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A GEOMETRIC HEAT FLOW FOR ONE-FORMS ON THREE DIMENSIONAL MANIFOLDS

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1. Introduction

In this paper, we introduce a geometrically motivated heat flow for one-forms on 3-manifolds. Throughout, $(\mathbf{M}^3, g)$ is a Riemannian, compact, orientable 3-manifold and

$$\Omega_s^1(\mathbf{M}^3) \overset{\text{def}}{=} \left\{\alpha \in \Omega^1(\mathbf{M}^3) \middle| \, |\alpha|^2 = 1\right\}.$$

In §3 we prove:

1.1. THEOREM. *Let $\alpha \in \Omega_s^1(\mathbf{M}^3)$, $\beta \in \Omega^1(\mathbf{M}^3) \times \mathbf{R}^+$. The weakly parabolic system*

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df); \quad f \overset{\text{def}}{=} *(\alpha \wedge d\beta + \beta \wedge d\alpha),$$
$$\beta(\cdot, 0) = \alpha(\cdot) \tag{1.1}$$

has a unique, smooth solution for $t \in [0, \infty)$.

The evolution for the function f is also weakly parabolic and has the form

$$\frac{\partial f}{\partial t} = \Delta_\alpha f + \nabla_X f \tag{1.2}$$

where Δ_α is essentially the Laplacian on the null space of α and $X \in \mathfrak{X}(\mathbf{M}^3)$ is a smooth, time independent vector field. Let $d_\alpha(p,q)$ be the distance between $p, q \in \mathbf{M}^3$ restricted to the null space of α (see (2.1)). In §4 we prove a version of the strong maximum principle:

1.2. THEOREM. *Let f be a solution to* (1.2) *on* $\mathbf{M}^3 \times [0, T]$. *If $f(\cdot, 0) \geq 0$ and if $\exists q \in \mathbf{M}^3$ such that $f(q, 0) > 0$, then $f(p,t) > 0$ for all $t \in (0, T]$ and for all p such that $d_\alpha(p,q) < \infty$.*

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Let $\phi \in \Omega^1_s(\mathbf{M}^3)$ with $\Phi = \text{null}(\phi) \subset \mathbf{TM}^3$. If $(\phi \wedge d\phi)(p) = 0\ \forall p \in U \subset \mathbf{M}^3$, then ϕ is said to be a *foliation form* in U. The Frobenius integrability theorem asserts that the distribution Φ is integrable in U. The antithesis of foliation forms are contact forms. If $(\phi \wedge d\phi)(p) \neq 0$ at $p \in \mathbf{M}^3$, then the hyperplane distribution Φ may not be integrated near p to give submanifolds of $\mathbf{M}^3$ and ϕ is said to be a *contact form* at p. See [Ar], [B] for a more detailed treatment.

By studying the highest order term in (1.2), one may see that where α defines a foliation, $f(\cdot, t)$ diffuses along the leaves of $A = \text{null}(A)$. Where α is contact, $f(\cdot, t)$ propagates transversally to A by flowing out along circular, integral curves of A. Note that $f(\cdot, 0)$ is a measure of the non-integrability of α.

For convenience, we make the following definition.

1.3. Definition. The space $\mathfrak{C} \subset \Omega^1_s(\mathbf{M}^3)$ of "conductive one-forms" is the set of $\alpha \in \Omega^1(\mathbf{M}^3)$ satisfying:

(i) (weakly contact[1]) $*(\alpha \wedge d\alpha) \geq 0$;

(ii) (heat conductor) $\forall p \in \mathbf{M}^3$, $\exists q \in \mathbf{M}^3$ such that $*(\alpha \wedge d\alpha)(q) > 0$ and $d_\alpha(p, q) < \infty$.

It is not difficult to construct an element of $\mathfrak{C}$ which is not strictly contact (§5).

1.4. Theorem. *$\mathfrak{C}$ is a non-empty set for every compact, orientable 3-manifold.*

Then, for $\varepsilon \in R^+$ we consider the small perturbation of α given by

$$\eta(\cdot, t) = \frac{\alpha(\cdot) + \varepsilon\beta(\cdot, t)}{|\alpha(\cdot) + \varepsilon\beta(\cdot, t)|}. \tag{1.3}$$

A computation gives

$$(\eta \wedge d\eta)(\cdot, t) = \frac{\alpha \wedge d\alpha + \varepsilon(\alpha \wedge d\beta + \beta \wedge d\alpha)(\cdot, t) + \varepsilon^2(\beta \wedge d\beta)(\cdot, t)}{|\alpha(\cdot) + \varepsilon\beta(\cdot, t)|^2}. \tag{1.4}$$

The middle term of (1.4) is precisely our function f. We do not attempt to control the quantity $*(\beta \wedge d\beta)$ during the evolution. Though seemingly an

[1]It has been brought to our attention that such an α may be referred to as a "confoliation".

anathema, we observe that this term has an extra ε in front of it. Then, the above theorems have the following immediate consequence.

1.5. COROLLARY (EXISTENCE OF CONTACT FORMS). *Let* $\alpha \in \mathfrak{C}$ *and* η *be as above. Then,* $\forall t_0 > 0$, $\exists \varepsilon = \varepsilon(t_0) > 0$ *making* $\eta(\cdot, t_0)$ *strictly contact, i.e.,* $*(\eta \wedge d\eta)(\cdot, t_0) > 0$.

In three dimensions, the topological obstructions to finding contact forms vanish. The existence of a contact one-form was first demonstrated by Lutz and Martinet using a surgery decomposition for three-manifolds [L], [M]. Subsequently, a much shorter proof was given by Thurston and Winkelnkemper using the open book decomposition of three-manifolds [TW] (see also [Gn], [Gz1]).

A further application of the flow was kindly pointed out to us by V. L. Ginzburg.

1.6. COROLLARY ([Gz2]). *Consider the direct product* $\Sigma_g \times S^1$, *where* Σ_g *is a surface of genus* $g > 0$, *foliated by fibers of the projection* $p\colon \Sigma_g \times S^1 \to S^1$. *Then* $\forall r$ *there exists a contact structure* C^r*-close to this foliation.*

Proof. For Σ_1 one may use $\eta = \sin(z)\,dx + \cos(z)\,dy + K\,dz$ where $K \in R^+$ is a large constant. Martinet's lemma (see [Gn]) allows us to "flatten" the contact structure along a section S^1 of p. Thus we obtain a contact structure on the product $H \times S^1$ of the handle $H = T^2 \setminus D^2$ and S^1, which degenerates near the boundary becoming a foliation. For $g > 1$, we may insert $H \times S^1$ into $\Sigma_g \times S^1$ and use the contact structure on $H \times S^1$ as a heat source. Theorem 1.2 shows that $\Sigma_g \times S^1$ becomes contact instantaneously. Q.E.D.

Finally, we mention that geometric heat flows associated to either strictly foliated or strictly contact manifolds have been studied. See, for example, the work of [NRT], [CL], [CH]. In contrast to these flows, our evolution equation allows for arbitrary integrability conditions in the directions of diffusion. Our flow is somewhat similar, in spirit, to the Yang-Mills heat flow for connections on circle bundles over surfaces where f is viewed as a measure of curvature.

