

TANGENTIAL-EXCEPTIONAL SETS FOR HARDY-SOBOLEV SPACES¹

CARME CASCANTE AND JOAQUÍN M. ORTEGA

Section 1. Introduction

Let B denote the unit ball in \mathbb{C}^n , and S its boundary. Let $d\sigma$ be the normalized Lebesgue measure on S . For $\alpha > 0$ and $0 < p < +\infty$, the Hardy-Sobolev space H_α^p is the space of holomorphic functions f in B so that $R^\alpha f \in H^p(B)$, where if $f = \sum_k f_k$ is its homogeneous expansion, $R^\alpha f = \sum_k (k+1)^\alpha f_k$. It is well known that for $\alpha p > n$, this space consists of Lipschitz functions.

In recent years there has been a great number of works dealing with the convergence along tangential approach regions of functions in the space of Poisson integrals of Bessel potentials of H^p functions. It turns out that the "tangentiality" of the approach region depends on n , α and p (see [N-R-S], [N-S] and [A-N]), and it flattens as the order of regularity increases. In the unit ball case, and for $p > 1$, it is easy to see, following [N-R-S] how these approach regions look.

If $f \in H_\alpha^p$, $n - \alpha p > 0$ and $\zeta \in S$, then

$$|f(z)| \leq C \left[|1 - z\bar{\zeta}|^{n/p} (1 - |z|)^{\alpha - n/p} M_p R^\alpha f(\zeta) + M_1 R^\alpha f(\zeta) \right] \quad (1)$$

where M_1 denotes the Hardy-Littlewood maximal function, and where $M_p f = (M_1 |f|^p)^{1/p}$.

In the extreme case $\alpha p = n$, the above pointwise estimate is replaced by

$$|f(z)| \leq C \left[|1 - z\bar{\zeta}|^{n/p} \left(\log \frac{1}{1 - |z|} \right)^{1-1/p} M_p R^\alpha f(\zeta) + M_1 R^\alpha f(\zeta) \right]. \quad (2)$$

Now, if we define

$$\Omega(\zeta) = \{z \in B; |1 - z\bar{\zeta}| \leq (1 - |z|)^{1-\alpha p/n}\},$$

Received September 28, 1992.

1991 Mathematics Subject Classification. Primary 32A40.

¹Partially supported by DGICYT grants.

respectively

$$\mathcal{E}(\zeta) = \left\{ z \in B; |1 - z\bar{\zeta}| < \frac{1}{\left(\log \frac{1}{1 - |z|}\right)^{(p-1)/n}} \right\},$$

and denote by Mf , respectively Pf , the corresponding maximal operators, (1) and (2) say that they are of weak type (p, p) . In particular if $f \in H_\alpha^p$, there exists $\lim f(z)$, as z approaches ζ , z in the tangential region, for almost every $\zeta \in S$.

Our purpose is to study the size of the exceptional set where the limit of a function in H_α^p along some intermediate tangential regions fails to exist. For the real case, the first result is in [A-N], and problems in the same direction for a class of holomorphic functions has been obtained in [Su] (see [Ci-Do-Su] and [Do] for related results).

Before stating more precisely our results, we need some more definitions. For $0 < \delta \leq +\infty$, and $\omega(t)$ a non-decreasing function in $[0, +\infty]$, vanishing at zero and satisfying $\omega(2t) \leq c\omega(t)$, and for $E \subset S$,

$$H_\delta^\omega(E) = \inf \left\{ \sum_j \omega(\delta_j); E \subset \cup B(\zeta_j, \delta_j), \delta_j \leq \delta \right\},$$

where $B(\zeta_j, \delta_j)$ is a non-isotropic ball. The non-isotropic Hausdorff measure is then defined by

$$H^\omega(E) = \lim_{\delta \rightarrow 0} H_\delta^\omega(E).$$

It is well known that H_∞^ω and H^ω have the same zero sets.

