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FUNCTIONAL INEQUALITIES, JACOBI PRODUCTS, AND QUASICONFORMAL MAPS

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1. Introduction

The special function (cf. (2.1))

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K), \tag{1.1}$$

where $K \in (0,\infty)$, $r \in (0,1)$, is closely related to geometric properties of quasiconformal mappings. Some examples of such geometric properties are the quasiconformal Schwarz lemma [LV, p. 64] and the study of the Beurling-Ahlfors extension of quasisymmetric functions [AH], [L], [LV]. We first recall two earlier explicit estimates for the function $\varphi_K(r)$ and then give our main results, which yield new identities and inequalities for this frequently occurring function. The basic inequality

$$r^{1/K} < \varphi_K(r) < 4^{1-1/K} r^{1/K} \tag{1.2}$$

for $K \in (1,\infty)$ and $r \in (0,1)$, has been known for more than thirty years. This inequality was recently sharpened [AVV3] to

$$\frac{1}{\operatorname{ch}\left(\frac{1}{K}\operatorname{arch}\left(\frac{1}{r}\right)\right)} < \varphi_K(r) < \operatorname{th}(\operatorname{arth} r + (K-1)\mu(r')), \tag{1.3}$$

for $K \in (1,\infty)$, $r \in (0,1)$ with $r' = \sqrt{1-r^2}$.

1.4. THEOREM. *For $K \in (0,\infty)$, let $f\colon [0,1] \to R$, be defined by*

$$f(r) = \frac{1-\varphi_{1/K}(r)}{(1-r)^{1/K}} \quad \textit{for } 0 \le r < 1,$$

and $f(1) = 8^{1-1/K}$. Then f is strictly increasing if $K > 1$ and strictly decreasing

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if $K < 1$. *In particular, for* $K \in (1, \infty)$, $r \in (0,1)$,

$$(1) \qquad 1-(1-r)^K < \varphi_K(r) < 1-8^{1-K}(1-r)^K,$$

$$(2) \qquad 1-8^{1-1/K}(1-r)^{1/K} < \varphi_{1/K}(r) < 1-(1-r)^{1/K}.$$

This theorem answers affirmatively a question raised by R. Kühnau (see the remark in Section 2 following the proof of Theorem 1.4). Applying C.G.J. Jacobi's formulas we obtain an infinite product expansion for $\varphi_K(r)$. This expression for $\varphi_K(r)$ follows immediately from Jacobi's work, but has apparently been overlooked in the literature.

1.5. THEOREM. *For* $K > 0$ *and* $0 < r < 1$ *we have*

$$(1) \quad \varphi_K(r) = 4\exp(-\mu(r)/K)\prod_{n=1}^{\infty}\left(\frac{1+q^{2n/K}}{1+q^{(2n-1)/K}}\right)^4,$$

$$(2) \quad \varphi_K(r) = \prod_{n=1}^{\infty}(\mathrm{th}((2n-1)K\mu(r'))^4)$$

$$= \sqrt{1-\prod_{n=1}^{\infty}\left(\mathrm{th}((2n-1)\mu(r)/K)^8\right)},$$

where $q = \exp(-2\mu(r))$.

The functional inequality

$$(1.6) \qquad \varphi_K(ab) \le \varphi_K(a)\varphi_K(b)$$

for $K \ge 1$, $a, b \in (0,1)$ was proved in [AVV1, 3.13]. Our next theorem gives a majorant for the right hand side of (1.6).

1.7. THEOREM. *Let* $K \ge 1$, $a, b \in (0,1)$. *Then*

$$(1) \qquad \varphi_K(ab) \le \varphi_K(a)\varphi_K(b) \le \varphi_K(\sqrt{ab})^2 \le \varphi_{K^2}(ab),$$

$$(2) \qquad \varphi_K(a)\varphi_K(b) \le \varphi_K(a^{1/K})\varphi_K(b)^{1/K} \le \varphi_{K^2}(ab),$$

with equality when $K = 1$.

1.8. THEOREM. *For* $K > 1$, $r \in (0,1)$ *the function*

$$f\colon (0,\infty) \to (r^{1/K}, 1)$$

defined by

$$f(p) = (\varphi_K(r^p))^{1/p},$$

is strictly decreasing. In particular,

$$\varphi_K(r^p) \le (\varphi_K(r))^p, \; p \ge 1,$$
$$\varphi_K(r^p) \ge (\varphi_K(r))^p, \; p \in (0,1].$$

Theorems 1.5, 1.7, and 1.8 together give interesting and perhaps new inequalities for infinite products. We shall give various applications of the function $\varphi_K(r)$ to quasiconformal mappings. By the quasiconformal counterpart of the Schwarz lemma [LV, p. 63] the function $\varphi_K(r)$—r measures the deviation of a K—quasiconformal automorphism of the unit disk from the identity map. For this function we obtain an explicit majorant in Section 3 and correct an error in an earlier result of the same kind in the literature.

The boundary correspondence of a quasiconformal mapping of the upper half-plane onto itself can be characterized as a homeomorphism of the real axis onto itself that satisfies the Beurling-Ahlfors ρ-condition [BA], [AH]. The ρ-condition is frequently used in the theory of Teichmüller spaces [L] and it has been extensively studied in its own right by W.K. Hayman and A. Hinkkanen [HH], [HI].

P. Tukia [T] showed recently that such homeomorphisms, often called quasisymmetric functions, can change the Hausdorff dimension of a set in a very peculiar way. We show here with a quantitative estimate that quasisymmetric functions approach linear maps when $\rho \to 1$. This result improves the qualitative estimate in [L, p. 32]. See also the interesting recent results of F.P. Gardiner and D.P. Sullivan in [GS]. Our results also complement and improve the earlier growth estimates of these maps in [HH] and [HI]. Our proof makes use of the quasiconformal extension of such a map and the function $\varphi_K(r)$. For integer values of K the function $\varphi_K(r)$ occurs also in number theory, namely in the study of modular equations [BB, pp. 102–109], [BE1], [BE2] and singular values associated with complete elliptic integrals. Bounds for the function $\varphi_K(r)$, such as those in Theorem 1.4, also yield bounds for such singular values. An example is the following corollary to Theorem 1.4 (2).

1.9. COROLLARY. *The pth singular value k_p (for definition see 3.18 below) admits the estimate*

$$(1 - 1/\sqrt{2})^{1/\sqrt{p}} < 1 - k_p < 8^{1-1/\sqrt{p}}(1 - 1/\sqrt{2})^{1/\sqrt{p}}$$

for $p = 1, 2, 3, \ldots$.

Some conjectures concerning the function $\varphi_K(r)$ are given at the end of Section 2.

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2. Proofs

The notation and terminology will be as in [LV]. The hyperbolic cosine and tangent and their inverse functions are denoted by ch, th, arch, and arth, respectively. The function $\mu(r)$, $0 < r < 1$, in (1.1) is given by [LV, (2.2), p. 60]

$$(2.1)\quad \mu(r) = \frac{\pi}{2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}, \qquad \mathscr{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

where

$$\mathscr{K}'(r) = \mathscr{K}(r'), \qquad r' = \sqrt{1-r^2}.$$

We shall need the following differentiation formula from [AVV1, 3.27]

$$(2.2)\qquad \frac{ds}{dr} = K\frac{s(s')^2\mathscr{K}^2(s)}{r(r')^2\mathscr{K}^2(r)},$$

where $s = \varphi_{1/K}(r)$, $K \in (0,\infty)$, $r \in (0,1)$, $r' = \sqrt{1-r^2}$, $s' = \sqrt{1-s^2}$.

