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CLOSED RANGE PROPERTY OF $\bar{\partial}$ ON NONPSEUDOCONVEX DOMAINS

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1. Introduction

The $\bar{\partial}$ problem on strictly pseudoconvex domains was solved by Kohn [7] with boundary regularity via the solution of the $\bar{\partial}$-Neumann problem. Later, Hörmander [6] solved the problem on pseudoconvex domains by introducing weighted norms, which bypassed the question of boundary regularity. On domains that are not necessarily pseudoconvex the $\bar{\partial}$ problem on (p, q) forms can still be solved if the Levi-form has $n - q$ positive eigenvalues at every point on the boundary ([2], [4], [6], [8] and others). There remains the question of solving the problem when the $n - q$ eigenvalues of the Levi-form are allowed to be zero. In a previous paper [5] we proved that we can solve $\bar{\partial}$ (in the space of $\bar{\partial}$-closed (p, r) form with coefficients locally L^2 in Ω and $r \geq q$) if the domains are so called weakly q-convex, which are domains that at every point on the boundary the sum of any $n - q$ eigenvalues of the Levi-form is non-negative. The above condition of weak q-convexity on the one hand makes the improvement of allowing the eigenvalues of the Levi-form to be zero, but on the other hand it requires the sum of the eigenvalues instead of the individual eigenvalue to be nonnegative. Hence, by considering the case of $(p, n - 1)$ forms, we can see that the weak q-convexity is not the optimal condition imposed on the Levi-form that we can solve $\bar{\partial}$. In this paper we try to further improve the requirement on the Levi-form. For $(p, n - 1)$ forms we only require that at every point on the boundary there is one holomorphic vector field whose Levi-form is nonnegative. This is the expected minimal condition. However, we can only prove that $\bar{\partial}$ has closed range. We also attempt to extend the result to (p, q) forms, with a partial success. We assume that the Levi-form can be split into blocks with an $(n - q)$ minor being positive semi-definite. Adding some extra assumptions we prove similarly that $\bar{\partial}$ has closed range.

2. Notations

Let Ω be a smooth domain in $\mathbb{C}^n$ and ρ a C^∞ defining function of Ω so that $\rho < 0$ in Ω and $|\partial \rho| = 1$ on the boundary. $T_x^{1,0}$ and $T_x^{0,1}$ represents the

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holomorphic and anti-holomorphic vectors at x respectively, and $T_x^{1,0}(b\Omega)$ represents the vector in $T_x^{1,0}$ which are tangential to the boundary. Throughout this paper we will assume that $L_1, \dots, L_n$ are holomorphic vector field with C^∞ coefficients and $L_1, \dots, L_{n-1}$ are tangential to $b\Omega$. Let $\omega_1, \dots, \omega_n$ be $(1,0)$ forms dual to $L_1, \dots, L_n$. Then

$$\rho_{ij} = \partial\bar{\partial}\rho\left(L_i, \bar{L}_j\right) \quad i, j = 1, 2, \dots, n-1$$

is the Levi-form associated to ρ. Also, if ϕ is a smooth function on Ω then we write

$$\phi_{ij} = \partial\bar{\partial}\phi\left(L_i, \bar{L}_j\right), \quad i, j = 1, 2, \dots, n.$$

We use $C_0^k(\Omega)$ to denote functions compactly supported in Ω that are k times differentiable, $L^2_{(p,q)}(\Omega, \phi)$ to denote the (p, q) forms with coefficients in L^2 with respect to the weight function $e^{-\phi}$, $L^2_{(p,q)}(\Omega, \text{loc})$ to denote the (p, q) forms with locally square integrable coefficients and $C^k_{(p,q)}(\Omega)$ to denote the (p, q) forms with C^k coefficients in Ω where k can be a positive integer or ∞. Let

$$\langle u, v\rangle_\phi = \int u\bar{v}e^{-\phi}\, dV \text{ and } \|u\|_\phi = \langle u, u\rangle_\phi.$$