2. Notation

The set of smooth vector fields on $\mathbf{M}^3$ will be denoted by $\mathfrak{X}(\mathbf{M}^3)$ and the set of smooth p-forms will be written as $\Omega^p(\mathbf{M}^3)$. As above, $\Omega^1_s(\mathbf{M}^3) \overset{\text{def}}{=} \{\alpha \in \Omega^1(\mathbf{M}^3) \mid |\alpha|^2 = 1\}$.

In local coordinates $\{x^i\}$, we denote the metric by $g = g_{ij}\, dx^i \otimes dx^j$ and the volume form by

$$\mu = \tfrac{1}{6}\mu_{ijk}\, dx^i \wedge dx^j \wedge dx^k.$$

For a tensor $T = T^i_{jk}$, we denote its length by $|T|^2 = T^i_{jk} T^p_{qr} g_{ip} g^{jq} g^{kr}$. When convenient, we conserve notation by using the extended Einstein summation convention. For example: $T_{ij}U_j = T_{ij}U_k g^{jk}$.

For $\alpha \in \Omega^1(\mathbf{M}^3)$ and $\alpha \neq 0$, we define the "plane field metric" and the "plane field Laplacian" to be

$$\gamma_{ij} \stackrel{\text{def}}{=} g_{ij} - \alpha_i \alpha_j \quad \text{and} \quad \Delta_\alpha \stackrel{\text{def}}{=} \gamma_{ij} \nabla_i \nabla_j. \tag{2.1}$$

We denote by $d(p, q)$ the usual distance between two points. Given a nonsingular $\alpha \in \Omega^1(\mathbf{M}^3)$, the distance $d_\alpha(p, q)$ along null(α) is defined to be

$$d_\alpha(p,q) \stackrel{\text{def}}{=} \begin{cases} \inf\left\{\text{length}(\gamma) | \gamma(s) \colon [0,1] \to \mathbf{M}^3,\ \gamma(0) = p,\ \gamma(1) = q,\ \alpha\left(\frac{\partial \gamma}{\partial s}\right) = 0 \right. \\ \infty \quad \text{if no integral curve exists.} \end{cases} \tag{2.2}$$

All space derivatives are with respect to the Levi-Cevita connection, hence $\nabla_p g_{ij} = 0$ and $\nabla_p \mu_{ijk} = 0$. When appropriate, subscripts will be used to denote differentiation. For example, a_r is used to denote differentiation in the radial direction for the function a. The Lie derivative of a volume form μ is given by $\mathfrak{L}_V \mu = d \cdot i_V \mu + i_V \cdot d\mu = d \cdot i_V \mu$ where $V \in \mathfrak{X}(\mathbf{M}^3)$ and i_V is the interior product.

The k-norm of a time dependent tensor T will be defined to be

$$\|T(\cdot, t)\|_k \stackrel{\text{def}}{=} \sup_{p \in \mathbf{M}^3} |\nabla^k T|^2(\cdot, t) \tag{2.3}$$

where the norm of the kth repeated derivative is given by

$$|\nabla^k T|^2 = \langle \underbrace{\nabla \cdots \nabla}_{k \text{ times}} T, \underbrace{\nabla \cdots \nabla}_{k \text{ times}} T \rangle.$$

3. The flow

The Cross Term Energy. Let g be a Riemannian metric with volume form μ on $\mathbf{M}^3$. A relative energy of a one-form $\beta \in \Omega^1(\mathbf{M}^3)$ with respect to a fixed one-form $\alpha \in \Omega^1(\mathbf{M}^3)$ may be given by

$$E(\alpha,\beta) = \int f^2 \mu \quad \text{where } f = *(\alpha \wedge d\beta + \beta \wedge d\alpha). \tag{3.1}$$

Notice that f represents the cross-terms in (1.4) produced when computing $\eta \wedge d\eta$ with $\eta = \alpha + \beta$. This type of energy is analogous to the Dirichlet energy functional.

3.1. Proposition. *The path of steepest descent for $E(\alpha,\beta)$ is given by*

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df) - 2f * d\alpha, \qquad \beta(\cdot, 0) = \alpha(\cdot) \in C^\infty. \tag{3.2}$$

Proof. The first variation of the energy is:

$$E'(\alpha,\beta) = 2\int f(\alpha \wedge d\beta' + \beta' \wedge d\alpha)$$

$$= 2\int (\beta' \wedge d(f\alpha) + \beta' \wedge f d\alpha).$$

We have made use of the identity $d(\alpha \wedge \beta') = d\alpha \wedge \beta' - \alpha \wedge d\beta'$. Expanding gives

$$E'(\alpha,\beta) = 2\int (\beta' \wedge df \wedge \alpha + 2\beta' \wedge f d\alpha).$$

In dimension 3, for a p-form, $*^2 = (-1)^{p(3-p)}$ so $*^2 = +1$. Using the metric, if $\tau \in \Omega^1(\mathbf{M}^3)$ and $\sigma \in \Omega^2(\mathbf{M}^3)$, then $*(\sigma \wedge \tau) = \langle \sigma, *\tau \rangle = \langle *\sigma, \tau \rangle$. Hence, the fastest descent for the energy is given by

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df) - 2f * d\alpha. \quad \text{Q.E.D.} \tag{3.3}$$

We will refer to the flow given in (3.2) as the *cross term flow*. Fixed points of the cross term flow satisfy $\alpha \wedge df = 2f d\alpha$. Wedging this with α tells us that $f\alpha \wedge d\alpha = 0$. Thus, if α is contact, f is identically zero.

Evolution Equations.

3.2. Definition. The highest order term of (3.2) will be referred to as the *contact flow*:

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df); \qquad f \overset{\text{def}}{=} *(\alpha \wedge d\beta + \beta \wedge d\alpha). \tag{3.4}$$

This simpler flow induces a beautiful evolution for f which contains no zero-order terms. For this reason, we will choose to carry our computations on this simpler flow. Actually, all of the subsequent results which we will prove about this flow essentially carry over to the gradient descent of the energy functional.

One might ask what properties fixed points of the contact flow satisfy. At a fixed point of this flow, $\alpha \wedge df = 0$. Therefore $df = h\alpha$ where h is some differentiable function. Differentiating this gives $dh \wedge \alpha + hd\alpha = 0$. Wedging this with α gives $h\alpha \wedge d\alpha = 0$. Thus, if $\alpha \wedge d\alpha > 0$, then $h = 0$ and the flow must converge to a solution where f is constant.

The combination of wedge product and Hodge star in (3.4) is essentially the three-dimensional cross-product. Hence, this flow preserves the inner product

$$\langle \beta(\cdot, t), \alpha(\cdot)\rangle = 1 \quad \text{for all } t \geq 0. \tag{3.5}$$

The evolution for β is quite degenerate. The evolution for f, however, is almost strictly parabolic. It is therefore expedient to consider a system with β and f as independent variables. We may assume, without loss of generality, that $|\alpha|^2 = 1$. This assumption merely simplifies the computations. We will use now the notation given in §2.