2.3. *Proof of Theorem* 1.4. We need only prove the result for $K \in (1,\infty)$, since the other case follows by inversion. Differentiating and using (2.1) and (2.2), we obtain

$$K\cdot(1-r)^{1+1/K}\cdot f'(r) = (1-s)\left[1 - \frac{s(1+s)\mathscr{K}'^2(s)}{r(1+r)\mathscr{K}'^2(r)}\right],$$

which is positive since $t\mathscr{K}'^2(t)$ is an increasing function of t on $(0,1)$ and $0 < s < r < 1$ [AVV2, 2.2(3)]. Finally, it follows from [LV, p. 65] and [AVV1, (3.4)] that $\lim_{r\to 1} f(r) = f(1)$. □

Theorem 1.4 is related to the conjecture [LV, p. 68] that a K-quasiconformal automorphism of B^2 with $f(0) = 0$ satisfies

$$(*) \qquad |f(x) - f(y)| \le 16^{1-1/K}|x - y|^{1/K}.$$

For $K \ge 1$, $r \in (0,1)$ let g be the extremal K-quasiconformal mapping [LV, p. 65] with $gB^2 = B^2$, $g(0) = 0$, and $g(r) = \varphi_{1/K}(r)$, $g(1) = 1$. If $x = r$, $y = 1$, then

$$|g(x) - g(y)| = 1 - \varphi_{1/K}(r) < 8^{1-1/K}|1 - x|^{1/K}.$$

Hence in this case $(*)$ holds even with a smaller constant.

2.4. *Proof of Theorem* 1.7. (1) The first inequality is (1.6). To prove the second one we show that for a fixed $a \in (0,1)$ the function

$$f(x) = \log \varphi_K(a^2) + \log \varphi_K(x^2) - 2\log \varphi_K(ax)$$

is increasing on $(0, a)$ and decreasing on $(a, 1)$, so that $f(x) < f(a) = 0$ for all $x \in (0, a) \cup (a, 1)$. If we write $s = \varphi_K(x^2)$, $t = \varphi_K(ax)$, $u = \varphi_K(x)$ and use the differentiation formula in [AVV2, Lemma 2.1] we get

$$f'(x) = \frac{2}{xK}(g(x^2) - g(ax))$$

with $g(x) = (1 - u^2)\mathscr{K}(u)^2/((1 - x^2)\mathscr{K}(x)^2)$. Since $g(x)$ is decreasing by [AVV1, Lemma 3.12] it follows that $f'(x) > 0$ for $x \in (0, a)$ and $f'(x) < 0$ for $x \in (a, 1)$. The third inequality follows from (2).

(2) The proof of part (2) will follow from Theorem 2.22. □

In connection with the study of quasisymmetric functions of the real line and their extension to quasiconformal mappings of the plane [BA], the function

$$(2.5) \qquad \lambda(K) = \left[\frac{\varphi_K\left(\frac{1}{\sqrt{2}}\right)}{\varphi_{1/K}\left(\frac{1}{\sqrt{2}}\right)}\right]^2, \qquad K > 0,$$

has found many applications [L], [LV]. We now consider the following generalization of $\lambda(K)$:

$$\eta_K(x) = \lambda(K, r) = \left[\frac{\varphi_K(r)}{\varphi_{1/K}(r')}\right]^2, \tag{2.6}$$

for $0 < K < \infty$, $0 < r < 1$, $r' = \sqrt{1-r^2}$ and $x = (r/r')^2$. S. Agard [A] introduced this function in the study of quasisymmetric functions of the real axis.

2.7. THEOREM. (1) *For* $K \in [1, \infty)$, *and* $r \in [1/\sqrt{2}, 1)$, $r' = \sqrt{1-r^2}$,

$$\lambda(K)\left(\frac{r}{r'}\right)^2 \le \lambda(K, r) \le \lambda(K)\left(\frac{r}{r'}\right)^{2K},$$

with equality if and only if $K = 1$ *or* $r = 1/\sqrt{2}$.

(2) *For* $K \in [1, \infty)$, $r \in (0, 1)$, $r' = \sqrt{1-r^2}$,

$$\lambda(K, r) \ge K^4\left(\frac{r}{r'}\right)^2,$$

with equality if and only if $K = 1$.

(3) *For* $K \in [1, \infty)$, $r \in (0, 1)$ *let* $s = \varphi_K(r)$, $r' = \sqrt{1-r^2}$, *and* $f_1(r) = rs'/(r's)$.
Then f_1 *is decreasing from* $(1/\sqrt{2}, 1)$ *onto* $(0, 1/\sqrt{\lambda(K)})$.

(4) *For* $K \in [1, \infty)$, $t \in [1, \infty)$ *let* $r = \sqrt{t/(1+t)}$, $s = \varphi_K(r)$, *let*

$$f_2(t) = (\eta_K(t) - t)/t = \left(\frac{r's}{rs'}\right)^2 - 1,$$

Then f_2 *is increasing from* $(1, \infty)$ *onto* $(\lambda(K) - 1, \infty)$. *In particular*,

$$\eta_K(t) \ge \lambda(K)t.$$

(5) *For* $K \in [1, \infty)$, $t \in (0, \infty)$ *let* $f_3(t) = \eta_K(t)/t^K$. *Then* f_3 *is decreasing from* $(0, \infty)$ *onto* $(16^{K-1}, \infty)$. *In particular*,

$$\eta_K(t) \ge 16^{K-1}t^K,$$

for all $t \in (0, \infty)$.

Proof. Since the cases of equality in (1) and (2) are obvious, we only need to prove the strict inequalities. With $s = \varphi_K(r)$, using (2.1) and (2.2), we obtain

$$\frac{ds}{dr} = \frac{1}{K}\frac{s(s')^2\mathscr{K}^2(s)}{r(r')^2\mathscr{K}^2(r)} = K\frac{s(s')^2\mathscr{K}'^2(s)}{r(r')^2\mathscr{K}'^2(r)},$$

whence

$$\frac{ds}{dr} < K\frac{s(s')^2}{r(r')^2},$$

for $1 < K < \infty$, $0 < r < 1$. Now let $1/\sqrt{2} < r < 1$. Then $\mu^{-1}(\pi/2K) < s < 1$. Hence

$$\int_{\mu^{-1}(\frac{\pi}{2K})}^{s}\frac{dt}{t(t')^2} < K\int_{1/\sqrt{2}}^{r}\frac{dt}{t(t')^2},$$

which yields

$$\log\left(\frac{s}{s'}\right) - \frac{1}{2}\log\lambda(K) < K\log\left(\frac{r}{r'}\right),$$

so that

$$\frac{s}{s'} < \sqrt{\lambda(K)}\left(\frac{r}{r'}\right)^K,$$

and the upper bound in (1) follows.