In Section 2 we begin with a characterization of the Hardy-Sobolev spaces H_α^p , $p > 1$, obtaining a representation in terms of a "fractional Cauchy-type" transform of functions in $L^p(d\sigma)$. As an immediate corollary we deduce from this representation and the results in [Su], the desired size of the tangential-exceptional set of H_α^p functions. The case $p \leq 1$ follows directly from the methods in [A].

The third section deals with the extreme case $\alpha p = n$. We obtain a necessary condition for $E \subset S$ to be an exceptional set with respect to tangential regions $\mathcal{E}_\mu^\rho(\zeta)$ of exponential type. Similar results for the real case were obtained by [Do] (see also [Ci-Do-Su] for the other results in this line).

On the other hand, we also include two examples which give some information about the sharpness of the necessary condition.

As a final remark on notation, we adopt the usual convention writing by the same letter various absolute constants which values may differ in each occurrence.

Finally, we would like to thank P. Ahern and J. Bruna for some helpful conversations on parts of this paper. We also thank the referee for some suggestions.

Section 2

We begin this section with an integral representation for the Hardy-Sobolev spaces H_α^p , $p > 1$. Such representation can be viewed as a holomorphic nonisotropic version of the classical Calderon's identity between Sobolev and potential spaces. One of the inclusions is established in [A-Co]. Since to our knowledge there is no written proof of the other one, we include it here.

Let $1 < p < +\infty$ and $0 < \alpha < n$. For $f \in L^p(d\sigma)$, define

$$C_\alpha f(z) = \int_S \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}} d\sigma(\zeta),$$

and

$$C_\alpha^p = C_\alpha L^p(d\sigma), \quad \text{normed by } \|F\|_{C_\alpha^p} = \inf_{F=C_\alpha f} \|f\|_{L^p(d\sigma)}.$$

THEOREM 2.1. $H_\alpha^p = C_\alpha^p$, with equivalence of norms provided $1 < p, \alpha < n$.

Proof of Theorem 2.1. As we have already said, Lemma 2.2 in [A-Co] gives $\|C_\alpha f\|_{\alpha, p} \leq C\|f\|_{L^p(d\sigma)}$. Hence, we just need to prove that the map $C_\alpha : L^p \rightarrow H_\alpha^p$ is onto.

Let P_k be a homogeneous polynomial of degree k . Using Lemma 2.1 in [A-Co] it is easy to check (see page 433 in same work)

$$C_\alpha P_k(z) = \frac{\Gamma(n)\Gamma(n-\alpha+k)}{\Gamma(n-\alpha)\Gamma(n+k)} P_k(z). \quad (1)$$

Suppose first that α is an integer. Then $\Gamma(n+k)/\Gamma(n-\alpha+k)$ is a polynomial in k of degree α , and it can be written as

$$\frac{\Gamma(n+k)}{\Gamma(n-\alpha+k)} = a_\alpha(k+1)^\alpha + a_{\alpha-1}(k+1)^{\alpha-1} + \cdots + a_0.$$

Since $R^\alpha P_k(z) = (k+1)^\alpha P_k(z)$, the above formula together with (1), shows that

$$C_\alpha (a_\alpha R^\alpha + a_{\alpha-1} R^{\alpha-1} + \cdots + a_0 \text{Id}) P_k = \frac{\Gamma(n)}{\Gamma(n-\alpha)} P_k. \quad (2)$$

For $f \in H_\alpha^p$, define

$$Tf = \frac{\Gamma(n - \alpha)}{\Gamma(n)} (a_\alpha R^\alpha f + a_{\alpha-1} R^{\alpha-1} f + \cdots + a_0 f).$$

Then, T is a bounded operator from H_α^p to H^p (see [Gr] and [Kr]) and, by (2), for each $f \in H_\alpha^p$, $C_\alpha T f = f$, that is, C_α has a right inverse.