The operator $\bar{\partial}$ is easily defined for (p, q) forms with coefficients in $C^1(\Omega)$. T denotes the closed operator which is the maximal extension of the $\bar{\partial}$ operator from $L^2_{(p,q-1)}(\Omega, \phi)$ to $L^2_{(p,q)}(\Omega, \phi)$, similarly S is the corresponding operator from $L^2_{(p,q)}(\Omega, \phi)$ to $L^2_{(p,q+1)}(\Omega, \phi)$ and T^* is the adjoint of T. We use D_T and N_T to denote the domain and null space of the operator T respectively. $\mathfrak{D}_{(p,q)}(U \cap \bar{\Omega})$ denotes the space of smooth (p, q) forms in D_T with coefficients compactly supported in $U \cap \bar{\Omega}$. To simplify the notations, we will prove the theorems for $(0, q)$ forms. The theorem for (p, q) forms follows since we can consider (p, q) forms as $(0, q)$ forms where coefficients are $(p, 0)$ forms.

We will use ε^K_{kJ} to denote the sign of the permutation taking kJ to K. The symbol $'$ in the summation means that the summation is over strictly increasing indices. Finally, C and const in this paper are constants that may vary from line to line.

3. $\bar{\partial}$ problem on $(n-1)$ forms

THEOREM 1. *Let Ω be a relatively compact subset in $\mathbb{C}^n$ with smooth boundary. Suppose that at every point $z \in b\Omega$, there exists a neighborhood U of z and a smooth vector field L defined in U with values in $T^{1,0}(b\Omega \cap U)$ such* *that the Levi-form of L satisfies $\partial\bar{\partial}\,\rho(L,\bar{L}) \geq 0$ on the boundary. Then the range of $\bar{\partial}$ from $L^2_{(p,n-2)}(\Omega)$ to $L^2_{(p,n-1)}(\Omega)$ is closed and $N = N_{T^*} \cap N_s$ is finite dimensional.*

We note that if we can prove that for some $\phi \in C(\bar{\Omega})$ the operator from $L^2_{(p,n-2)}(\Omega,\phi)$ to $L^2_{(p,n-1)}(\Omega,\phi)$ has a closed range, then the same is true for other choices of ϕ. This is due to the simple reason that different ϕ's will give equivalent norms on the space $L^2_{(p,q)}(\Omega)$. Before proving Theorem 1, we need to prove the following lemma.

LEMMA. *Let $z \in b\Omega$. Assume that in a neighborhood U of z there exists a smooth vector field L with values in $T^{1,0}(b\Omega \cap U)$ such that $\partial\bar{\partial}\rho(L,\bar{L}) \geq 0$ on the boundary. Then for a sufficiently large number K and some $0 < \eta < 1$ by setting $\phi = -(-\rho)^\eta + K|z|^2$ we have for all $u \in \mathfrak{D}_{(p,n-1)}(U \cap \bar{\Omega})$,*

$$C\|u\|^2_\phi \leq \|Su\|^2_\phi + \|T^*u\|^2_\phi + \|gu\|^2_\phi \tag{1}$$

for some $C > 0$ and $g \in C_0^\infty(\Omega)$. Moreover we can make C arbitrarily large by setting K to be large. (g depends on K.)

Proof. Let $L_1,\dots,L_n$ be smooth vector fields that form a basis for $T_x^{1,0}(U \cap \bar{\Omega})$ at each point $x \in U \cap \bar{\Omega}$, $L = L_1$ and $L_1,\dots,L_{n-1}$ tangential to the boundary. Let $\omega_1,\dots,\omega_n$ be (1,0) forms dual to $L_1,\dots,L_n$.

If $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$, we define an operator $\delta_j w = e^\phi L_j(we^{-\phi})$ on functions $w \in C^1(U \cap \Omega)$.

Consider a $(0, n-1)$ form $u \in \mathfrak{D}_{(0,n-1)}(U \cap \bar{\Omega})$. We can write u as

$$u = \sum_{i=1}^{n} u_{\hat{i}}\bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \cdots \wedge \hat{\bar{\omega}}_i \wedge \cdots \wedge \bar{\omega}_n.$$

where ^ means that the term is missing in the expression and $u_{\hat{i}}$ denotes $u_{12\ldots\hat{i}\ldots n}$. Note that $u \in D_{T^*}$ implies that $u_{\hat{i}} = 0$ on $b\Omega$ for $i \neq n$.