Next, for the lower bound in (1), let $f(r) = (sr')/(s'r)$. Then differentiation gives

$$\begin{aligned} r(r')^2\frac{f'(r)}{f(r)} &= \frac{1}{K}\left[\frac{(s')^2\mathscr{K}^2(s)}{\mathscr{K}^2(r)} + \frac{s^2\mathscr{K}^2(s)}{\mathscr{K}^2(r)}\right] - 1 \\ &= \frac{\mathscr{K}(s)\mathscr{K}'(s)}{\mathscr{K}(r)\mathscr{K}'(r)} - 1 > 0, \end{aligned}$$

since $\mathscr{K}(x)\mathscr{K}'(x)$ is increasing on $[1/\sqrt{2}, 1]$ and $1/\sqrt{2} \le r < s < 1$ [AVV2, Theorem 2.2(8)]. Hence $f(r) \ge f(1/\sqrt{2}) = \sqrt{\lambda(K)}$, and the lower bound in (1) follows.

For (2), from the above argument, using the facts that $x'\mathscr{K}(x)^2$ is decreasing and $x\mathscr{K}'(x)^2$ is increasing on $(0, 1)$ [AVV2, Theorem 2.2 (3)], we get

$$K\left(\frac{s'}{r'}\right)^2 < \frac{ds}{dr} < \frac{1}{K}\frac{ss'}{rr'}.$$

Hence

$$\lambda(K, r) = \left(\frac{s}{s'}\right)^2 > K^4\left(\frac{r}{r'}\right)^2.$$

For (3) we obtain

$$\begin{aligned}
\frac{f_1'(r)}{f_1(r)} &= \frac{1}{r} + \frac{r}{(r')^2} - \left(\frac{s}{(s')^2} + \frac{1}{s}\right)\frac{ds}{dr} \\
&= \frac{1}{r(r')^2} - \frac{1}{s(s')^2}\frac{1}{K}\frac{s(s')^2\mathscr{K}(s)^2}{r(r')^2\mathscr{K}(r)^2} \\
&= \frac{1}{r(r')^2K}\left[K - \frac{\mathscr{K}(s)^2}{\mathscr{K}(r)^2}\right] \\
&= \frac{1}{r(r')^2K}\left[\frac{\mathscr{K}'(r)\mathscr{K}(s)}{\mathscr{K}(r)\mathscr{K}'(s)} - \frac{\mathscr{K}(s)^2}{\mathscr{K}(r)^2}\right] \\
&= \frac{\mathscr{K}(s)}{r(r')^2K\mathscr{K}'(s)\mathscr{K}(r)^2}[\mathscr{K}(r)\mathscr{K}'(r) - \mathscr{K}(s)\mathscr{K}'(s)].
\end{aligned}$$

Since $s \geq r$ it follows from [AVV2, 2.2 (8)] that $f_1'(r) < 0$ on $(1/\sqrt{2}, 1)$. The limiting values of f_1 are clear.

Part (4) follows easily from (3).

Finally, for the proof of part (5), let $g(r) = (r')^K s/(r^K s')$. Then

$$\begin{aligned}
\frac{g'(r)}{g(r)} &= \frac{1}{(s')^2 s}\frac{(s')^2 s\mathscr{K}(s)^2}{Kr(r')^2\mathscr{K}(r)^2} - \frac{K}{r(r')^2} \\
&\frac{1}{Kr(r')^2}\left[\frac{\mathscr{K}(s)^2}{\mathscr{K}(r)^2} - K^2\right] < 0.
\end{aligned}$$

Thus f_3 is decreasing from $(0, \infty)$ onto $(16^{K-1}, \infty)$. □

The second inequality (1) in Theorem 2.7 is reversed if $r \in (0, 1/\sqrt{2}]$. Moreover, since $\lambda(1/K, r')\lambda(K, r) = 1$, one can obtain an analog of 2.7 for $K \in (0, 1]$.

In terms of the notation $\tau(s) = \pi/\mu(1/\sqrt{1+s})$, $s > 0$, the function $\lambda(K, r)$ can also be written as

$$\lambda(K, r) = \frac{\varphi_K(r)^2}{1 - \varphi_K(r)^2} = \tau^{-1}\left(\frac{1}{K}\tau(s)\right), \qquad r = \sqrt{\frac{s}{1+s}}.$$

This alternate form of the function $\lambda(K,r)$ is sometimes used in the literature.

2.8. COROLLARY. *For $1 \le K < \infty$ we have*

$$(1) \qquad \lambda(K)s \le \tau^{-1}\left(\frac{1}{K}\tau(s)\right) \le \lambda(K)s^K \quad \text{for } 1 \le s < \infty,$$

$$(2) \qquad \lambda(K)(1/s) \le 1/\tau^{-1}(K\tau(s)) \le \lambda(K)(1/s)^K \quad \text{for } 0 < s \le 1,$$

$$(3) \qquad K^4 t \le \tau^{-1}\left(\frac{1}{K}\tau(t)\right) \quad \text{for } 0 < t < \infty.$$

Proof. The result follows immediately from Theorem 2.7. □

2.9. THEOREM. *For each $r \in (0,1)$, let $f\colon [1,\infty) \to R$ be defined by*

$$f(K) = \frac{\log\left(\lambda(K,r)(r'/r)^2\right)}{K-1}$$

if $K > 1$ and

$$f(1) = \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r).$$

Then f is strictly decreasing and

$$f[1,\infty) = \left(\frac{\pi\mathscr{K}(r)}{\mathscr{K}'(r)}, \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\right].$$

In particular, for $K \in [1,\infty)$, $r \in (0,1)$,

$$(2.10) \qquad \exp(\pi(K-1)\mathscr{K}(r)/\mathscr{K}'(r)) \le \lambda(K,r)\cdot\left(\frac{r'}{r}\right)^2 \le \exp\left(\frac{4(K-1)}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\right),$$

with equality if and only if $K = 1$.

Proof. From (2.1) and (2.2), with $s = \varphi_K(r)$, we obtain

$$\frac{ds}{dK} = \frac{4}{\pi^2 K^2}\mu(r)s(s')^2\mathscr{K}^2(s),$$

and

$$\frac{ds'}{dK} = -\frac{4}{\pi^2 K^2}\mu(r)s^2 s' \mathscr{K}^2(s).$$

Hence

$$\frac{d}{dK}\left(\log\left(\frac{s}{s'}\right)\right) = \frac{1}{s}\frac{ds}{dK} - \frac{1}{s'}\frac{ds'}{dK}$$
$$= \frac{4\mu(r)\mathscr{K}^2(s)}{\pi^2 K^2} = \frac{\mathscr{K}'^2(s)}{\mu(r)}.$$

Thus $(d/dK)(\log(s/s'))$ is positive and decreasing on $[1,\infty)$, hence $\log(\lambda(K,r))$ is increasing and concave on $[1,\infty)$. Hence by the Monotone l'Hopital's Rule [AVV3, Lemma 2.2] $f(K)$ is decreasing. Finally,

$$\lim_{K\to 1} f(K) = \lim_{s\to r}\frac{2\mathscr{K}'^2(s)}{\mu(r)} = \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r),$$

and

$$\lim_{K\to\infty} f(K) = \lim_{s\to 1}\frac{2\mathscr{K}'^2(s)}{\mu(r)} = \frac{\pi\mathscr{K}(r)}{\mathscr{K}'(r)}. \quad \square$$

2.11. COROLLARY (cf. [BA], [AVV1, Theorem 1.1]). *The function* $\log\lambda(K)/(K-1)$ *is strictly decreasing from* $(1,\infty)$ *onto* (π, a), *where* $a = (4/\pi)\mathscr{K}^2(1/\sqrt{2}) = \lambda'(1) = 4.37688\ldots$ *In particular,*

$$e^{\pi(K-1)} < \lambda(K) < e^{a(K-1)},$$

for $1 < K < \infty$.