For general α , the asymptotic development in [T-E] and Stirling's formula, give that there exist $\lambda_i(\alpha, n)$, $i \in \mathbb{N}$, so that for each $r > 0$,

$$\lim_{k \rightarrow +\infty} (k+1)^r \left[1 - \frac{\Gamma(n+k-\alpha)}{\Gamma(n+k)} (\lambda_0(k+1)^\alpha + \cdots + \lambda_{r-1}(k+1)^{\alpha-r+1}) \right] = \lambda_r.$$

Let

$$b_k = 1 - \frac{\Gamma(n+k-\alpha)}{\Gamma(n+k)} (\lambda_0(k+1)^\alpha + \cdots + \lambda_{r-1}(k+1)^{\alpha-r+1});$$

the above convergence says that there exists k_0 so that for $k \geq k_0$,

$$|b_k| < \frac{2|\lambda_r| + 1}{(k+1)^r}. \quad (3)$$

Let T be the operator defined by

$$T(P_k) = \begin{cases} \frac{\Gamma(n-\alpha)\Gamma(n+k)}{\Gamma(n)\Gamma(n-\alpha+k)} P_k & \text{if } k < k_0, \\ \frac{\Gamma(n-\alpha)}{\Gamma(n)} ((k+1)^\alpha \lambda_0 + \cdots + (k+1)^{\alpha-r+1} \lambda_{r-1}) P_k & \text{if } k \geq k_0, \end{cases}$$

where P_k is a homogeneous polynomial of degree k .

Again as a consequence of [Gr] and [Kr], for each $r \in \mathbb{N}$, $T: H_\alpha^p \rightarrow H^p$, and we will see that provided r and k_0 are chosen big enough ($r = z$ is sufficient), the operator $I - C_\alpha T: H_\alpha^p \rightarrow H_\alpha^p$ has norm strictly less than one. This is equivalent to say that if we let $T_1 = R^\alpha(I - C_\alpha T)$, there exists $\varepsilon < 1$ such that

$$\|T_1 f\|_{H^p} \leq \varepsilon \|f\|_{p, \alpha}.$$

Now, if $f \in H_\alpha^p$, let $f = \sum_{k \geq 0} P_k$ be its homogeneous expansion. Then

$$T_1 f = \sum_{k \geq k_0} (k+1)^\alpha b_k P_k.$$

First suppose $1 < p \leq 2$. Integrating on slices (see [Ru, page 15]) we get

$$\begin{aligned} \|T_1 f\|_2^2 &= \int_S \frac{1}{2\pi} \int_0^{2\pi} |T_1 f(e^{i\theta}\zeta)|^2 d\theta d\sigma(\zeta) \\ &= \sum_{k \geq k_0} (k+1)^{2\alpha} |b_k|^2 \int_S |P_k(\zeta)|^2 d\sigma(\zeta) \\ &\leq c \sum_{k \geq k_0} \frac{1}{(k+1)^{2r}} \int_S |R^\alpha P_k(\zeta)|^2 d\sigma(\zeta), \end{aligned}$$

where in last inequality we have used (3).

Now applying Theorem 2.1 in [A-B] to each $R^\alpha P_k$, we obtain

$$\|T_1 f\|_2^2 \leq c \sum_{k \geq k_0} \frac{1}{(k+1)^{2r-2}} \|R^\alpha f\|_1^2 \leq \varepsilon'^2 \|R^\alpha f\|_p^2,$$

where $\varepsilon' < 1$, if r and k_0 are chosen big enough.

Finally, since $p \leq 2$, we deduce that there exists $\varepsilon < 1$ so that

$$\|T_1 f\|_p < \varepsilon \|f\|_{p,\alpha}.$$

If $p > 2$, integrating again on slices, and applying Theorem 6.3 in [Du] to each one of the slices we obtain

$$\begin{aligned} \|T_1 f\|_p^p &= \frac{1}{2\pi} \int_S \int_0^{2\pi} |T_1 f(e^{i\theta}\zeta)|^p d\theta d\sigma(\zeta) \\ &\leq \sum_{k \geq k_0} \int_S (k+1)^{p-2+\alpha p} |b_k|^p |P_k(\zeta)|^p d\sigma(\zeta) \\ &\leq c \sum_{k \geq k_0} \int_S \frac{1}{(k+1)^{r p - p + 2}} |R^\alpha P_k(\zeta)|^p d\sigma(\zeta) \\ &\leq c \sum_{k \geq k_0} \frac{1}{(k+1)^{p(r-2)+2}} \|R^\alpha f\|_1^p < \varepsilon' \|R^\alpha f\|_p^p \end{aligned}$$

provided K_0 and r are big enough.