From Hörmander [6] we have the following two integration by parts formulas:

For $v, w \in C_0^1(U \cap \bar{\Omega})$,

$$\langle v, \bar{L}_j w\rangle_\phi = -\langle \delta_j v, w\rangle_\phi + \langle \phi_j v, w\rangle_\phi, \quad 1 \leq j \leq n-1 \tag{2}$$

where σ_j is some C^∞ function in $\bar{U} \cap \bar{\Omega}$, and for $1 \le j, k \le n-1$,

$$\langle \bar{L}_k w, \bar{L}_j w \rangle_\phi \ge \langle \delta_j w, \delta_k w \rangle_\phi - \langle \phi_{jk} w, w \rangle_\phi - \int_{b\Omega \cap U} \rho_{jk} |w|^2 e^{-\phi}\, dS + O(\|w\|_\phi \||w\||_\phi) \tag{3}$$

where

$$\||w\||_\phi = \sum_{j=1}^{n} \left\| \bar{L}_j w \right\|_\phi + \|w\|_\phi .$$

We know from Hörmander [6] that

$$\left| \sum_{i,j} \|\bar{L}_j u_{\hat{i}}\|_\phi^2 + \sum_{j,k,K} \langle \phi_{jk} u_{jK}, u_{kK} \rangle_\phi + \sum_{j=1}^{n-1} \sum_K \int_{b\Omega \cap U} \rho_{jj} |u_{jK}|^2 e^{-\phi}\, dS - \|Su\|_\phi^2 - \|T^* u\|_\phi^2 \right|$$
$$\le C \|u\|_\phi \left(\|Su\|_\phi + \|T^* u\|_\phi + \||u\||_\phi \right).$$

Using (3) to integrate by parts on the term $\|\bar{L}_j u_{\hat{i}}\|_\phi^2$ for the above expression for $2 \le j \le n-1$ with $j \ne i$ we get

$$\sum_{i=1}^{n} \left(\left\| \bar{L}_1 u_{\hat{i}} \right\|_\phi^2 + \left\| \bar{L}_n u_{\hat{i}} \right\|_\phi^2 \right) + \sum_{i=2}^{n-1} \left\| \bar{L}_i u_{\hat{i}} \right\|_\phi^2$$
$$+ \sum_{\substack{2 \le j \le n-1 \\ i \ne j}} \left\| \delta_j u_{\hat{i}} \right\|_\phi^2 + \int_{b\Omega \cap U} \rho_{11} |u_{\hat{n}}|^2 e^{-\phi}\, dS$$
$$+ \int_{\Omega \cap U} \left(\sum_K \phi_{11} |u_{1K}|^2 + \sum_K \phi_{nn} |u_{nK}|^2 + \sum_{\substack{1 \in K \text{ and } n \in L \\ \text{or } 1 \in L \text{ and } n \in K}} \phi_{jk} u_K \bar{u}_L \right) e^{-\phi}\, dV$$
$$+ O\left(\sum \|u\|_\phi \||u\||_\phi \right) - \|Su\|_\phi^2 - \|T^* u\|_\phi^2$$
$$\le C\left(\|Su\|_\phi^2 + \|T^* u\|_\phi^2 + \|u\|_\phi^2 + \|u\|_\phi \||u\||_\phi \right)$$

The terms $\|u\|_\phi \||u\||_\phi$ of the above inequality can be absorbed into the second and the third term of the left hand side by using the small constant, large constant method and invoking (2) to absorb terms $\langle \bar{L}_j u_{\hat{i}}, u_{\hat{k}} \rangle$ whenever $i \neq j$. Finally we get

$$\sum_{i=1}^{n} \left(\|\bar{L}_1 u_{\hat{i}}\|_\phi^2 + \|\bar{L}_n u_{\hat{i}}\|_\phi^2 \right) + \sum_{i=2}^{n-1} \|\bar{L}_i u_{\hat{i}}\|_\phi^2$$

$$+ \sum_{\substack{2 \le j \le n-1 \\ i \neq j}} \|\delta_j u_{\hat{i}}\|_\phi^2 + \int_{b\Omega \cap U} \rho_{11} |u_{\hat{n}}|^2 e^{-\phi} \, dS$$

$$+ \int_{\Omega \cap U} \left(\sum_K \phi_{11} |u_{1K}|^2 + \sum_K \phi_{nn} |u_{nK}|^2 + \sum_{\substack{1 \in K \text{ and } n \in L \\ \text{or } 1 \in L \text{ and } n \in K}} \phi_{jk} u_K \bar{u}_L \right) e^{-\phi} \, dV$$

