}+1}\int_0^1\int_J\int_{R^{n_1}}\int_{R^{n_2}}\left|(B|y-sy'|)^l\partial_y^{\tilde{\beta}}\widehat{K_j}^1(\xi,y-sy')\right|^2 dy$$
$$\times|\hat{a}^1(\xi,y')|^2\,d\xi\,dy'\,ds.$$

By a change of variables on the last inequality, one has

(14)

$$\int_{{}^c\tilde{R}_r^2} \left|(B|y|)^l T_j a\right|^2 dx\,dy \leq C|J|^{2\frac{\lambda_2}{n_2}+1} \int_J \int_{R^{n_1}} \int_{R^{n_2}} \left|(B|y|)^l \partial_y^{\tilde{\beta}} \widehat{\tilde{K}_j}^1(\xi, y)\right|^2 dy |\hat{a}^1(\xi, y')|^2 d\xi\, dy'.$$

Let $B = 2^{-j(a_2-1)}$. As in the inequality (13), one constructs a similar inequality

$$(15)\quad \sup_{\xi\in R^{n_1}} \left(\int_{|\eta|\approx 2^j} \left|(B\partial_\eta)^\delta\left(\eta^{\tilde{\beta}}\widehat{K_j}(\xi, \eta)\right)\right|^2 d\xi\, d\eta\right)^{1/2} \leq C2^{j(\lambda_2-b_2+n_2/2)}$$

by using the hypothesis (3).

Hence, applying Plancherel's Theorem on the integral $\int_{R^{n_2}} |\cdots|^2\, dy$ on (14) and applying (15), the inequality (14) is not bigger than

$$|J|^{2\lambda_2/n_2+1} \int_{R^{n_2}} \int_{R^{n_1}} 2^{2j(\lambda_2-b_2)+jn_2} |\hat{a}^1(\xi, y')|^2 d\xi\, dy' \leq C2^{j(2(\lambda_2-b_2)+n_2)} |I|^{2(1/2-1/p)} |J|^{2(1/2-1/p)+2\lambda_2/n_2+1}.$$

Therefore,

$$\int_{{}^c\tilde{R}_r^2} |T_j a|^p\, dx\,dy \leq C2^{j((a_2-1)l+\lambda_2-b_2+n_2/2)p} r^{-lp+n_2(\frac{2-p}{2})} |J|^{-\frac{l}{n_2}p+\frac{\lambda_2}{n_2}p+\frac{p}{2}}.$$

This is (6). Lemma A is proved.

Proof of Theorem A. As in Lemma A, since m is supported on E_1, without loss of generality, we assume $m_{ij}(\xi) = 0$ if $i < 0$ or $j < 0$. Let us write

$$\int_{{}^c\tilde{R}_r} |Ta|^p = \int_{{}^c\tilde{R}_r^1} \cdots + \int_{{}^c\tilde{R}_r^2} \cdots + \int_{{}^c\tilde{R}_r^3} \cdots$$

and recall $k = [n_1(1/p - 1/2)] + 1$, $l = [n_2(1/p - 1/2)] + 1$. Then there exists $\sigma > 0$ such that

$$\max\left\{\left(-k\left(\frac{2p}{2-p}\right) + n_1\right), \left(-l\left(\frac{2p}{2-p}\right) + n_2\right)\right\} \leq -\left(\frac{2}{2-p}\right)\sigma.$$

Hence

$$r^{(-k(\frac{2p}{2-p})+n_1)(\frac{2-p}{2})} + r^{(-l(\frac{2p}{2-p})+n_2)(\frac{2-p}{2})} \le 2r^{-\sigma},$$

if $r \ge 2$. For each rectangle $I \times J$, there exist $i_0, j_0 \in Z$ such that $|I|^{1/n_1} \approx 2^{-i_0}$, $|J|^{1/n_2} \approx 2^{-j_0}$. Therefore, if $0 < p \le 1$,

(16)
$$\int_{{}^c\tilde{R}_r^3} |Ta|^p \le \sum_{ij} \int_{{}^c\tilde{R}_r^3} |T_{ij}a|^p = \sum_{i \ge i_0} \sum_{j \ge j_0} + \sum_{i \ge i_0} \sum_{j < j_0} + \sum_{i < i_0} \sum_{j \ge j_0} + \sum_{i < i_0} \sum_{j < j_0}.$$

We are going to apply (5) in Lemma A by choosing the distinct λ_1 and λ_2 on the distinct terms of sums on (16). That is to say, (i) in the sums $\Sigma_{i \ge i_0} \Sigma_{j \ge j_0}$ one picks $\lambda_1 = \lambda_2 = 0$, (ii) in the sums $\Sigma_{i \ge i_0} \Sigma_{j < j_0}$ on takes $\lambda_1 = 0$, $\lambda_2 = l = [n_2(1/p - 1/2)] + 1$, (iii) in the sums $\Sigma_{i < i_0} \Sigma_{j \ge j_0}$ one lets $\lambda_1 = k = [n_1(1/p - 1/2)] + 1$, $\lambda_2 = 0$ and (iv) in the sums $\Sigma_{i < i_0} \Sigma_{j < j_0}$ on sets $\lambda_1 = k = [n_1(1/p - 1/2)] + 1$, $\lambda_2 = l = [n_2(1/p - 1/2)] + 1$. Hence, from (16),