Hence $\|T_1 f\|_p \leq \varepsilon \|R^\alpha f\|_p = \varepsilon \|f\|_{p,\alpha}$, and $C_\alpha T$ is invertible in H_α^p .

In particular there exists $S: H_\alpha^p \rightarrow H_\alpha^p$ with $C_\alpha TS = Id$. Since $T: H_\alpha^p \rightarrow H^p$, we are done. ■

Before stating a result concerning the size of the tangential-exceptional sets for H_α^p , we need some definitions. Assume $n - \alpha p > 0$, and let $\zeta \in S$, $\tau \geq 1$ and $\beta > 0$. Define the tangential approach region

$$\Omega_\tau(\zeta) = \Omega_{\tau,\beta}(\zeta) = \{z \in B; |1 - z\bar{\zeta}|^\tau < \beta(1 - |z|)\},$$

and if $f: B \rightarrow \mathbb{C}$, let

$$M_\tau f(\zeta) = M_{\tau,\beta} f(\zeta) = \sup_{z \in \Omega_\tau(\zeta)} |f(z)|$$

be the corresponding maximal function. Notice that if $\tau = 1$, $\Omega_1(\zeta) = D(\zeta)$ is the usual admissible region, and we will denote $M_1 f$ by Nf .

THEOREM 2.2. *Suppose $0 < p < +\infty$, $\tau > 1$, $\alpha p < n$ and $m = \tau(n - \alpha p)$. Let ν be a positive Borel measure on S so that*

$$\nu(B(\zeta, \delta)) = O(\delta^m) \quad \text{for all } \zeta \in S, \delta > 0.$$

Then there exists $C > 0$ such that for each $f \in H_\alpha^p$,

$$\|M_\tau f\|_{L^p(d\nu)} \leq C \|f\|_{p,\alpha}.$$

Proof of Theorem 2.2. The case $p > 1$ follows immediately from theorem 3.8 in [Su], where the same conclusion is proved for C_α^p , and the previous Theorem 2.1.

The remaining case $0 < p \leq 1$ can be shown using the same methods of Theorem 1.1 in [A], where the admissible case, $\tau = 1$ is considered. ■

Theorem 2.2 together with a non-isotropic Frostman type theorem in [Co], and the same kind of construction of holomorphic functions in H_α^p with prescribed ‘‘tangential’’ exceptional sets in [A-Co], lead to the following characterization of such exceptional sets.

COROLLARY 2.1. *Let $E \subset S$ compact, $0 < p < +\infty$, $\alpha p < n$ and $m = \tau(n - \alpha p)$ with $\tau > 1$. Then $E = E(f)$ for some $f \in H_\alpha^p$ if and only if $H^m(E) = 0$, where*

$$E(f) = \{\zeta \in S; \nexists \lim f(z), z \rightarrow \zeta \in \Omega_\tau(\zeta)\}.$$

Remark 2.1. Theorem 2.2 and its corollary still remains valid for $F = P[K_\alpha * g]$, $g \in L^p(d\sigma)$, where $\alpha \in \mathbb{N}$ and $0 < \alpha < n$, $0 < n - \alpha p$, and K_α

is the non-isotropic Riesz kernel given by:

$$K_\alpha(z, \zeta) = \frac{1}{|1 - z\bar{\zeta}|^{n-\alpha}} \quad z, \zeta \in S,$$

and where $P(z, \zeta)$ is the Poisson-Szegö kernel. The proof can still be used, since if F is such an \mathcal{H} -harmonic function, then $N(R^\alpha F) \in L^p(d\sigma)$ (see [A-Ca]).