3.3. PROPOSITION. *Assume that $\alpha \in \Omega^1_s(\mathbf{M}^3)$ and $|\alpha|^2 = 1$. The contact flow* (3.4) *is equivalent to the system*

($\diamond$)

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df), \qquad \beta(\cdot, 0) = \beta_0(\cdot) \in \Omega^1(\mathbf{M}^3),$$

$$\frac{\partial}{\partial t}f = \Delta_\alpha f - \operatorname{div}(\alpha)\langle \alpha, \nabla f\rangle + \langle \nabla_\xi \alpha, \nabla f\rangle, \quad f(\cdot, 0) = f_0(\cdot) \in C^\infty(\mathbf{M}^3)$$

In local coordinates, the system may be written as

($\diamond$)

$$\frac{\partial}{\partial t}\beta_i = (\alpha_j \nabla_k f)\mu_{ijk}, \qquad \beta(\cdot, 0) = \beta_0(\cdot) \in \Omega^1(\mathbf{M}^3),$$

$$\frac{\partial}{\partial t}f = \Delta_\alpha f - \alpha_k \nabla_j \alpha_j \nabla_k f + \alpha_j \nabla_j \alpha_k \nabla_k f, \quad f(\cdot, 0) = f_0(\cdot) \in C^\infty(\mathbf{M}^3)$$

Proof. In local coordinates, the contact flow may then be written as:

$$\frac{\partial \beta_i}{\partial t} = (\alpha_j \nabla_k f)\mu_{ijk}.$$

Hence, for $f = *(\alpha \wedge d\beta + \beta \wedge d\alpha)$ we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= \left(\frac{\partial \beta_p}{\partial t}\nabla_q \alpha_r\right)\mu_{pqr} + \left(\alpha_p \nabla_q \frac{\partial \beta_r}{\partial t}\right)\mu_{pqr} \\ &= \left((\alpha_j \nabla_k f)\mu_{pjk}\nabla_q \alpha_r\right)\mu_{pqr} + \left(\alpha_p \nabla_q(\alpha_j \nabla_k f \mu_{rjk})\right)\mu_{pqr} \\ &= (\alpha_j \nabla_k f \nabla_q \alpha_r)\mu_{pjk}\mu_{pqr} - \left(\alpha_r \nabla_q(\alpha_j \nabla_k f)\right)\mu_{pjk}\mu_{pqr}.\end{aligned}$$

We may make use of the identity $\mu_{pjk}\mu_{pqr} = g_{jq}g_{kr} - g_{jr}g_{kq}$ to simplify the above expression:

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\left(\alpha_r \nabla_q(\alpha_j \nabla_k f)\right)(g_{jq}g_{kr} - g_{jr}g_{kq}) + (\alpha_j \nabla_k f \nabla_q \alpha_r)(g_{jq}g_{kr} - g_{jr}g_{kq}) \\ &= -\alpha_k \nabla_j(\alpha_j \nabla_k f) + \alpha_j \nabla_k(\alpha_j \nabla_k f) + \alpha_j \nabla_k f \nabla_j \alpha_k - \alpha_j \nabla_k f \nabla_k \alpha_j.\end{aligned}$$

Recall that $|\alpha|^2 = 1$ implies $\alpha_j \nabla_k \alpha_j = 0$. Expanding the expression above, we obtain

$$\begin{aligned}\frac{\partial f}{\partial t} &= |\alpha|^2 \nabla_p \nabla_p f - \alpha_k \alpha_j \nabla_j \nabla_k f + \alpha_j \nabla_k \alpha_j \nabla_k f - \alpha_k \nabla_j \alpha_j \nabla_k f \\ &\quad + \alpha_j \nabla_k f \nabla_j \alpha_k - \alpha_j \nabla_k f \nabla_k \alpha_j \\ &= \Delta f - \alpha_k \alpha_j \nabla_j \nabla_k f - \alpha_k \nabla_j \alpha_j \nabla_k f + \alpha_j \nabla_k f \nabla_j \alpha_k \quad \text{Q.E.D.}\end{aligned}$$

Short Time Existence. Since the evolution of f is not a fully parabolic equation, it is not entirely clear that the system has short time existence. We will prove that the system does indeed have solutions for a small time interval by regularizing the system and proving estimates independent of the added term.

We will now consider the following regularized system:

$$(\diamond)_\varepsilon \qquad \begin{aligned}&\frac{\partial}{\partial t}\beta_\varepsilon = *(\alpha \wedge df_\varepsilon) \\ &\frac{\partial}{\partial t} f_\varepsilon = \Delta_\alpha f_\varepsilon - \operatorname{div}(\alpha)\langle \alpha, \nabla f_\varepsilon \rangle + \langle \nabla_\xi \alpha, \nabla f_\varepsilon \rangle + \varepsilon \Delta f_\varepsilon \\ &\beta_\varepsilon(\cdot, 0) = \beta_0(\cdot) \in \Omega^1(\mathbf{M}^3) \\ &f_\varepsilon(\cdot, 0) = f_0(\cdot) \in C^\infty(\mathbf{M}^3)\end{aligned}$$

where β_ε and f_ε will be viewed as independent variables, $\varepsilon > 0$, and $|\alpha|^2 = 1$.

3.4. THEOREM. *Let $(\beta_\varepsilon, f_\varepsilon)$ be a solution to $(\diamond)_\varepsilon$ with time interval $[0, T_\varepsilon)$. There exist constants $a_{kl} = a_{kl}(\alpha, f_0, g_{ij})$ and $b_{kl} = b_{kl}(\alpha, f_0, g_{ij})$, independent of ε, such that*

$$|\partial_t^l \nabla^k f_\varepsilon|^2 \leq \|f_0\|_{k+2l} e^{a_{kl}t}$$

$$|\partial_t^l \nabla^k \beta_\varepsilon|^2 \leq \|\alpha\|_{k+2l} e^{b_{kl}t}$$

for $t \in [0, T_\varepsilon)$.

Proof. For any $\varepsilon > 0$, standard parabolic theory gives a solution for the time interval $[0, T_\varepsilon)$. The estimates will follow from an induction argument on the number k of spatial derivatives. The full theorem follows since one time derivative may be bounded by two space derivative. Let $X^k = (-\alpha_l \nabla_j \alpha_j + \alpha_j \nabla_j \alpha_l) g^{lk} \in \mathfrak{X}(\mathbf{M}^3)$.

$k = 0$. Notice that the evolution equation for f_ε depends only on g_{ij} and α. Therefore, we will consider the evolution of f_ε first. The equation

$$\frac{\partial f_\varepsilon}{\partial t} = \gamma_{ij} \nabla_i \nabla_j f_\varepsilon + X_i \nabla_i f_\varepsilon + \varepsilon \Delta f_\varepsilon$$

gives

$$\frac{\partial}{\partial t} f_\varepsilon^2 = \Delta_\alpha f_\varepsilon^2 - 2\gamma_{ij}(\nabla_i f_\varepsilon)(\nabla_j f_\varepsilon) + X_i \nabla_i f_\varepsilon^2 + \varepsilon\left(\Delta f_\varepsilon^2 - 2|\nabla f_\varepsilon|^2\right). \quad (3.6)$$

Thus, the weak maximum principle implies that $f_\varepsilon^2(\cdot, t) \leq f_\varepsilon^2(\cdot, 0) \leq \|f_0\|_0$ for all $t \geq 0$.