$$\le C\left(\|Su\|_\phi^2 + \|T^* u\|_\phi^2 + \|u\|_\phi^2 \right) \tag{4}$$

We will focus our attention on the fifth term of the left hand side of (4) which arises from the weight function ϕ. Now consider the function $\phi = -(-\rho)^\eta + K|z|^2$ for some $0 < \eta < 1$. Clearly $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$. The function $-(-\rho)^\eta$ has been introduced by Diederich and Fornaess [1] on pseudoconvex domains. However, it can still give the right growth in the direction L_1 as well as in the normal direction in the present case. We easily obtain

$$\begin{aligned} \phi_{ij} &= \partial\bar{\partial}\phi\left(L_i, \bar{L}_j\right) \\ &= \eta(-\rho)^{\eta-1} \partial\bar{\partial}\rho\left(L_i, \bar{L}_j\right) - \eta(\eta - 1)(-\rho)^{\eta-2} \\ &\quad \times (L_i \rho)\left(\bar{L}_j \rho\right) + K \partial\bar{\partial}|z|^2\left(L_i, \bar{L}_j\right). \end{aligned}$$

We consider different cases of i and j.

Using $\partial\bar{\partial}\rho(L_1, \bar{L}_1) \ge 0$ on the boundary we get

$$\begin{aligned} \phi_{11} &\ge \eta(-\rho)^{\eta-1} \partial\bar{\partial}\rho\left(L_1, \bar{L}_1\right) + K \partial\bar{\partial}|z|^2\left(L_1, \bar{L}_1\right) \\ &\ge \text{const}\left((-\rho)^{\eta-1}(\rho C) + K\right) \\ &\ge \text{const } K \end{aligned} \tag{5}$$

if we choose the neighborhood U to be small enough. (We may assume that this neighborhood is same as the original U because of the compactly supported function g to be chosen later on.) Next

$$\begin{aligned}\phi_{nn} &= \eta(-\rho)^{\eta-1}\partial\bar{\partial}\rho(L_n,\bar{L}_n) - \eta(\eta-1)\\ &\quad\times(-\rho)^{\eta-2}|L_n\rho|^2 + K\partial\bar{\partial}|z|^2(L_n,\bar{L}_n)\\ &\geq \text{const}\big((-\rho)^{\eta-2} - (-\rho)^{\eta-1} - K\big). \qquad (6)\end{aligned}$$

For i, j not both 1 and n we simply use

$$|\phi_{ij}| \leq \text{const}\big((-\rho)^{\eta-1} + K\big). \qquad (7)$$

From (7) we have

$$|\phi_{ij}u_{1J}\bar{u}_{nL}| \leq \text{const}\Big(|u_{1J}|^2 + \big((-\rho)^{2\eta-2} + K^2\big)|u_{nL}|^2\Big).$$

Putting (5), (6), and (7) into (4) get

$$\begin{aligned}(K - \text{const})\sum_J \|u_{1J}\|_\phi^2 &+ \sum_J \int_{\Omega\cap U}\big((-\rho)^{\eta-2} - (-\rho)^{2\eta-2}\\ &-(-\rho)^{\eta-1} - K^2\big)|u_{nJ}|^2 e^{-\phi}\,dV \leq C\big(\|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|u\|_\phi^2\big).\end{aligned}$$

Choosing K large enough the first term on the right hand side above can be made positive, in fact can be made arbitrarily large. For fixed K, if ρ is small enough the second term on the right hand side is positive. Thus we get for $u \in \mathfrak{D}_{(0,n-1)}(U \cap \bar{\Omega})$

$$C\|u\|_\phi^2 \leq \|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|gu\|_\phi^2$$

for some $g \in C_0^\infty(\Omega)$.

Proof of Theorem 1. Since $b\Omega$ is compact, we can find finitely many points $x_1, \ldots, x_n \in b\Omega$ and corresponding neighborhoods $U_{x_1}, \ldots, U_{x_n}$ so that (1) is true in these neighborhoods.