Section 3

We have seen in the previous section that the “wideness” of the tangential approach regions for functions in H_α^p , flattens as $n - \alpha p$ goes to zero. On the other hand, if $f \in H_\alpha^p$, and $n - \alpha p < 0$, f is a continuous function up to the boundary. So one would expect that in the limit case $n = \alpha p$ (as it happens in the real case), the convergence of a function in H_α^p exists within a much wider region.

Let $1 < p < +\infty$, $\mu \geq 1$ and define for $\zeta \in S$ and $C > 0$

$$\mathcal{E}_\mu(\zeta) = \mathcal{E}_{\mu,p}^C(\zeta) = \left\{ z; |1 - z\bar{\zeta}| < \frac{C}{\left(\log \frac{1}{1 - |z|}\right)^{(p-1)\mu/n}} \right\},$$

and for f defined on B , let $P_\mu f$ be the corresponding maximal function.

THEOREM 3.1. *Let ν be a positive Borel measure on S satisfying*

$$\nu(B(\zeta, \delta)) = O\left(\delta^{n/\tau} \left(\log \frac{1}{\delta}\right)^{1-q}\right), \quad \text{where } q > p, \mu > 1.$$

Then there exists $C > 0$ so that if $f \in H_\alpha^p$, $\alpha p = n$,

$$\int_S |P_\mu f(\zeta)|^p d\nu(\zeta) \leq C \|f\|_{p,\alpha}^p.$$

Proof of Theorem 3.1. For $f \in H_\alpha^p$ we write

$$f(z) = c(\alpha) \int_0^1 \left(\log \frac{1}{t}\right)^{\alpha-1} g(tz) dt,$$

where $g = R^\alpha f$. Thus the theorem will be proved once we show

$$\int_S |P_\mu f(\zeta)|^p d\nu(\zeta) \leq C \|Ng\|_{L^p(d\sigma)}^p.$$

Breaking the integral defining f in two pieces, from 0 to 1/2 and from 1/2 to 1, it is enough to show that if $g \in H^p$, and

$$G(z) = \int_{1/2}^1 (1-t)^{\alpha-1} g(tz) dt,$$

then

$$\int_S |P_\mu G(\zeta)|^p d\nu(\zeta) \leq C \|Ng\|_{L^p(d\sigma)}^p.$$

By Hölder's inequality

$$\begin{aligned} |G(z)| &\leq \left\{ \int_{1/2}^1 (1-t)^{-1} \left(\log \frac{1}{1-t} \right)^{(1-p)p'/p} \left(\log \log \frac{1}{1-t} \right)^{(1-q)p'/p} dt \right\}^{1/p'} \\ &\quad \cdot \left\{ \int_{1/2}^1 (1-t)^{n-1} \left(\log \frac{1}{1-t} \right)^{p-1} \right. \\ &\quad \left. \times \left(\log \log \frac{1}{1-t} \right)^{q-1} |g(tz)|^p dt \right\}^{1/p}. \end{aligned}$$

Since $q > p > 1$, the first integral converges. Since $N|g|^p \in L^1(d\sigma)$, and

$$\|N|g|^p\|_{L^1(d\sigma)} = \|N|g|\|_{L^p(d\sigma)}^p,$$

we may apply Lemma 2.1 in [A-N], and write

$$|g(z)|^p \leq \sum_{k \geq 1} \lambda_k a_k(z), \quad z \in B,$$

where

$$\sum_{k \geq 1} \lambda_k \leq C \|Ng\|_{L^p(d\sigma)}^p,$$

and each a_k is a non-negative α -atom satisfying:

(a) there exists $\zeta_k \in S$, $\delta_k > 0$ so that

$$\text{supp } a_k \subset T(B(\zeta_k, \delta_k)),$$

where the tent $T(B(\zeta_k, \delta_k)) = B \setminus \cup D(\eta)$, and the union is over all $\eta \in S \setminus B(\zeta_k, \delta_k)$;

(b) $a_k(z) \leq \delta_k^{-n}$, for all $z \in B$.