Proof. Put $r = 1/\sqrt{2}$ in (2.10). $\square$

The constant a in Corollary 2.11 can also be written as $\Gamma(\frac{1}{4})^4/(4\pi^2)$. This constant occurs also in connection with the sharp form of the Schottky theorem due to J. Hempel [HA, p. 702].

2.12. THEOREM. *For each* $K \in (1,\infty)$, *let* $s = \varphi_K(r)$, $0 < r < 1$. *Then the function*

$$g(r) = \frac{s}{s'} - \frac{r}{r'}$$

is strictly increasing from $(0,1)$ *onto* $(0,\infty)$, *where* $r' = \sqrt{1-r^2}$, $s' = \sqrt{1-s^2}$.

Proof. Differentiation with respect to r yields

$$\begin{aligned} g'(r) &= \frac{1}{(s')^3 K} \frac{s(s')^2 \mathscr{K}^2(s)}{r(r')^2 \mathscr{K}^2(r)} - \frac{1}{(r')^3} \\ &= \frac{1}{s'(r')^2} \left[\frac{s\mathscr{K}(s)\mathscr{K}'(s)}{r\mathscr{K}(r)\mathscr{K}'(r)} - \frac{s'}{r'} \right] > 0, \end{aligned}$$

since $0 < r < s < 1$ and since the function $x\mathscr{K}'(x)$ is increasing on $(0,1)$ [AVV2, Theorem 2.2(3)]. Finally, the limiting values are obvious. □

2.13. Corollary. *For each* $K \in (1,\infty)$, *with notation as in* (2.6), *the function* $h(x) = \eta_K(x) - x$ *is strictly increasing from* $(0,\infty)$ *onto* $(0,\infty)$.

Proof. With $x = (r/r')^2$, $s = \varphi_K(r)$, we have

$$h(x) = \left[\frac{s}{s'} + \frac{r}{r'}\right]\left[\frac{s}{s'} - \frac{r}{r'}\right],$$

hence the result follows from Theorem 2.12. □

In a recent work [VVW] on quasiconformal maps in R^n, $n \geq 2$, certain special functions have found application to the geometric study of quasiconformal mappings. In the plane case these functions coincide with the one in Corollary 2.13.

2.14. *Proof of Theorem* 1.5. For the proof of part (1) we use Jacobi's product formula [J, p. 146]

$$k = 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^4$$

with $q = \exp(-\pi\mathscr{K}'(k)/\mathscr{K}(k)) = \exp(-2\mu(k))$ and $0 < k < 1$. If we write $y = \mu(k)$, this identity yields

$$\mu^{-1}(y) = 4\exp(-y) \prod_{n=1}^{\infty} \left(\frac{1+\exp(-4ny)}{1+\exp(-(4n-2)y)} \right)^4.$$

Part (1) follows on putting $y = \mu(r)/K$.

For the proof of part (2) we use another formula of Jacobi [J, p. 146], namely

$$k' = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 = \prod_{n=1}^{\infty}(\mathrm{th}((2n-1)\mu(k)))^4$$

so that with $\mu(k) = y$ we have

$$k' = \sqrt{1-(\mu^{-1}(y))^2} = \prod_{n=1}^{\infty}\mathrm{th}^4((2n-1)y).$$

Since $\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1$ both equalities in 1.5 (2) have been established. □

In view of the definition (1.1) of $\varphi_K(r)$ a remarkable feature about Theorem 1.5 is that the infinite products involve only the function $\mu(r)$ and not its inverse. Theorems 1.4, 1.5, and 1.7 together with (1.2), (1.3) yield an interesting series of sharp inequalities for infinite products. We can derive further functional inequalities for $\varphi_K(r)$ if we use the sharp inequality

$$\begin{aligned} &B(\varphi_{1/K}(r), \varphi_{1/K}(s)) \\ &\quad \le \varphi_K(B(r,s)) \le B(\varphi_K(r), \varphi_K(s)), B(r,s) = \frac{r+s}{1+rs}, \end{aligned}$$

for $K \ge 1, r, s \in (0,1)$, or other inequalities from [AVV1]. The composition property $\varphi_A(\varphi_B(r)) = \varphi_{AB}(r)$, $A, B > 0$, together with the fact that $\varphi_2(r) = 2\sqrt{r}/(1+r)$ enables one to evaluate $\varphi_{2^n}(r)$ by recursion. Thus for $K = 2^n$ we obtain explicit evaluations of the infinite products in 1.5(1) and (2) in terms of algebraic functions. We give such a formula for $n = 1$ at the end of Section 3 in 3.17.

Theorem 1.5 immediately yields

$$\sqrt{1-(\mathrm{th}(\mu(r)/K))^8} < \varphi_K(r) < (\mathrm{th}(K\mu(r')))^4 \tag{2.15}$$

for $K > 0$, $0 < r < 1$, which seems to be a new inequality. Note that (2.15) is not sharp when $K = 1$, whereas (1.3) is sharp in this case. Computer experiments together with (2.15) when $K = 1$ yield the approximate identity for $r \in (0,1)$

$$1 < (\mathrm{th}(\mu(r)))^8 + (\mathrm{th}(\mu(r')))^8 < 1.006.$$

For our next result we recall from [AVV3, Theorem 1.11(1)] that

$$\varphi_K(r) < \operatorname{th}\left(2\operatorname{arth}\left(2^{1-1/K}A(r)^{1/K}\right)\right), \qquad A(r) = \frac{r}{1+r'}, \tag{2.16}$$

for $K > 1$ and for $r \in (0, r_K)$ with $r_K = 2^{2-K}/(1+4^{1-K})$.

2.17. THEOREM. *For $K \in (1,\infty)$, $r \in (0,1)$,*

$$\varphi_K(r) < \operatorname{th}\left(K^2 4^{1-1/K}\operatorname{arth}(r^{1/K})\right), \tag{1}$$

$$\varphi_{1/K}(r) > \left(\operatorname{th}(K^{-2}4^{1/K-1}\operatorname{arth} r)\right)^K. \tag{2}$$

Proof. We need only prove (1) since (2) follows from (1) by inversion. Denoting $c(K) = K^2 4^{1-1/K}$, $s = \varphi_K(r)$, let

$$f(r) = \operatorname{arth}(s) - c(K)\operatorname{arth}(r^{1/K}).$$

Then $f(0) = 0$ and for $0 < r < 1$,

$$f'(r) = \frac{s\mathscr{K}^2(s)}{Kr(r')^2\mathscr{K}^2(r)} - \frac{c(K)r^{1/K}}{Kr(1-r^{2/K})} < 0$$

iff

$$c(K) > \frac{s}{r^{1/K}}K^2\frac{\mathscr{K}'^2(s)}{\mathscr{K}'^2(r)}\frac{1-r^{2/K}}{1-r^2},$$

which is true by (1.2). Thus f is strictly decreasing and $f(r) < f(0)$, hence (1) follows. □