$k = 1$. We may also obtain the evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f_\varepsilon|^2 = {} & 2\nabla_p f_\varepsilon(\gamma_{ij} \nabla_i \nabla_j \nabla_p f_\varepsilon + \nabla_p \gamma_{ij} \nabla_i \nabla_j f_\varepsilon + X_i \nabla_i \nabla_p f_\varepsilon + \nabla_p X_i \nabla_i f_\varepsilon) \\ & + 2\nabla_p f_\varepsilon \gamma_{ij} R_{pijq} \nabla_q f_\varepsilon + 2\varepsilon \nabla_p f_\varepsilon(\Delta \nabla_p f_\varepsilon - R_{pq} \nabla_q f_\varepsilon).\end{aligned}$$

Gathering terms, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f_\varepsilon|^2 = {} & \Delta_\alpha |\nabla f_\varepsilon|^2 - 2\gamma_{ij}(\nabla_i \nabla_p f_\varepsilon)(\nabla_j \nabla_p f_\varepsilon) + 2\nabla_p \gamma_{ij} \nabla_i \nabla_j f_\varepsilon \nabla_p f_\varepsilon \\ & + X_i \nabla_i |\nabla f_\varepsilon|^2 + 2\nabla_p X_i \nabla_i f_\varepsilon \nabla_p f_\varepsilon + 2\gamma_{ij} R_{pijq} \nabla_q f_\varepsilon \nabla_p f_\varepsilon \\ & + \varepsilon\left(\Delta |\nabla f_\varepsilon|^2 - 2|\nabla^2 f_\varepsilon|^2 - 2R_{pq} \nabla_p f_\varepsilon \nabla_q f_\varepsilon\right). \quad (3.7)\end{aligned}$$

If we are to get rid of all of the second derivative terms, we must better understand the structure of $\nabla\gamma$. Choose an orthonormal frame so that at the point $x \in \mathbf{M}^3$, $\alpha = (0, 0, 1)$. It is easy to see that $\gamma_{33}(x) = 0$. Recall that we are assuming $|\alpha|^2 = 1$. Thus $\alpha_i \nabla_p \alpha_i(x) = \nabla_p \alpha_3(x) = 0$. So

$$\nabla_p \gamma_{ij} \nabla_i \nabla_j f_\varepsilon \nabla_p f_\varepsilon(x) = 0 \quad \text{if } i = j = 3. \tag{3.8}$$

Using $2ab \leq \delta a^2 + \delta^{-1} b^2$ with $\delta > 0$, we see that this term may be dominated by

$$-2\gamma_{ij}(\nabla_i \nabla_p f_\varepsilon)(\nabla_j \nabla_p f_\varepsilon)$$

at the expense of a term $\delta^{-1}|\nabla f_\varepsilon|^2$.

We may now use the fact that $|\nabla X|^2$ and the curvature tensor R_{ijkl} are independent of time and are bounded to conclude that $\exists C_1 > 0$ where $C_1 = C_1(X, R_{ijkl})$ such that

$$\frac{\partial}{\partial t}|\nabla f_\varepsilon|^2 \leq \Delta_\alpha |\nabla f_\varepsilon|^2 + X_i \nabla_i |\nabla f_\varepsilon|^2 + C_1 |\nabla f_\varepsilon|^2 + \varepsilon\left(\Delta |\nabla f_\varepsilon|^2 + 2C_1 |\nabla f_\varepsilon|^2\right). \tag{3.9}$$

Thus, if we consider the combination $W = |\nabla f_\varepsilon|^2 + C_1 f_\varepsilon^2$,

$$\frac{\partial}{\partial t} W \leq \Delta_\alpha W + X_i \nabla_i W + C_1 W + \varepsilon \Delta W \tag{3.10}$$

and the weak maximum principle implies

$$|\nabla f|^2(t) \leq W(t) \leq W(0) e^{C_1 t}. \tag{3.11}$$

$k > 1$. The bounds for higher derivatives of f_ε follow in a similar fashion. That is, $\exists \{C_i\}_{i=0}^{k-1}$ where $C_i = C_i(X, R_{ijkl}) > 0$, such that

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla^k f_\varepsilon|^2 \leq \Delta_\alpha |\nabla^k f_\varepsilon|^2 + X_i \nabla_i |\nabla^k f_\varepsilon|^2 + \sum_{i=0}^{k} C_i |\nabla^i f_\varepsilon|^2 \\ + \varepsilon\left(\Delta |\nabla^k f_\varepsilon|^2 + 2C_k |\nabla^k f_\varepsilon|^2\right). \end{aligned} \tag{3.12}$$

Note that, as in (3.9), terms of the form

$$(\nabla_{p_1} \cdots \nabla_{p_k} f_\varepsilon) \nabla_{p_1} \gamma_{ij} (\nabla_i \nabla_j \nabla_{p_2} \cdots \nabla_{p_k} f_\varepsilon)$$

have been dominated by cross terms coming from the Laplacian

$$\gamma_{ij}(\nabla_i\nabla_{p_1}\cdots\nabla_{p_k})(\nabla_j\nabla_{p_1}\cdots\nabla_{p_k}).$$

By induction, the terms $\Sigma_{i=0}^{k-1}C_i|\nabla^i f_\varepsilon|^2$ are bounded and W may be chosen as above to be a linear combination of the $|\nabla^i f_\varepsilon|^2$. Again, one obtains the inequality

$$\frac{\partial}{\partial t}W \leq \Delta_\alpha W + X_i\nabla_i W + CW + \varepsilon\Delta W \tag{3.13}$$

and the full result follows.

The evolution of β is zero order and essentially only depends on first derivatives of f_ε. Hence, the corresponding result for β_ε is trivial since the estimates for f_ε have been established. Q.E.D.

We now wish to make the following comments.

3.5. *Remarks.* (i) The estimates of Theorem 3.4 depend strongly upon the initial conditions since the equation is ostensibly diffusing in only 2 out of the 3 directions.

(ii) Since transverse diffusion is no longer a local phenomenon in a foliated region, point-wise techniques may not be appropriate to obtain stronger estimates.

(iii) The divergence term in the evolution equation of f has the natural interpretation of measuring the "mean curvature" of the distribution null(α).

The above estimates actually tell us that f and all its derivatives are still bounded at the time T_ε. Therefore, one may extend the flow past this time.

3.6. COROLLARY. *The estimates above actually imply that* $T_\varepsilon = \infty$.

This does not mean, of course, that the flows converge at $t = \infty$. We now may state the main result which allows us to construct solutions to $(\diamond)$ as a limit of solutions to $(\diamond)_\varepsilon$ as $\varepsilon \to 0$.

3.7. THEOREM. *System* $(\diamond)$ *has a unique solution* (β, f) *on* $\mathbf{M}^3 \times [0,\infty)$. *Furthermore, if* $\beta_0(\cdot) = \alpha(\cdot)$ *and* $f_0(\cdot) = *(\alpha \wedge d\beta + \beta \wedge d\alpha)(\cdot, 0)$, *then*

$$f(\cdot, t) = *(\alpha \wedge d\beta + \beta \wedge d\alpha)(\cdot, t).$$

Proof. Uniform convergence of $(\beta_\varepsilon, f_\varepsilon)$ as $\varepsilon \to 0$ follows directly from Theorem 3.4 and the Arzela-Ascoli theorem. The uniqueness of solutions follows easily from the weak maximum principle of parabolic equations. The fact that f_ε is algebraically related to β_ε when $\varepsilon \to 0$ follows also from uniqueness. Q.E.D.