Thus we only need to prove that

$$\int_S \sup_{z \in \mathcal{E}_\mu(\zeta)} \int_0^1 (1-t)^{n-1} \left(\log \frac{1}{1-t} \right)^{p-1} \left(\log \log \frac{1}{1-t} \right)^{q-1} a(tz) dt d\nu(\zeta) \quad (1)$$

is bounded independently of the atom a , which is supported in $T(B(\zeta_0, \delta))$.

Since a supported in $T(B(\zeta_0, \delta))$, the inner integral is in fact, from $1 - c\delta$ to 1 for some positive constant c .

First, suppose

$$\delta < \frac{C}{\left(\log \frac{1}{\delta} \right)^{(p-1)\mu/n}}.$$

Then $a(tz)$ is zero unless $\zeta \in \tilde{B}$, where

$$\tilde{B} = B \left(\zeta_0, \frac{C}{\left(\log \frac{1}{\delta} \right)^{(p-1)\mu/n}} \right).$$

Consequently, the inner integral above (1) is bounded by

$$\begin{aligned} & \int_{1-c\delta}^1 (1-t)^{n-1} \left(\log \frac{1}{1-t} \right)^{p-1} \left(\log \log \frac{1}{1-t} \right)^{q-1} \delta^{-n} \chi_{\tilde{B}}(\zeta) dt \\ & \leq C \left(\log \frac{1}{\delta} \right)^{p-1} \left(\log \log \frac{1}{\delta} \right)^{q-1} \chi_{\tilde{B}}(\zeta). \end{aligned}$$

Integrating with respect to ν and using the hypothesis, we deduce that (1) is bounded by

$$C \left(\log \log \frac{1}{\delta} \right)^{q-1} \left(\log \left(\frac{\left(\log \frac{1}{\delta} \right)^{(p-1)\mu/n}}{C} \right) \right)^{1-q} \leq C.$$

Finally, if

$$\delta \geq \frac{C}{\left(\log \frac{1}{\delta}\right)^{(p-1)\mu/n}},$$

δ is bounded from below, and $a(tz)$ is zero unless $\zeta \in B(\zeta_0, \delta)$. Hence (1) is bounded by

$$C \left(\log \frac{1}{\delta}\right)^{p-1} \left(\log \log \frac{1}{\delta}\right)^{q-1} \delta^{n/\tau} \left(\log \frac{1}{\delta}\right)^{1-q},$$

which is also bounded.

Remark 3.1. Theorem 3.1 can be used to prove the existence of limits of H_α^p function within “exponential” tangential regions \mathcal{E}_μ , along varieties. For instance, if Γ is a smooth curve and ν is the arc-length measure on it, it is well known that if Γ is transverse, $\nu(B(\zeta, \delta)) = O(\delta)$, whereas if it is complex-tangential $\nu(B(\zeta, \delta)) = O(\delta^{1/2})$. Thus if Γ is transverse (respectively, complex-tangential), $f \in H_\alpha^p$, $\alpha p = n$, and $\mu > n$ (respectively, $\mu > 2n$), $\lim f(z)$ exists as z approaches ζ , $z \in \mathcal{E}_\mu(\zeta)$ for almost every $\zeta \in \Gamma$ (with respect to arc-length). Note that in the transverse case, the tangential region is wider than in the complex-tangential case.

THEOREM 3.2. *Let $1 < p, \mu > 1$, $\alpha p = n$. Then there exists $C > 0$ so that*

$$H_\infty^{n/\mu}(\{P_\mu f(\zeta) > t\}) \leq C \frac{\|f\|_{p,\alpha}^p}{t^p},$$

for each $f \in H_\alpha^p$.

To prove Theorem 3.2 we need the following lemma.

LEMMA 3.1. *Let $1 < p$, $\alpha p = n$. There exists $C > 0$ so that for each $g \in L^p(d\sigma)$, $z_0 \in S$ and $z \in \mathcal{E}_\mu(z_0)$,*

$$\left| \int_S \frac{g(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}} d\sigma(\zeta) \right| \leq CT_{n-n/\mu} g(z_0),$$

where

$$T_{n-n/\mu} g(z_0) = \sup_{z_0 \in Q} \left(\frac{1}{|Q|^{1/\mu}} \int_Q |g(\zeta)|^p d\sigma(\zeta) \right)^{1/p},$$

(here Q denotes non-isotropic balls in S).