$$\begin{aligned}
\int_{{}^c\tilde{R}_r^3} |Ta|^p &\le \sum_{i \ge i_0} \Big(\sum_{j \ge j_0} + \sum_{j < j_0} \Big) + \sum_{i < i_0} \Big(\sum_{j \ge j_0} + \sum_{j < j_0} \Big) \\
&\le Cr^{-\sigma} \sum_{i \ge i_0} |I|^{(\frac{1}{2} - \frac{1}{p})p + (-k(\frac{2p}{2-p})\frac{1}{n_1} + 1)(\frac{2-p}{2}) + \frac{p}{2}} 2^{i((a_1 - 1)k - b_1 + \frac{n_1}{2})p} \\
&\quad \cdot \Bigg\{ \sum_{j < j_0} |J|^{(\frac{1}{2} - \frac{1}{p} + \frac{\lambda_2}{n_2})p + (-l(\frac{2p}{2-p})\frac{1}{n_2} + 1)(\frac{2-p}{2}) + \frac{p}{2}} 2^{j((a_2 - 1)l + \lambda_2 - b_2 + \frac{n_2}{2})p} \\
&\qquad + \sum_{j \ge j_0} |J|^{(\frac{1}{2} - \frac{1}{p})p + (-l(\frac{2p}{2-p})\frac{1}{n_2} + 1)(\frac{2-p}{2}) + \frac{p}{2}} 2^{j((a_2 - 1)l - b_2 + \frac{n_2}{2})p} \Bigg\} \\
&\quad + Cr^{-\sigma} \sum_{i < i_0} \cdots \Big(\sum_{j \ge j_0} \cdots + \sum_{j < j_0} \cdots \Big) \\
&\le Cr^{-\sigma} \sum_{i \ge i_0} |I|^{(\frac{1}{2} - \frac{k}{n_1})p} 2^{i(-k + \frac{n_1}{2})p} \Bigg\{ \sum_{j < j_0} 2^{-j_0 n_2 p/2} 2^{j n_2 p/2} \\
&\qquad + \sum_{j \ge j_0} 2^{-j_0 n_2 (\frac{1}{2} - \frac{l}{n_2})p} 2^{j(-l + \frac{n_2}{2})p} \Bigg\} + Cr^{-\sigma} \sum_{i < i_0} \cdots \\
&\le Cr^{-\sigma} \Bigg\{ \sum_{i \ge i_0} 2^{-i_0 n_1 (\frac{1}{2} - \frac{k}{n_1})p} 2^{i(-k + \frac{n_1}{2})p} + \sum_{i < i_0} \cdots \Bigg\} \\
&\le Cr^{-\sigma}
\end{aligned}$$

where the last two inequalities are obtained by the fact, $n_2/2 < l$ and $n_1/2 < k$.

On the other hand, for the boundedness of the integrals

$$\int_{{}^c\tilde{R}_r^1} |Ta|^p \quad \text{and} \quad \int_{{}^c\tilde{R}_r^2} |Ta|^p,$$

these can be proved by following the same ideas as the proof in the above case, applying (6), (7) in Lemma A and the next two inequalities, respectively,

$$\int_{{}^c\tilde{R}_r^2} |Ta|^p \le \sum_j \int_{{}^c\tilde{R}_r^2} \left| \sum_i T_{ij} a \right|^p \quad \text{and} \quad \int_{{}^c\tilde{R}_r^1} |Ta|^p \le \sum_i \int_{{}^c\tilde{R}_r^1} \left| \sum_j T_{ij} a \right|^p.$$

Theorem A is proved.

Proof of Theorems B, C, D. As in the proof of Theorem A, one can prove Theorems B, C, D by establishing the corresponding lemmas. In the proof of Lemma A, the equations have nothing to do with the "signs" of a_1, a_2 except (13) and (15). The existences of (13) and (15) depend on the signs of i and a_1 (j and a_2), in particular, on $ia_1 > 0$ ($ja_2 > 0$). Therefore, we omit those proofs.

Proof of Theorem 2. Taking a smooth function ψ on R^1 with compact support $\{t \mid |t| < 2\}$ and $\psi(t) = 1$ if $|t| \le 1$, let

(17)
$$\begin{aligned} m(\xi,\eta) &= (1-\psi(\xi))(1-\psi(\eta))m(\xi,\eta) + (1-\psi(\xi))\psi(\eta)m(\xi,\eta) \\ &\quad + \psi(\xi)(1-\psi(\eta))m(\xi,\eta) + \psi(\xi)\psi(\eta)m(\xi,\eta) \\ &\equiv m_1(\xi,\eta) + m_2 + m_3 + m_4. \end{aligned}$$

Then one applies m_i, $i = 1,2,3,4$, to Theorems A, B, C, D, respectively. Theorem 2 is followed by setting $a_1 = a_2 = 0$ in Theorems A, B, C, D.

Proof of Theorem 3. Again, we borrow the decomposition (17) of m on the proof of Theorem 2. Then the boundedness of T is obtained by setting $a_1 = a_2 = 1$ on Theorem A, $a_1 = 1$, $a_2 = 0$ on Theorem B, $a_1 = 0$, $a_2 = 1$ on Theorem C and $a_1 = a_2 = 0$ on Theorem D.

Proof of Theorem 4. Since the inverse Fourier transform of m has compact support, there exists a smooth function ϕ such that

$$m(\xi,\eta) = \int_{R^{n_1}\times R^{n_2}} \phi(\xi-\xi',\eta-\eta')m(\xi',\eta')\,d\xi'\,d\eta'.$$

Theorem 4 is proved by applying Theorem 3.

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