4. Strong maximum principle

We wish to show f becomes positive instantly. Intuitively, the "heat" generated by regions where $f > 0$ should travel infinitely fast over finite distances of the foliation and cause the function f to become positive. Of course, as in the case of the standard heat equation, we cannot expect the temperature to rise at distances infinitely far away from a source.

We will prove a version of the strong maximum principle for weakly parabolic equations. See Bony [Bn] for an excellent presentation in the case of degenerate elliptic equations. We have modeled our proof on [PW, chapter 3] and [GT].

4.1. Definition. We will say that an operator of the form

$$L(u) = a_{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b_i\frac{\partial u}{\partial x^i} + cu - \frac{\partial u}{\partial t}$$

is *uniformly weakly parabolic* (*U.W.P.*) with respect to $\xi \in \mathfrak{X}(\mathbf{R}^3)$ if the matrix $a_{ij} \geq 0$ satisfies

(i) $a(\xi, \xi) = 0$ and

(ii) $\lambda |X|^2 \leq a(X, X) \leq \Lambda |X|^2$ for $X \perp \xi$ and $\lambda, \Lambda \in R^+$.

4.2. Hopf Lemma. *Let $E \subset \mathbf{R}^3 \times \mathbf{R}^+$ and let L be U.W.P. (w.r.t. ξ) with $f \geq 0$ satisfying*

$$L(f) = a_{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + b_i\frac{\partial f}{\partial x^i} + cf - \frac{\partial f}{\partial t} \leq 0 \tag{4.1}$$

where the coefficients a_{ij}, b_i, and c are bounded. Let p be a point on ∂E where f is zero and assume that at p a tangent ball B_1 to ∂E can be constructed such that $B_1 \subset E$ and $f > 0$ in B_1. Suppose further that the inward normal $\partial/\partial\nu$ of B_1 at p is not parallel to any vector in span$\{\partial/\partial t, \xi\}$. Then $(\partial/\partial\nu)f(p) > 0$.

Proof. Assume for simplicity that the center of B_1 is the origin in E. Also, we may assume that $f > 0$ on ∂B_1 except at p. Let $R = r$ be the radius of B_1 and define

$$v(x, y, z, t) = e^{-KR^2} - e^{-K(x^2+y^2+z^2+t^2)} \quad \text{for } K > 0. \tag{4.2}$$

Let B_2 be a ball of radius less than R centered around p (see Figure 4.1). We will denote the spatial position vector by $\vec{r} = (x, y, z)$ and $\vec{r}_i$ will denote the ith component of $\vec{r}$. Then, for large enough K

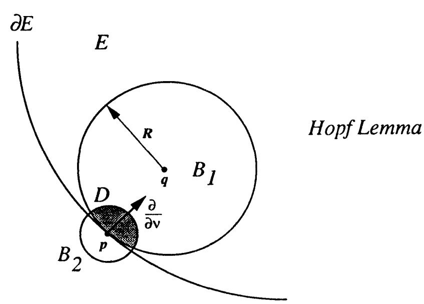
$$L(v) = -2Ke^{-K(x^2+y^2+z^2+t^2)}\left(2Ka_{ij}\vec{r}_i\vec{r}_j - a_{ii} + b_i\vec{r}_i + t\right) - cv < 0 \quad (4.3)$$

in the compact region $D = B_1 \cap B_2$ as long as $\vec{r}$ is not collinear with ξ and $\partial/\partial t$ and $|\vec{r}|^2 > 0$. The conclusion follows exactly as in [PW] by considering $w = f + \varepsilon v$ for small ε. Q.E.D.

Note, we will identify ξ with its dual 1-form in $\mathbf{R}^3$ so that, with a slight abuse of notation, we may use the definition in (2.2) for $d_\xi(p, q)$.

4.3. THEOREM (STRONG MAXIMUM PRINCIPLE). *Let f be a solution on $\mathbf{R}^3 \times [0, T]$ to the operator $L(f) \leq 0$ as given above. If $f \geq 0$ at $t = 0$ and if $\exists q \in \mathbf{M}^3$ such that $f(q, 0) > 0$, then $f(p, t) > 0$ for all $t \in (0, T]$ and for all p such $d_\xi(p, q) < \infty$.*

By continuity of the solution, $\exists \delta > 0$ such that $f(q, t) > \delta$ for $\forall t \leq t_0$. If $d_\xi(p, q) < \infty$, then let $\gamma(s)$ be a path from p to q which is an integral curve of the plane field $\xi^\perp$. Choose a coordinate chart (x, y, z, t) in a neighbor-



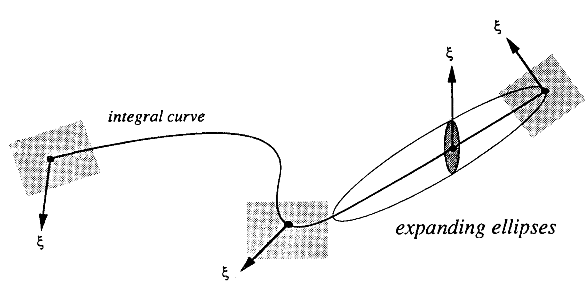


FIG. 4.1

hood of γ such that $\partial/\partial z = \xi$, $\partial/\partial x \perp \xi$, and $(s, 0, 0, t_0) = \gamma(s)$. In this coordinate chart, one may then apply directly the arguments given in [PW] to finish the theorem.

An alternative method of proof (see Figure 4.1), suggested to us by G. Huisken, is the following. Expand a ball B_R around q until it touches a zero of f at a radius of B_{R_0}. The Hopf Lemma implies that this point will be somewhere on the y, z, t equator. Now, deform the ball into a family of ellipses E_a given by $(ax)^2 + y^2 + z^2 + t^2 = R_0$ for $a \geq 1$.

If E touches a zero before it reaches p, or if E actually reaches p, the zero must be away from the equator. The Hopf Lemma then yields a contradiction to the fact that the derivative of f at such a point must vanish. Q.E.D.

5. Conductive one-forms

For Corollary 1.5, we wish to show that the set $\mathfrak{C}$ of "conductive" one forms is non-empty. The procedure is to find a divergence free vector field V on $\mathbf{M}^3$. Recurrence properties of the flow lines of V allow us to decompose the manifold into solid tori. In order to construct $\alpha \in \mathfrak{C}$, we glue in standard contact forms,[2] called "propellers," along the cores of the tori. Between the propellers, the forms will meld into foliation forms. We wish to express our gratitude to W. Thurston for his suggestions in this section.

5.1. THEOREM. *$\mathfrak{C}$ is a non-empty set for every compact, orientable 3-manifold.*

Proof. We first establish a result about nonvanishing, divergence free vector fields. It was brought to our attention that the following lemma was first demonstrated by Asimov [As] and Gromov [Gv] for manifolds of any dimension with zero Euler characteristic. For the sake of completeness we sketch an elementary proof.