2.18. *Proof of Theorem* 1.8. The limiting values follow from l'Hopital's rule. Let $s = f(p)$, $u = r^p$, $v = s^p$. Then $\mu(v) = \mu(u)/K$, $v > u$, and

$$\frac{dv}{dp} = \frac{v(v')^2\mathscr{K}^2(v)}{Ku(u')^2\mathscr{K}^2(u)}\frac{du}{dp}.$$

Now

$$\frac{du}{dp} = u\log r,$$

$$\frac{dv}{dp} = s^p\log s + ps^{p-1}\frac{ds}{dp} = v\left(\log s + \frac{p}{s}\frac{ds}{dp}\right).$$

Hence,

$$\log s + \frac{p}{s}\frac{ds}{dp} = \frac{(v')^2\mathscr{K}^2(v)\log r}{(u')^2\mathscr{K}^2(u)K} = (v')^2\mathscr{K}^2(v)\frac{\mu(v)\log r}{(u')^2\mathscr{K}^2(u)\mu(u)}$$
$$= \frac{(v')^2\mathscr{K}(v)\mathscr{K}'(v)\log r}{(u')^2\mathscr{K}(u)\mathscr{K}'(u)}.$$

Writing $m(r) = (2/\pi)(r')^2\mathscr{K}(r)\mathscr{K}'(r)$ this yields

$$\frac{p}{s}\frac{ds}{dp} = \frac{1}{m(u)}\left[m(u)\log\left(\frac{1}{s}\right) - m(v)\log\left(\frac{1}{r}\right)\right],$$

so that

$$\frac{p^2}{s}\frac{ds}{dp} = \frac{1}{m(u)}\left[m(u)\log\left(\frac{1}{v}\right) - m(v)\log\left(\frac{1}{u}\right)\right] < 0,$$

since $m(x)/\log(1/x)$ is increasing on $(0,1)$ [AVV2, Lemma 4.2 (2)]. □

2.19. *Conjectures.* For $K \in [1,\infty)$ and $r \in (0,1)$, the inequality

(2.20) $$\operatorname{th}(c(K)\operatorname{arth}(r^{1/K})) \geq \varphi_K(r) \geq \operatorname{th}(d(K)\operatorname{arth}(r^{1/K})),$$

where

$$c(K) = \max\{K, 4^{1-1/K}\},\ d(K) = \min\{K, 4^{1-1/K}\},$$

holds, with equality iff $K = 1$ or $K = 2$. The two particular cases $K = 1$ or $K = 2$ are clear with equality in (2.20) for all $r \in (0,1)$. Our computational experiments suggest that

(2.21) $$\varphi_K(r) \geq \operatorname{th}\left(2^{2-1/K}\operatorname{arth}\left(A(r)^{1/K}\right)\right),$$

for $K > 1$ and $r \in (0,1)$ where $A(r) = r/(1+r')$ is as in (2.16). If (2.21) indeed holds, then it is very close to the upper bound in (2.16).

2.22. THEOREM. *For $K, L \geq 1$, $a, b \in (0,1)$,*

(1) $$\varphi_K(a^{1/L})(\varphi_L(b))^{1/K} \leq \varphi_K(a^{1/L}\varphi_L(b)) \leq \varphi_K(\varphi_L(ab)) = \varphi_{KL}(ab),$$

(2) $$\prod_{j=1}^{n}\varphi_{K_j}(a_j) \leq \varphi_K(a),$$

where $a = a_1\ldots a_n$, $K = K_1\ldots K_n$, $a_i \in (0,1)$, $K_i \geq 1$.

Proof. The inequalities in (1) follow from the fact that $\varphi_K(a)/a^{1/K}$ is decreasing [H] while (2) follows by induction. □

The next theorem shows that the inequality (2.20) holds for certain values of K.

2.23. THEOREM. *For $p = 1, 2, \ldots, K = 2^p$, and for all $r \in [0, 1]$,*

$$\varphi_K(r) \le \operatorname{th}\big(K \operatorname{arth}(r^{1/K})\big). \tag{1}$$

For $K \ge 2$, and for all $r, u \in (0, 1)$ we have

$$\varphi_K(r) \le \varphi_{K^2}(r^2), \tag{2}$$

$$\varphi_K(r)\varphi_K(u) \le \varphi_{K^2}(ru)^2. \tag{3}$$

Proof. For (1) recall first that $\varphi_{AB}(r) = \varphi_A(\varphi_B(r))$ and that [LV, p. 64]

$$\varphi_2(r) = \frac{2\sqrt{r}}{1+r} = \operatorname{th}\big(2 \operatorname{arth}(\sqrt{r}\,)\big),$$

so that the result is true for $p = 1$. Next, suppose that the result holds for a certain integer p. Then

$$\begin{aligned}\varphi_{2^{p+1}}(r) &= \varphi_2(\varphi_{2^p}(r)) = \operatorname{th}\Big(2 \operatorname{arth}\Big(\sqrt{\varphi_{2^p}(r)}\,\Big)\Big) \le \operatorname{th}\big(2 \operatorname{arth}(\varphi_{2^p}(\sqrt{r}\,))\big)\\ &\le \operatorname{th}\big(2^{p+1} \operatorname{arth}(r^{2^{-p-1}})\big),\end{aligned}$$

as desired. In the second last step the inequality (1.6) was applied.

For (2), let $s = \varphi_K(r)$, $t = \varphi_{K^2}(r^2)$. Then $\mu(s) = \mu(r)/K$ and $\mu(t) = \mu(r^2)/K^2$. By [AVV2, 4.3 (4)] the function $\mu(r)/\log(1/r)$ in increasing, and thus $\mu(r^2) \le 2\mu(r)$. Hence

$$\mu(t) \le 2\mu(r)/K^2 = 2\mu(s)/K \le \mu(s),$$

and thus $t \ge s$ as desired.

For (3) from Theorem 1.7 and (2) we have

$$\varphi_K(r)\varphi_K(u) \le \varphi_K(\sqrt{ru}\,)^2 \le \varphi_{K^2}(ru)^2. \quad \square$$

2.24. *Remark.* From the proof of Theorem 1.5 one can read off various identities and inequalities for $\mu^{-1}(y)$ and $\mu(r)$. For instance we see that for $y > 0$,

$$\mu^{-1}(y) = \sqrt{1 - \prod_{n=1}^{\infty} \mathrm{th}^8((2n-1)y)}\,. \tag{2.25}$$

The identities

$$\mu(r)\mu(r') = \pi^2/4 \quad \text{and} \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \pi^2/2$$

imply that

$$(\mu^{-1}(y))^2 + \left(\mu^{-1}\left(\frac{\pi^2}{4y}\right)\right)^2 = 1, \tag{2.26}$$

$$\frac{1-\mu^{-1}(y)}{1+\mu^{-1}(y)} = \mu^{-1}\left(\frac{\pi^2}{2y}\right), \tag{2.27}$$

for $y > 0$. Now (2.25) and (2.26) yield the following Pythagorean type identity for $y > 0$,

$$\prod_{n=1}^{\infty} \mathrm{th}^8((2n-1)y) + \prod_{n=1}^{\infty} \mathrm{th}^8\left((2n-1)\frac{\pi^2}{4y}\right) = 1, \tag{2.28}$$

whereas (2.25) and (2.27) yield

$$\frac{\prod_{n=1}^{\infty} \mathrm{th}^8((2n-1)y)}{\left(1 + \prod_{n=1}^{\infty} \mathrm{th}^4\left((2n-1)\frac{\pi^2}{4y}\right)\right)^2} = \prod_{n=1}^{\infty} \mathrm{th}^4\left((2n-1)\frac{y}{2}\right). \tag{2.29}$$