If χ is a function, we know that $[S, \chi]$ and $[T^*, \chi]$ are also of order zero. Since C can be made arbitrarily large, by using a partition of unity argument, we get for $u \in D_{T^*} \cap C^\infty_{(0,n-1)}(\bar{\Omega})$

$$\|u\|_\phi^2 \leq \text{const}\big(\|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|gu\|_\phi^2\big) \qquad (8)$$

for some $g \in C_0^\infty(\Omega)$. Since $D_{T^*} \cap C^\infty_{(0,n-1)}(\bar{\Omega})$ is dense in $D_{T^*} \cap D_S$, (8) is also true for $u \in D_{T^*} \cap D_S$.

Now Lemma (3.4.2) of Hörmander [6] says that

$$B = \left\{f : f \in D_S \cap D_{T^*} : \|Sf\|_\phi^2 + \|T^*f\|_\phi^2 + \|f\|_\phi^2 < 1\right\}$$

is relatively compact in $L^2_{(p,q)}(\Omega, \mathrm{loc})$. By this lemma, for any sequence $g_k \in D_{T^*} \cap D_S$ with $\|g_k\|_\phi$ bounded and both $T^*g_k \to 0$ and $Sg_k \to 0$, g_k has a convergent subsequence in $L^2_{(p,n-1)}(\Omega, \mathrm{loc})$. By (8), g_k has a convergent subsequence in $L^2_{(p,n-1)}(\Omega, \phi)$. Hence the following theorem from Hörmander [6] implies that $\bar\partial$ has closed range and N is finite dimensional.

THEOREM H. *Assume for every sequence $g_k \in D_{T^*} \cap D_S$ with $\|g_k\|_{L^2}$ bounded and both $T^*g_k \to 0$ and $Sg_k \to 0$, g_k has a strongly convergent subsequence. Then T has closed range and N is finite dimensional.*

4. $\bar\partial$ problem for (p, q) forms

THEOREM 2. *Let Ω be a relatively compact subset of $\mathbb{C}^n$ with smooth boundary. Assume that for every $z \in b\Omega$ there exists a neighborhood U of z and smooth vector fields $L_1, \dots, L_n \in T^{1,0}(U \cap \overline{\Omega})$ with $L_1, \dots, L_{n-1}$ tangential and there is a C^2 function χ on $\overline{\Omega}$ with the following properties in $U \cap \overline{\Omega}$ for some integer $m < n$:*

(i) *The Levi form $(\rho_{ij})_{i \le i, j \le m}$ is positive semidefinite on $b\Omega \cap U$, and $\rho_{ij} = 0$ for $1 \le i \le m$ and $m < j < n$.*

(ii) *The commutators $[L_j, \overline{L}_i] = 0$ for $i \in \{1, \dots, m, n\}$ and $m < j < n$.*

(iii) (a) *$(\chi_{ij})_{1 \le i, j \le m}$ is positive definite near the boundary.*

(b) *$\chi_{ij} = 0$ for $1 \le i \le m$, $m < j < n$.*

Then the range of $\bar\partial$ from $L^2_{(p,q-1)}(\Omega)$ to $L^2_{(p,q)}(\Omega)$ is closed for $q \ge n - m$.

Proof. Following the proof of Theorem 1, we only need to prove that the corresponding statement for $(0, q)$ forms in the lemma still holds. That is to say, assume (i) and (ii) of the theorem holds in a neighborhood U of z, we will prove that for some $0 < \eta < 1$ and large M, by setting $\phi = -(-\rho)^\eta + M\chi$ we have

$$C\|u\|_\phi^2 \le \|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|gu\|_\phi^2 \tag{9}$$

for all $u \in \mathfrak{D}_{(p,q)}(U \cap \overline{\Omega})$ where $C > 0$ is some constant and $g \in C_0^\infty(\Omega)$. Also we can make C as large as we please by setting M large.

In the present case of $(0, q)$ forms the expansion of $\|Su\|_\phi^2 + \|T^*u\|_\phi^2$ is more difficult.