5.2. LEMMA (DIVERGENCE FREE VECTOR FIELDS). *On every compact, orientable 3-manifold, $\exists V \in \mathfrak{X}(\mathbf{M}^3)$ and volume form $\mu \in \Omega^3(\mathbf{M}^3)$ such that $V \neq 0$ and $\mathfrak{L}_V \mu = 0$.*

5.3. *Remark* (petitio principii). If we are willing to assume the existence of a contact form η, then the canonical vector $T \in \mathfrak{X}(\mathbf{M}^3)$ of η is divergence free for the volume form $\mu = \eta \wedge d\eta$. Recall that T is defined by $\eta(T) = 1$ and $i_T d\eta = 0$.

---

[2] It has been brought to our attention that this procedure may be referred to as "Lutz twisting".

SKETCH OF LEMMA. Recall $\mathfrak{L}_V \mu = d \cdot i_v \mu$. Let $\mu = v \cdot r dr\, d\theta\, ds$, $v > 0$, be a volume form on a torus T_R or cylinder C_R

$$T_R = \{(r, \theta, s) | r \leq R; \theta, s \in [0, 2\pi)\}$$
$$C_R = \{(r, \theta, s) | r \leq R; \theta \in [0, 2\pi) s \in [0, 1]\}$$

embedded in $\mathbf{M}^3$. Note that the vector field

$$Z = (e(r, \theta)/v) \frac{\partial}{\partial s}$$

is divergence free.

5.4. DEFINITION. (i) A *whirlpool* $\mathfrak{W} = \{T_R, Z\}$ is a vector field

$$Z = e(r, \theta, s) \frac{\partial}{\partial s}$$

with $e \geq 0$, $\text{supp}(e) \subset\subset T_R$ and $\mathfrak{L}_Z \mu = 0$.

(ii) A *rip current* $\mathfrak{R} = \{C_R, Z\}$ is a vector field

$$Z = e(r, \theta, s) \frac{\partial}{\partial s}$$

with $e \geq 0$, $\text{supp}(e) \subset\subset C_R$ and $\mathfrak{L}_Z \mu = 0$ for $s \in [0.1, 0.9]$.

Now, let $V \in \mathfrak{X}(\mathbf{M}^3)$ be a gradient-like vector field [Mn] for a Morse function $h: \mathbf{M}^3 \to \mathbf{R}$. Let $B^i_m \subset \mathbf{M}^3$ be tiny coordinate balls around $p^i_m \in \mathbf{M}^3$ where p^i_m is the mth singularity $1 \leq m \leq n_i$ of index $i \in \{0, \cdots, 3\}$. Note that $\text{degree}(V(p^i_m)) = (-1)^i$. Let $S^i_m = \partial B^i_m$ and $B = \cup B^i_m$. Since the Euler characteristic $\chi(\mathbf{M}^3) = 0$, $\Sigma(n_i(-1)^i) = 0$.

On $\mathbf{M}^3 \setminus B$, it is not hard to construct a volume form μ for which V is divergence free. This is accomplished by dragging surface measures defined on S^0_m by the flow generated by V until they are swallowed into the spheres S^3_m. A simple cut and paste argument may be employed if a singularity of index 1 or 2 is encountered (see Figure 5.1). We may extend μ to be a volume form on all of $\mathbf{M}^3$.

The divergence theorem tells us that the total flux across the balls vanishes:

$$\int_{\partial(\mathbf{M}^3 \setminus B)} i_V \mu = \int_{\mathbf{M}^3 \setminus B} d \cdot i_V \mu = \int_{\mathbf{M}^3 \setminus B} \mathfrak{L}_V \mu = 0.$$

Since $\chi(\mathbf{M}^3) = 0$, a system of rip currents $\mathfrak{R}$, starting and ending in B and

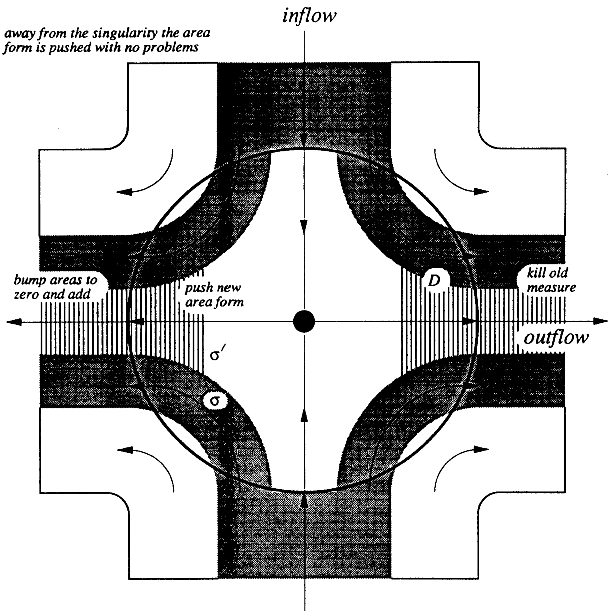


FIG. 5.1

transverse to V, may be installed to redistribute flux so that

$$\operatorname{flux}(V, S^i_m) = \int_{S^i_m} i_V \mu = (-1)^i = \operatorname{degree}(V(p^i_m)).$$

Hence, in $\mathbf{M}^3 \setminus B, (V+Z) \neq 0$ and $\mathfrak{L}_{V+Z} = 0$. We rename the vector field again by V.

Since $\chi(\mathbf{M}^3) = 0$, we may completely pair singularities of opposite degrees. Let $\{p, q\}$ be one such pair with surrounding balls $\{B_p, B_q\}$. For $\mathfrak{R} = \{C_R, Z\}$, a fast moving rip current connecting the two balls, let B_{p-q} be a barbell shaped, smooth approximation of the set $B_p \cup C_{R/2} \cup B_q$ (see Figure 5.2). Now V, restricted to $S_{p-q} = \partial B_{p-q}$, is a non-vanishing, degree 0 vector field. The flux of V across S_{p-q} is zero and we may assume V is divergence free in a collar neighborhood $N_{p-q} \subset B_{p-q}$ of S_{p-q}.

Note that $d \cdot i_V \mu$ is a closed form on B_{p-q}. By Stokes' theorem,

$$\int_{B_{p-q}} d \cdot i_V \mu = \int_{S_{p-q}} i_V \mu = 0$$

and by de Rham's theorem, $d \cdot i_V \mu = d\tau$ where $\tau \in \Omega^2(B_{p-q})$ has support