We also get

$$\sqrt{1 - \mathrm{th}^8 y} < \mu^{-1}(y) < \mathrm{th}^4\left(\frac{\pi^2}{4y}\right), \tag{2.30}$$

and

$$(\mathrm{th}\,\mu(r))^4 > r'$$

which yields

$$\operatorname{arth}(r'^{1/4}) < \mu(r) < \frac{\pi^2}{4\operatorname{arth}(r^{1/4})}. \tag{2.31}$$

3. Some applications

3.1. COROLLARY. *Let f be a K-quasiconformal mapping of the unit disk B^2 onto itself with $f(0) = 0$ and let $x \in B^2$. Then*

$$\varphi_{1/K}(|x|) \le |f(x)| \le \varphi_K(|x|), \tag{1}$$

$$8^{1-K}(1 - |x|)^K \le 1 - |f(x)| \le 8^{1-1/K}(1 - |x|)^{1/K}. \tag{2}$$

Proof. Part (1) is the quasiconformal Schwarz lemma [LV, p. 64, (3.4)], while (2) follows from Theorem 1.4. □

Our next application deals with the function $\varphi_K(r) - r$, $K \in (1, \infty)$, $r \in (0, 1)$ for which P.P. Belinskii [B, p. 16, formula (19)] gives the inequality

(3.2)
$$\varphi_K(r) - r < r\left[\left(\frac{4}{r}\right)^{1-1/K} - 1\right] < \left(1 - \frac{1}{K}\right) r \log\frac{4}{r} < (K - 1) r \log\frac{4}{r}.$$

The first inequality in (3.2) follows from (1.2). But the second inequality is false, since by [LV, p. 65]

$$\lim_{r \to 0} \frac{\varphi_K(r)}{r^{1/K}} = 4^{1-1/K}.$$

We now obtain a corrected form of (3.2) which also sharpens a result of J. Zajac in [Z].

3.3. THEOREM. *For $K \in (1, \infty)$, $t \in (0, 1)$, and $t' = \sqrt{1 - t^2}$ we have*

(1)
$$\frac{K-1}{K} t^{(K+1)/(2K)} \log(1/t)$$
$$< \varphi_K(t) - t < \min\left\{t', \frac{1}{K}\right\}(K - 1)4^{(K-1)/K} t^{1/K} \log(4/t)$$

(2)
$$(K - 1) t^{(K+1)/2} \log(1/t) < t - \varphi_{\frac{1}{K}}(t) < (K - 1)4^{1-1/K} t (t')^{1/K} \cdot \log(4/t).$$

In particular, if $1 < K < (\log 8)/2$, *then*

$$\varphi_K(t) - t < (K-1)(\log 4)(4^{1-1/K})t' t^{1/(4K)}.$$

Proof. We only prove (1), since the proof of (2) is similar. Fix $t \in (0,1)$ and let $s = f(K) = \varphi_K(t)$. Then $\mu(s) = \mu(t)/K$ by (1.1), and by differentiating with respect to K we get

$$\frac{ds}{dK} = \frac{4\mu(t)s(s'\mathscr{K}(s))^2}{(\pi K)^2}.$$

By the mean value theorem there exists a number $K_0 \in (1, K)$ such that

$$s = f(K) = f(1) + (K-1)f'(K_0).$$

We now estimate $f'(K_0)$. We have

$$f'(K_0) = \frac{4\mu(t)s_0(s_0'\mathscr{K}(s_0))^2}{(\pi K_0)^2},$$

where $s_0 = f(K_0)$ and $s_0' = \sqrt{1-s_0^2}$. Note that $t < s_0 < s$.

Since $x'\mathscr{K}(x)$ is decreasing [AVV2, Theorem 2.2 (3)], we get

$$f'(K_0) \le \frac{4\mu(t)s(t'\mathscr{K}(t))^2}{\pi^2} = \frac{4(\pi/2)\mathscr{K}'(t)s(t'\mathscr{K}(t))^2}{\pi^2\mathscr{K}(t)} = sm(t),$$

say. Then

$$s - t = \varphi_K(t) - t = (K-1)f'(K_0) < (K-1)sm(t) < (K-1)st'\log\frac{4}{t}$$

by [AVV2, Lemma 4.2 (5)]. By (1.2) $s < (4^{1-(1/K)})t^{1/K}$, hence the first upper bound in (1) follows.

Now for the second upper bound in (1), we have

$$\begin{aligned}
\varphi_K(t) - t &< 4^{(K-1)/K}t^{1/K} - t \\
&= 4^{(K-1)/K}t^{1/K}\left(1 - \left(\frac{t}{4}\right)^{(K-1)/K}\right) \\
&< 4^{(K-1)/K}t^{1/K}\log\left(\frac{4}{t}\right)^{(K-1)/K} \\
&= 4^{(K-1)/K}\frac{K-1}{K}t^{1/K}\log\left(\frac{4}{t}\right).
\end{aligned}$$

Next, for the lower bound in (1) we have

$$\begin{aligned}\varphi_K(t) - t > t^{1/K} - t &= t^{1/K}(1 - t^{1-1/K}) \\ &> t^{1/K} t^{(K-1)/(2K)} \frac{K-1}{K} \log\left(\frac{1}{t}\right) \\ &= \frac{K-1}{K} t^{(K+1)/(2K)} \log\left(\frac{1}{t}\right).\end{aligned}$$

Finally, if $1 < K < (\log 8)/2$, then $h(t) = (t^{3/(4K)})\log(4/t)$ is increasing on $(0, 1]$, so that $h(t) < \log 4$, and the result follows. □

In [B, p. 80, Lemma 12], P.P. Belinskii gives the inequality

$$\lambda(K) < 1 + 12(K-1) \tag{3.4}$$

for $K > 1$ close to 1. Because the incorrect part of the inequality (3.2) was used in the proof of (3.4) the proof given in [B, pp. 80–82] for (3.4) is not valid. We observe that Corollary 2.11 yields the following improved form of (3.4).

3.5. Corollary. *For all* $K \in (1, \infty)$,

$$1 + \pi(K-1) < \lambda(K) < 1 + a(K-1)\exp(a(K-1)) \tag{3.6}$$

where $a = 4.37688\ldots$ *is as in Corollary* 2.11. *In particular, if* $K \in (1, 1 + 1/(2a))$, *then*

$$\lambda(K) < 1 + 8(K-1). \tag{3.7}$$

3.8. Theorem. *For* $t \in (0,1)$, $t' = \sqrt{1-t^2}$, *and* $K \in (1, \infty)$, *we have*

$$\varphi_K(t) - t < (t')^2 \operatorname{th}((K-1)\mu(t')) < (K-1)(t')^2 \log(4/t').$$

Proof. The first inequality follows immediately from the upper bound in (1.3). Since $(\operatorname{th} x)/x$ is decreasing on $(0, \infty)$, the second inequality is a consequence of the well known property [LV, p. 64, (2.10)] that $\mu(t') < \log(4/t')$. A slightly different final estimate follows if we use the inequalities in Remark 2.24. □

3.9. *The* ρ*-condition.* In [BA] Beurling and Ahlfors characterized the boundary correspondence of quasiconformal automorphisms of the upper half plane H^2 as those homeomorphisms $f\colon R \to R$ that satisfy the *ρ-condition*

$$\frac{1}{\rho} \le \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \le \rho \tag{3.10}$$

for all $x \in R$, $t > 0$ and for some constant $\rho \in (1, \infty)$. The boundary values of a K-quasiconformal map satisfy (3.10) with the constant $\rho = \lambda(K)$ which is sharp for each $K \ge 1$ [L, p. 34]. For each $\rho \ge 1$ let us denote by $K(\rho)$ the smallest constant K such that each homeomorphism $f\colon R \to R$ satisfying (3.10) has an extension to a K-quasiconformal mapping of the whole plane R^2 which agrees with f on the real axis. It is well-known by [BA], [AH] and by later results of M. Lehtinen [L, p. 34] that

$$K(\rho) \le \min\{2\rho - 1, \rho^{3/2}\}.$$

It seems to be an open problem whether this inequality is sharp for any $\rho > 1$.