Let $u = \sum_{|J|=q} u_J \bar{\omega}_J$ be in $\mathfrak{D}_{(p,q)}(U \cup \bar{\Omega})$. Then

$$Su = \sum_{j,J} \bar{L}_j u_J \bar{\omega}^j \wedge \bar{\omega}^J + \dots$$

and

$$T^* u = -\sum_{j,K} \delta_j u_{jK} \bar{\omega}^K + \dots$$

where ... means that there is no differentiation in u. $u \in D_{T^*}$ implies that $u_J = 0$ if $n \in J$ in our system of vector fields $L_1, \dots, L_n$. We will need to use the formula

$$\begin{aligned} \left[\delta_k, \bar{L}_j\right] w &= \left(\bar{L}_j L_k \phi\right) w + \left[L_k, \bar{L}_j\right] w \\ &= \left(\bar{L}_j L_k \phi\right) w + \sum_i c^i_{jk} L_i w - \sum_i \bar{c}^i_{kj} \bar{L}_i w \\ &= \phi_{kj} w + \sum_i c^i_{jk} \delta_i w - \sum_i \bar{c}^i_{kj} \bar{L}_i w \end{aligned} \tag{10}$$

where c^i_{jk} are coefficients independent of ϕ. Also note that the Levi-form $\rho_{kj} = c^n_{jk}$ (Folland-Kohn [3]).

Clearly

$$\begin{aligned} &\|Su\|^2_\phi + \|T^* u\|^2_\phi \\ &\geq \text{const}\left(\sum_K{}' \left\| \sum_{k,J} \varepsilon^K_{kJ} \bar{L}_k u_J \right\|^2_\phi + \sum_H{}' \left\| \sum_{k,J} \varepsilon^J_{kH} \delta_k u_J \right\|^2_\phi - O\left(\|u\|^2_\phi\right) \right) \\ &= \text{const}\left(\sum_K{}' \left(\left\| \sum_{k \in \{1,\dots,m,n\}, J} \varepsilon^K_{kJ} \bar{L}_k u_J \right\|^2_\phi + \left\| \sum_{m<k<n, J} \varepsilon^K_{kJ} \bar{L}_k u_J \right\|^2_\phi \right) \right. \\ &\quad + \sum_H{}' \left(\left\| \sum_{k \in \{1,\dots,m,n\}, J} \varepsilon^J_{kH} \delta_k u_J \right\|^2_\phi + \left\| \sum_{m<k<n, J} \varepsilon^J_{kH} \delta_k u_J \right\|^2_\phi \right) \\ &\quad + 2\,\mathrm{Re}\left(\sum_K{}' \left\langle \sum_{k \in \{1,\dots,m,n\}, J} \varepsilon^K_{kJ} \bar{L}_k u_J, \sum_{m<l<n, L} \varepsilon^K_{lL} \bar{L}_l u_L \right\rangle_\phi \right. \\ &\quad \left. \left. + \sum_H{}' \left\langle \sum_{k \in \{1,\dots,m,n\}, J} \varepsilon^J_{kH} \delta_k u_J, \sum_{m<l<n, L} \varepsilon^L_{lH} \delta_l u_L \right\rangle_\phi \right) - O\left(\|u\|^2_\phi\right) \right). \end{aligned} \tag{11}$$

We will first show how to simplify the second last term of the right hand side. In the following computation we will omit the range of k, l, L, and J. We should keep in mind that $k \in \{1, \ldots, m, n\}$, $m < l < n$, and J and L are some tuples with increasing indices corresponding to the indices k, l, K and H.