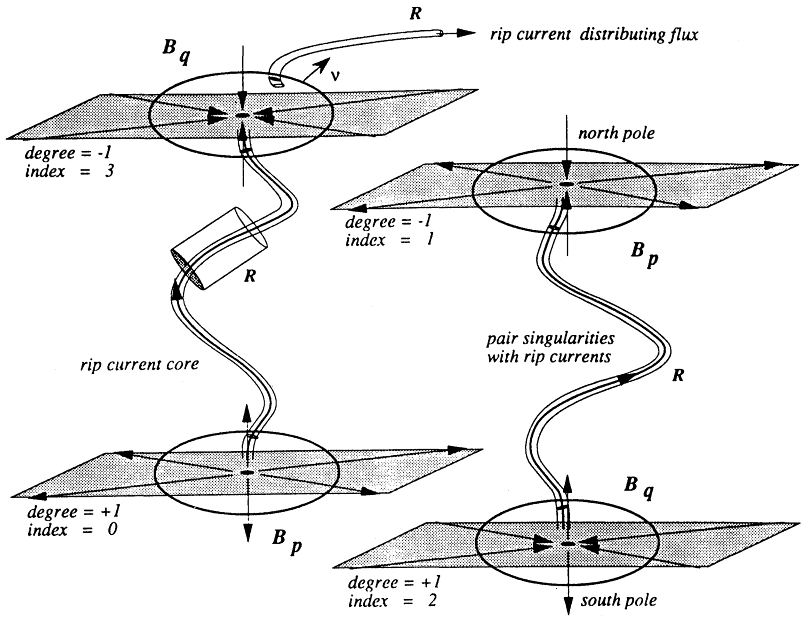


FIG. 5.2

in $B_{p-q} \setminus N_{p-q}$. Since

$$\mu: \Omega^2(B_{p-q}) \to \mathfrak{X}(B_{p-q})$$

is nondegenerate, there is a unique vector field W with support in $B_{p-q} \setminus N_{p-q}$ such that $i_W \mu = -\tau$. Then $X = V + W$ is divergence free in B_{p-q}, but may vanish in $B_{p-q} \setminus N_{p-q}$. Again, relabel $V + W$ as V.

We may now put in a thin whirlpool $\mathfrak{W}^1 = \{T^1, Z^1\}$ along which moves fluid from one pole of B_{p-q} to the other through $C_{R/2}$ and then recirculates, transversally to V, in $M \setminus B$ (see Figure 5.3). Now $T^2 = B_{p-q} \setminus (B_{p-q} \cap T^1)$ is again a solid torus and one may check that, by construction, V is transverse to the longitudinal lines of ∂T^2. We then finish the argument by putting in a complementary whirlpool $\mathfrak{W}^2 = \{T^2, Z^2\}$ which circulates fast enough that $V + Z^1 + Z^2 \neq 0$ in B_{p-q}. Finally, one may find a metric, conformal to the original metric, whose volume form agrees with the one constructed above.

Q.E.D.

Let $V \in \mathfrak{X}(\mathbf{M}^3)$, $V \neq 0$ be a volume preserving vector field. $\varepsilon = \varepsilon(g_{ij}, |\nabla V|) > 0$, to be chosen later, will be so small that for distances ε, V barely changes. Now, we choose a point $p_0 \in \mathbf{M}^3$ and a neighborhood U of

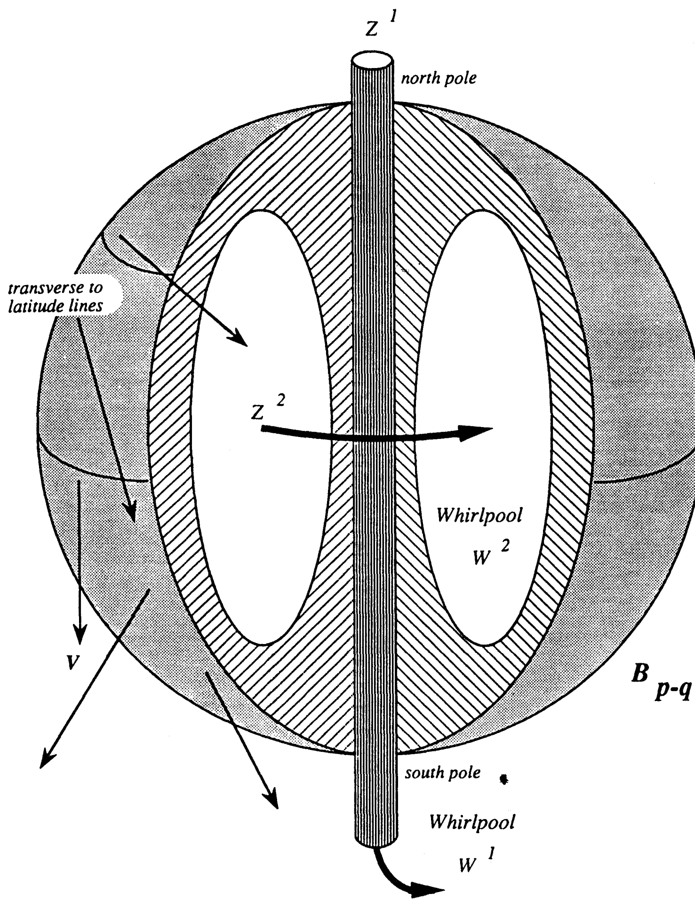


FIG. 5.3

p_0 of diameter $\varepsilon/10$. Since $\operatorname{div} V = 0$, the Poincaré recurrence lemma [Ar] implies that $\exists q_0 \subset U$ such that an integral curve γ_0 passing through q_0 eventually returns to $r_0 \in U$. Continue choosing points p_i and curves γ_i until $\forall p \in \mathbf{M}^3$, $\exists n$ such that $d(p, \gamma_n) < \varepsilon$. Since $\mathbf{M}^3$ is compact, we need only finitely many integral curve segments to accomplish this. Now, keeping the curves mutually disjoint, we close off the ends of the γ_i to give circles which are "nearly" integral curves of V.

It is not hard to show that for ε small enough, the Voronoi (or nearest neighbor) cell decomposition (see Figure 5.4) given by the γ_i decomposes $\mathbf{M}^3$ into the union of solid tori $\{T_i\}$. Let $\{T_i'\}$, $T_i' \subset T_i$, be smooth, solid tori with cores γ_i and let $A_i = T_i \setminus T_i'$.

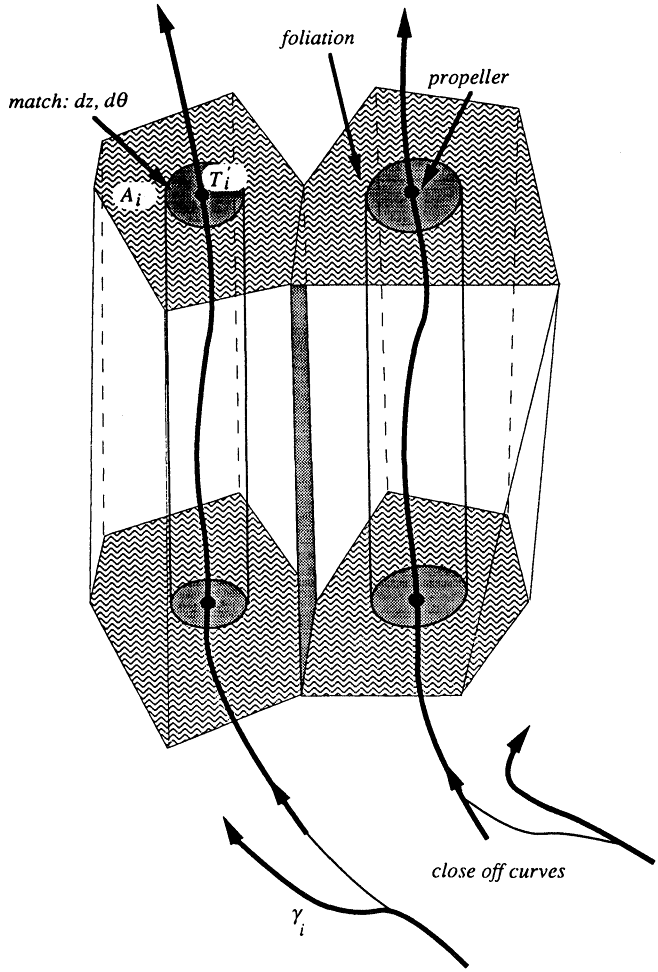


FIG. 5.4

Our model (see Figure 5.5) for a twisting contact form $\sigma \in \Omega^1(T)$ is a *propeller* [TW]. This is a generalization of the standard contact form $dz + xdy - ydx$ whose null space makes a quarter turn at infinite distance from the z-axis. The propeller is constructed on cylinders or solid tori with fixed radius r_0. In cylindrical coordinates (r, θ, z), σ will be written as:

$$\sigma = a(r, \theta, z)\, dz + b(r, \theta, z)\, d\theta \tag{5.1}$$

where $b \sim r^2$ for $r \sim 0$ near the origin.