W.K. Hayman and A. Hinkkanen have extensively studied functions satisfying (3.10) independent of quasiconformal extension [HH], [HI]. They obtained sharp bounds for the growth of a function satisfying (3.10) and normalized by the conditions $f(0) = 0$, $f(1) = 1$. That these conditions are mere normalizations follow from the fact that along with f also $h \circ f \circ g$ satisfies (3.10) with the same ρ whenever h, g are similarity maps.

Alternatively, growth estimates for the functions satisfying (3.10) can also be derived by using quasiconformal extension together with the result of Agard [A] that a K-quasiconformal map $f\colon R^2 \to R^2$ satisfies

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \frac{\varphi_K\left(\sqrt{\frac{t}{1+t}}\right)^2}{\varphi_{1/K}\left(1/\sqrt{1+t}\right)^2} = \lambda\left(K, \sqrt{\frac{t}{1+t}}\right) \tag{3.11}$$

for all distinct $x, y, z \in R^2$ with $t = |x-y|/|x-z|$. From (3.11) it also follows that

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \ge \lambda\left(1/K, \sqrt{\frac{t}{1+t}}\right).$$

3.12. THEOREM. *Let $f\colon R \to R$ be a homeomorphism satisfying the ρ-condition* (3.10) *and let $K = K(\rho)$. If $f(0) = 0$ and $f(1) = 1$ then for $y > 2$,*

$$f(y) - 1 \le \lambda\left(K, \sqrt{(y-1)/y}\right) \le \lambda(K)(y-1)^K.$$

Proof. Apply (3.11) with $z = 0$, $x = 1$, and use Theorem 2.7. □

As pointed out in [L, p. 32] it follows easily by normal family considerations that normalized functions satisfying (3.10) approach the identity when $\rho \to 1$. We now show that a quantitative majorant for the "speed" of convergence can be found if we use the Beurling-Ahlfors extension, the proof of 3.12 and Corollary 2.13.

3.13. THEOREM. *Let $h\colon \overline{R} \to \overline{R}$ be a homeomorphism satisfying the ρ-condition* (3.10) *normalized by* $h(0) = 0$, $h(1) = 1$, *and* $h(\infty) = \infty$. *Then with* $K = K(\rho)$ *we have*

$$(1)\quad (1-K)4^{1-1/K}x\sqrt{x-1}\log(16x) \le h(x) - x \le (K-1)4^{2K-1-1/K}x^{K-1/(2K)}(x-1)^{1/(2K)}\log(16x)$$

for all $x \ge 1$ *and*

$$(2)\quad (1-K)4^{1-1/K}x(1-x)^{1/(2K)}\log(16/x) \le \varphi_{1/K}(\sqrt{x})^2 - x \le h(x) - x \le \varphi_K(\sqrt{x})^2 - x \le (K-1)4^{1-1/K}x^{1/(2K)}\sqrt{1-x}\log(16/x)$$

for $x \in (0,1)$ *and*

$$(3)\quad \frac{4x}{\pi}(K-1)\mathscr{K}\left(\sqrt{|x|/(1+|x|)}\right)\mathscr{K}\left(1/\sqrt{1+|x|}\right) \le x - h(x) \le \frac{4(K-1)}{\pi}(16(|x|+1))^{K-1}\mathscr{K}\left(\sqrt{|x|/(1+|x|)}\right) \times \mathscr{K}\left(1/\sqrt{1+|x|}\right)|x|^{1/K}$$

for $x < 0$.

Proof. (1) From the definition of $K(\rho)$ it follows that f has an extension to a K-quasiconformal mapping of R^2. Using the fact that modulus of a curve family is quasi-invariant under a quasiconformal mapping applied to the family of all curves joining the segment $[0,1]$ with the set $[x,\infty)$ we get

$$\frac{1}{K}\tau(x-1) \le \tau(h(x)-1) \le K\tau(x-1),$$

$$1 + \tau^{-1}(K\tau(x-1)) \le h(x) \le 1 + \tau^{-1}\left(\frac{1}{K}\tau(x-1)\right).$$

Thus

$$1/\varphi_K^2(1/\sqrt{x}) \le h(x) \le 1/\varphi_{1/K}^2(1/\sqrt{x}),$$
$$1/\varphi_K^2(r) - r^{-2} \le h(x) - x \le 1/\varphi_{1/K}^2(r) - r^{-2},$$

where $r^2 = 1/x$, $0 < r < 1$. Now we estimate both the bounds by using Theorem 3.3. First,

$$\begin{aligned}1/\varphi_{1/K}^2(r) - r^{-2} &= \frac{1}{\varphi_{1/K}^2(r)r^2}(r - \varphi_{1/K}(r))(r + \varphi_{1/K}(r))\\ &\le \frac{2r}{16^{1-K}r^{2K+2}}(K-1)4^{1-1/K}r(r')^{1/K}\log(4/r)\\ &= (K-1)4^{2K-1-1/K}x^{K-1/(2K)}(x-1)^{1/(2K)}\log(16x).\end{aligned}$$

Next,

$$\begin{aligned}r^{-2} - 1/\varphi_K^2(r) &= \frac{1}{\varphi_K^2(r)r^2}(\varphi_K(r) - r)(\varphi_K(r) + r)\\ &\le \frac{2(K-1)}{r^{2+2/K}}4^{1-1/K}r'r^{1/K}\log(4/r)\\ &= (K-1)4^{1-1/K}\sqrt{x-1}\,x^{(K+1)/2K}\log(16x)\\ &\le (K-1)4^{1-1/K}x\sqrt{x-1}\log(16x).\end{aligned}$$

(2) The proof of (2) is similar to the proof of (1) and the details are omitted.