$$
\begin{aligned}
&\sum_K{}' \Big\langle \sum_{k,J} \varepsilon_{kJ}^K \bar{L}_k u_J, \sum_{l,L} \varepsilon_{lL}^K \bar{L}_l u_L \Big\rangle_\phi + \sum_H{}' \Big\langle \sum_{k,J} \varepsilon_{kH}^J \delta_k u_J, \sum_{l,L} \varepsilon_{lH}^L \delta_l u_L \Big\rangle_\phi \\
&\quad = -\sum_K{}' \sum_{l,L} \varepsilon_{lL}^K \Big\langle \sum_{k,J} \varepsilon_{kJ}^K \delta_l \bar{L}_k u_J, u_L \Big\rangle_\phi + O\left(\Big\| \sum_{k,J} \varepsilon_{kJ}^K \bar{L}_k u_J \Big\|_\phi \|u\|_\phi \right) \\
&\qquad - \sum_H{}' \sum_{l,L} \varepsilon_{lH}^L \Big\langle \sum_{k,J} \varepsilon_{kH}^J \bar{L}_l \delta_k u_J, u_L \Big\rangle_\phi + O\left(\Big\| \sum_{k,J} \varepsilon_{kH}^J \delta_k u_J \Big\|_\phi \|u\|_\phi \right) \\
&\quad = \sum_H{}' \sum_{l,k,L,J} \varepsilon_{kH}^J \varepsilon_{lH}^L \langle [\delta_k, \bar{L}_l] u_J, u_L \rangle_\phi \\
&\qquad + O\left(\left(\Big\| \sum_{k,J} \varepsilon_{kJ}^K \bar{L}_k u_J \Big\|_\phi + \Big\| \sum_{k,J} \varepsilon_{kH}^J \delta_k u_J \Big\|_\phi \right) \|u\|_\phi \right) \\
&\quad = \sum_H{}' \sum_{l,k} \langle \phi_{kl} u_{kH}, u_{lH} \rangle_\phi \\
&\qquad + \sum_H{}' \sum_{l,k} \left\langle \left(\sum_{i=1}^m c_{jk}^i \delta_i - \sum_{i=1}^m \bar{c}_{kj}^i \bar{L}_i \right) u_{kH}, u_{lH} \right\rangle_\phi \\
&\qquad + O\left(\left(\Big\| \sum_{k,J} \varepsilon_{kJ}^K \bar{L}_k u_J \Big\|_\phi + \Big\| \sum_{k,J} \varepsilon_{kH}^J \delta_k u_J \Big\|_\phi \right) \|u\|_\phi \right) \\
&\quad = \sum_H{}' \sum_{l,k} \langle \phi_{kl} u_{kH}, u_{lH} \rangle_\phi + O\left(\left(\Big\| \sum_{k,J} \varepsilon_{kJ}^K \bar{L}_k u_J \Big\|_\phi \right.\right. \\
&\qquad \left.\left. + \Big\| \sum_{k,J} \varepsilon_{kH}^J \delta_k u_J \Big\|_\phi + \sum_K{}' \sum_{i=1}^m \|\bar{L}_i u_K\|_\phi^2 \right) \|u\|_\phi \right) \qquad (12)
\end{aligned}
$$

where the first equality follows from (2), line 3 follows from the fact that $\varepsilon_{lL}^K \varepsilon_{kJ}^K = -\varepsilon_{iH}^J \varepsilon_{jH}^L$ (J and L on the two sides are given and $J \neq L$), the second equality follows from (10) and assumption (i) and (ii) of the theorem gives the index i is from 1 to m. Finally the last equality follows from integrating by parts on $\delta_i u_{kH}$ using (2).

We can apply the standard integration by parts computation to the first and the third term of the right hand side in (11) (the reader may refer to [3] or [6] for the details) to get

$$\sum_K{}' \Big\| \sum_{k\in\{1,\dots,m,n\},J} \varepsilon^K_{kJ} \bar L_k u_J \Big\|^2_\phi + \sum_H{}' \Big\| \sum_{k\in\{1,\dots,m,n\},J} \varepsilon^J_{kH} \delta_k u_J \Big\|^2_\phi$$

$$= \sum_K{}' \sum_{k\in\{1,\dots,m,n\}} \|\bar L_k u_K\|^2_\phi$$

$$+ \sum_K{}' \sum_{j,k\in\{1,\dots,m,n\}} \int_{U\cup\Omega} \phi_{jk} u_{jK} \bar u_{kK} e^{-\phi}\, dV$$

$$+ \sum_K{}' \sum_{1\le j,k\le m} \int_{U\cap b\Omega} \rho_{jk} u_{jK} \bar u_{kK} e^{-\phi}\, dV - O\big(\|u\|^2_\phi\big) \tag{13}$$