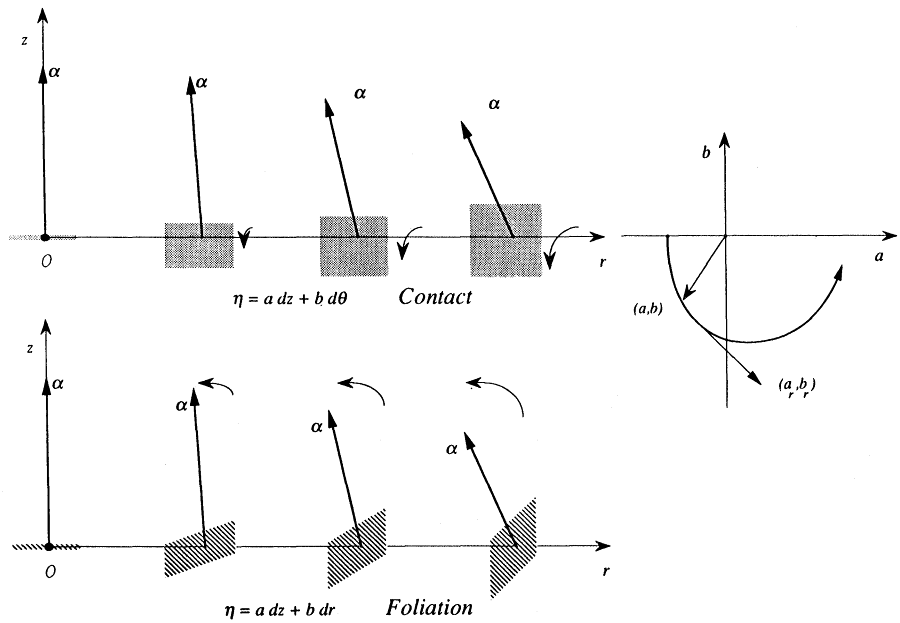


FIG. 5.5

Surprisingly,

$$\sigma \wedge d\sigma = (b_r a - a_r b) \cdot r^{-1} \text{ vol} \tag{5.2}$$

depends only on the derivatives in the r direction. This has the following geometric interpretation [TW]. For a fixed height z_0 and angle θ_0, $\sigma(r, \theta_0, z_0)$ is a contact form for $r \in [0, r_0]$ if the position vector $(a, b) \in R^2 \setminus \{(0,0)\}$ is not colinear with (a_r, b_r) and $(a_r, b_r) \neq 0$ (see Figure 5.1). We will ask that $a(r, \theta_0, z_0) = -1$ and that the position vector $(a(r, \theta_0, z_0), b(r, \theta_0, z_0))$ moves in a counter clockwise direction for $r \in [0, r_0]$. One may then meld the propeller into a foliation near $r = r_0$ by slowing down the parametrization until the derivatives vanish. The winding number of this curve is related to the notion of "overtwisted" contact structures [E].

We remark that twisting σ in the $d\theta$ direction gives a contact form while twisting in the dr direction adds "mean curvature" to the distribution. For example, the foliation form $dz + r^2\,dr$ has "bullet" shaped integral surfaces.

Now, along the boundary of the Voronoi cells ∂T_i, $V^\perp$ defines a plane field which may be pushed off the two-skeleton in such a way as to give a smooth foliation in A_i. This may be done in such a way that the defining one-form of this foliation, in cylindrical coordinates around $\partial T_i'$, has no dr component. Now, glue standard propellers into the T_i' and have their dz and $d\theta$ components match with α at $\partial T_i'$.

It clear that all points of $\mathbf{M}^3$ are either at a propeller or are a finite distance from one. We may assume that $|\alpha|^2 = 1$ since, for a differentiable function h, $\tilde{\alpha} = h\alpha$ gives $\tilde{\alpha} \wedge d\tilde{\alpha} = h^2\alpha \wedge d\alpha$. Q.E.D.

6. Conclusion and acknowledgements

Existence of contact forms in higher dimensions, for the case of open manifolds, is completely answered by Gromov (see [H]). However, for closed manifolds of dimension greater than 3, this question is still unresolved. It is our hope that methods similar to the ones discussed in here will prove useful.

Also, one could hope that the set of forms which one converges to have some special properties with respect to the metric. In fact, one could attempt to construct a global foliation by starting with an initial one form α containing a Reeb component. Reeb components seem to act as heat sinks since f must travel an infinite distance along the null space of α to reach the compact torus leaf (see Figure 6.1) and never become contact inside [Gr]. Note that this phenomenon indicates that our solutions are not analytic.

It is the author's great pleasure to thank Richard Hamilton for his help and suggestions. His encouragement and support helped bring this idea to fruition. Special thanks to Matt Grayson for early computer simulations which suggested that flows of this type would work and to V.L. Ginzburg for

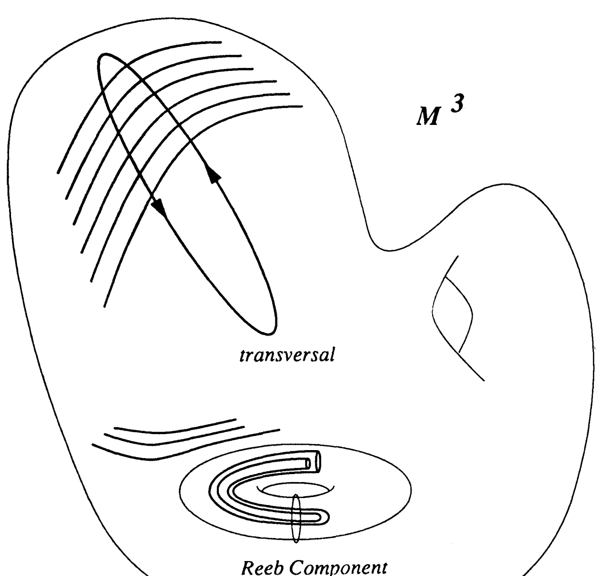


FIG. 6.1

Corollary 1.6. We wish to thank G. Huisken, W. Thurston and L.F. Wu for many fruitful discussions on this flow and to acknowledge discussions with Y. Eliashberg, M. Freedman, J. Lee, and L. Wang on related topics.

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