(3) For (3) we argue as in (1) but with the curve family joining $[x, 0]$ with $[1, \infty)$. Thus we get the following equivalent inequalities

$$\frac{1}{K}\tau\left(\frac{1}{|x|}\right) \le \tau\left(\frac{1}{|h(x)|}\right) \le K\tau\left(\frac{1}{|x|}\right)$$

$$\frac{1}{\tau^{-1}\left(\frac{1}{K}\tau\left(\frac{1}{|x|}\right)\right)} \le h(x) \le \frac{1}{\tau^{-1}\left(K\tau\left(\frac{1}{|x|}\right)\right)}$$

$$\frac{\varphi_{1/K}(r)^2}{\varphi_K(r')^2} \le |h(x)| \le \frac{\varphi_K(r)^2}{\varphi_{1/K}(r')^2}$$

with $r = \sqrt{|x|/(1+|x|)}$. The last inequality can be rewritten as

$$\lambda\left(\frac{1}{K}, r\right) \le |h(x)| \le \lambda(K, r). \tag{3.14}$$

To find more explicit estimates let $f(K) = \lambda(K, r) = (s/s')^2$ and $g(K) = \lambda(1/K, r) = (t/t')^2$, where $s = \varphi_K(r)$, $t = \varphi_{1/K}(r)$. Then

$$\begin{aligned}\frac{f'(K)}{f(K)} &= 2\left[\frac{1}{s}\frac{ds}{dK} + \frac{s}{(s')^2}\frac{ds}{dK}\right] \\ &= \frac{2}{s(s')^2}\frac{ds}{dK} = \frac{2}{s(s')^2}\left(\frac{4}{\pi^2}\right)\frac{s(s')^2\mathscr{K}(s)^2}{K^2}\mu(r) \\ &= \frac{8}{\pi^2K^2}\mathscr{K}(s)^2\mu(r) \le \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r),\end{aligned} \tag{3.15}$$

since $\mathscr{K}'(s) \le \mathscr{K}'(r)$. Next,

$$\begin{aligned}\frac{g'(K)}{g(K)} &= 2\left[\frac{1}{t}\frac{dt}{dK} + \frac{t}{(t')^2}\frac{dt}{dK}\right] = \frac{2}{t(t')^2}\frac{dt}{dK} \\ &= -\frac{8}{\pi^2}\mathscr{K}(t)^2\mu(r) \ge -\frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r).\end{aligned} \tag{3.16}$$

Hence by (3.15)

$$f'(K) \le \frac{4}{\pi}f(K)\mathscr{K}(r)\mathscr{K}'(r) \le 16^{K-1/K}\frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2/K}}{(r')^{2K}}.$$

Then by (3.16)

$$-g'(K) \le \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)g(K) \le \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2K}}{(r')^{2/K}}.$$

By the mean value theorem there exists $K_1 \in (1, K)$ such that

$$\begin{aligned}\lambda(K, r) - (r/r')^2 &= f(K) - f(1) = (K-1)f'(K_1) \\ &\le \frac{4(K-1)}{\pi}16^{K_1-1/K_1}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2/K_1}}{(r')^{2K_1}} \\ &\le \frac{4(K-1)}{\pi}16^{K-1/K}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2/K}}{(r')^{2K}}.\end{aligned}$$

Similarly, there exists $K_2 \in (1, K)$ such that

$$\begin{aligned}\lambda(1/K, r) - (r/r')^2 = g(K) - g(1) &= (K-1)g'(K_2)\\ &\geq -(K-1)\frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2K_2}}{(r')^{2/K_2}}\\ &\geq -(K-1)\frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^2}{(r')^2}.\end{aligned}$$

Thus we get

$$\begin{aligned}&-\frac{4}{\pi}(K-1)\mathscr{K}(r)\mathscr{K}'(r)\frac{r^2}{(r')^2}\\ &\quad\leq |h(x)| - |x| \leq \frac{4(K-1)}{\pi}16^{K-1/K}\mathscr{K}(r)\mathscr{K}'(r)\frac{r^{2/K}}{(r')^{2K}}.\end{aligned}$$

That is,

$$\begin{aligned}x - h(x) &\leq \frac{4(K-1)}{\pi}16^{K-1/K}\mathscr{K}\left(\sqrt{|x|/(1+|x|)}\right)\mathscr{K}\left(1/\sqrt{1+|x|}\right)\\ &\quad\times(|x|+1)^{K-1/K}|x|^{1/K}\\ &= \frac{4(K-1)}{\pi}(16(1+|x|))^{K-1/K}\mathscr{K}\left(\sqrt{|x|/(1+|x|)}\right)\\ &\quad\times\mathscr{K}\left(1/\sqrt{1+|x|}\right)|x|^{1/K}\end{aligned}$$

and

$$x - h(x) \geq \frac{4x(K-1)}{\pi}\mathscr{K}\left(\sqrt{|x|/(1+|x|)}\right)\mathscr{K}\left(1/\sqrt{1+|x|}\right),$$

for all $x < 0$. □

3.17. *Modular equations.* For integer values of p, solutions of equations of the form

$$\frac{\mathscr{K}'(r)}{\mathscr{K}(r)} = p\frac{\mathscr{K}'(s)}{\mathscr{K}(s)},$$

are given by the function $\varphi_{1/p}(s) = r$. This equation is called *the modular equation* of degree p. It has been extensively studied in number theory [BE1], [BE2], [BB, pp. 103–109]. S. Ramanujan and others have found dozens of identities satisfied by the solutions of the modular equation of degree p with $p = 3, 5, 7, 13, 17$ [BE1], [BE2], [BB]. Thus for instance the function $\varphi_3(r)$ satisfies the following identity for $r \in (0,1)$, $s = \varphi_3(r)$ [BB, pp. 103–109]:

$$\sqrt{r\,s} + \sqrt{r'\,s'} = 1.$$

We can solve this equation for $\varphi_3(r)$ if we use a symbolic computation program, and thus we can obtain formulas for $\varphi_{3^n}(r)$, $n = 1, 2, \ldots$. Further, using this formula together with the relations

$$\varphi_2(r) = \frac{2\sqrt{r}}{1+r}, \varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1$$

for $K > 0$, $0 < r < 1$, one can obtain formulas for instance for $\varphi_{3/2}(r)$, $\varphi_6(r)$. This last relation together with 1.5 (2) yields, for $r \in (0,1)$,

$$\frac{2\sqrt{r}}{1+r} = \prod_{n=1}^{\infty} (\operatorname{th}((4n-2)\mu(r')))^4.$$

3.18. *Singular values.* Given a positive integer $p = 1, 2, \ldots$ there exists a unique number $k_p \in (0,1)$ such that

$$\mu(k_p) = \frac{\pi \mathscr{K}'(k_p)}{2\mathscr{K}(k_p)} = \frac{\pi}{2}\sqrt{p}\,.$$

The number k_p is called the pth *singular value* (or also *singular modulus*) [BB, p. 139, 296]. A. Selberg and S. Chowla have proved [SC] that for several values of p, $\mathscr{K}(k_p)$ can be expressed in terms of the Euler Γ-function. Since $\mu(1/\sqrt{2}) = \pi/2$ we see that

$$k_p = \mu^{-1}\left(\sqrt{p}\,\mu(1/\sqrt{2})\right) = \varphi_{1/\sqrt{p}}(1/\sqrt{2})$$

and thus further by (2.5) we get

$$\lambda\left(1/\sqrt{p}\right) = \frac{k_p^2}{1-k_p^2} = 1/\lambda\left(\sqrt{p}\right) \tag{3.19}$$

for $p = 1, 2, \ldots$ The known values of k_p, $p = 1, 2, \ldots, 9$ [BB, p. 139] yield by (3.19) algebraic expressions for $\lambda(\sqrt{p})$, $p = 1, 2, \ldots, 9$. The numbers $\lambda(p)$ play also a crucial role in the multimillion decimal place calculation of π [BBB, Theorem 3, (5.13)] and [BB, Ch. 5].

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