Combining (11), (12), (13) and using the small constant, large constant technique to absorb the last term of (12) we get

$$\|Su\|^2_\phi + \|T^*u\|^2_\phi + \|u\|^2_\phi$$

$$\ge \text{const}\Bigg(\sum_K{}' \sum_{k\in\{1,\dots,m,n\}} \|\bar L_k u_K\|^2_\phi$$

$$+ \sum_K{}' \sum_{j,k\in\{1,\dots,m,n\}} \int_{U\cap\Omega} \phi_{jk} u_{jK} \bar u_{kK} e^{-\phi}\, dV$$

$$+ \sum_K{}' \sum_{1\le j,k\le m} \int_{U\cap b\Omega} \rho_{jk} u_{jK} \bar u_{kK} e^{-\phi}\, dV$$

$$+ \sum_K{}' \sum_{\substack{j\in\{1,\dots,m,n\}\\ m<k<n}} \int_{U\cap\Omega} \phi_{jk} u_{jK} \bar u_{kK} e^{-\phi}\, dV$$

$$+ \text{square terms}\Bigg)$$

Let us consider $\phi = -(-\rho)^\eta + M\chi$ where $0 < \eta < 1$, χ is the function in assumption (iii) of the theorem and M is a large number.

Since

$$\phi_{ij} = \eta(-\rho)^{\eta-1}\partial\bar\partial\rho\big(L_i, \bar L_j\big) - \eta(\eta-1)(-\rho)^{\eta-2}(L_i\rho)\big(\bar L_j\rho\big) + M\partial\bar\partial\chi\big(L_i, \bar L_j\big),$$

it is easy to see that

$$|\phi_{ij}| \le C\big((-\rho)^{\eta-1} + M\big), i = n, j \ne n,$$
$$\phi_{nn} \ge C\big((-\rho)^{\eta-2} - (-\rho)^{\eta-1} - M\big),$$
$$|\phi_{ij}| \le C \text{ for } 1 \le i \le m, m < j < n,$$

by assumptions (i) and (iii).

Hence for each $q-1$ tuple K, we have

$$\sum_{j,k=1}^{m} \phi_{jk} u_{jK} \bar{u}_{kK} \ge \text{const}\left(M \sum_{j=1}^{m} |u_{jK}|^2\right)$$
$$\phi_{nn}|u_{nK}|^2 \ge \text{const}\big((-\rho)^{\eta-2} - (-\rho)^{\eta-1} - M\big)|u_{nK}|^2$$
$$|\phi_{in} u_{iK} \bar{u}_{nK}| \le \text{const}\Big(|u_{iK}|^2 + \big((-\rho)^{2\eta-2} + M^2\big)|u_{nK}|^2\Big) \quad \text{for } i \ne n$$
$$|\phi_{jk} u_{jK} \bar{u}_{kK}| \le \text{const}\big(|u_{jK}|^2 + |u_{kK}|^2\big) \quad \text{for } 1 \le j \le m, m < k < n$$

Since $q \ge n - m$, every q-tuple J must contain an element in $\{1, \dots, m, n\}$. If we set M large enough putting the above inequalities into (14) we see that

$$\begin{aligned} &\|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|u\|_\phi^2 \\ &\quad \ge \text{const}\Bigg(M \sum_{n \notin J} \|u_J\|_\phi^2 \\ &\qquad + \sum_L \int_{U \cap b\Omega} \big((-\rho)^{\eta-2} - (-\rho)^{2\eta-2} - (-\rho)^{\eta-1} - M^2\big)|u_{nL}|^2 e^{-\phi}\, dV \end{aligned}$$

It follows that

$$\|u\|_\phi^2 \le \text{const}\big(\|Su\|_\phi^2 + \|T^*u\|_\phi^2 + \|gu\|_\phi^2\big)$$

for $u \in \mathfrak{D}_{(p,q)}(U \cap \overline{\Omega})$ where $g \in C_0^\infty(\Omega)$.

The rest of the proof goes through without changes as in the proof of Theorem 1.

Remarks. (1) The assumption (ii) and (iii) of theorem is true typically for domains where locally we can write the defining function as

$$\rho = \operatorname{Re} z_n + \rho_1(z_1, \dots, z_m) + \rho_2(z_{m+1}, \dots, z_{n-1})$$

where ρ_1 is a plurisubharmonic function, ρ_2 is any C^∞ function and $\rho_1(0) = \rho_2(0) = 0$.

(2) It seems that condition (ii) of Theorem 3 is difficult to satisfy. However, near points where there are m positive eigenvalues of the Levi form with $m \geq n - q$, the inequality (9) is satisfied in a neighborhood U without assumptions (ii) and (iii). So we really only need the assumptions near points where the positive eigenvalues of the Levi-form degenerate.

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