Proof of Lemma 3.1. Let $z_0 \in S$ and $z \in \mathcal{E}_\mu(z_0)$ and define $Q = \{\zeta; |1 - \zeta\bar{z}_0| < 4|1 - z\bar{z}_0|\}$. Then

$$\left| \int_S \frac{g(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}} d\sigma(\zeta) \right| \leq \int_Q \frac{|g(\zeta)|}{|1 - z\bar{\zeta}|^{n-\alpha}} d\sigma(\zeta) + \int_{Q^c} \frac{|g(\zeta)|}{|1 - z\bar{\zeta}|^{n-\alpha}} d\sigma(\zeta) \\ = \text{I} + \text{II},$$

and we will estimate both integrals separately.

In I we apply Hölder's inequality and we get

$$\text{I} \left(\int_Q |g(\zeta)|^p d\sigma(\zeta) \right)^{1/p} \left(\int_Q \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^{(n-\alpha)p'}} \right)^{1/p'} \\ \leq C \left(\int_Q |g(\zeta)|^p d\sigma(\zeta) \right)^{1/p} \left(\log \frac{1}{1 - |z|} \right)^{1/p'}.$$

Now, since $z \in \mathcal{E}_\mu(z_0)$, we have

$$\left(\log \frac{1}{1 - |z|} \right)^{1/p'} \leq |1 - z\bar{z}_0|^{-n/\mu p}.$$

Thus

$$\text{I} \leq C |1 - z\bar{z}_0|^{-n/\mu p} \left(\int_Q |g(\zeta)|^p d\sigma(\zeta) \right)^{1/p} \leq CT_{n-n/\mu} g(z_0).$$

In II, let $\delta = 4|1 - z\bar{z}_0|$. Then by Hölder's inequality

$$\text{II} \leq \sum_{z^k \delta < M} \frac{1}{(2^k \delta)^{n-n/p}} \cdot \left(\int_{|1 - \zeta\bar{z}_0| < 2^k \delta} |g(\zeta)|^p d\sigma(\zeta) \right)^{1/p} \cdot (2^k \delta)^{n/p} \\ \leq CT_{n-n/\mu} g(z_0). \quad \blacksquare$$

Proof of Theorem 3.2. Let $f \in H_\alpha^p$. By Theorem 2.1, $f = C_\alpha g$ with $g \in L^p(d\sigma)$. By Lemma 3.1,

$$P_\mu f(\zeta) \leq CT_{n-n/\mu} g(\zeta),$$

for any $\zeta \in S$, and since by Lemma 1 in [Do],

$$H_\infty^{n/\mu}(\{T_{n-n/\mu} g > t\}) \leq C \frac{\|g\|_p^p}{t^p},$$

we obtain the desired conclusion. \blacksquare

COROLLARY 3.1. *Let $1 < p$, $\alpha p = n$, $\mu > 1$. Then for each $f \in H_\alpha^p$ the limit $f(z)$ exists as z approaches $\zeta \in S$, z in $\mathcal{E}_\mu(\zeta)$, except for a set E with $H^{n/\mu}(E) = 0$.*

On the other direction we begin with the following construction

PROPOSITION 3.1. *Let $0 < m < n/\mu$, and $E \subset S$ compact so that $H^m(E) = 0$. Then there exists $p > 1$ and α with $\alpha p = n$, and $f \in H_\alpha^p$ so that for each $\zeta \in E$, the maximal function $P_\mu f(\zeta) = +\infty$. In particular,*

$$E \subset \{\zeta \in S; \nexists \lim f(z), z \rightarrow \zeta, z \in \mathcal{E}_\mu(\zeta)\}.$$

Proof of Proposition 3.1. Let $1 < p$ and $\alpha = n/p$ so that $n - \alpha < 1$. For each $z \in B$, let g_z be the function on S defined by

$$g_z(\zeta) = \frac{1}{|1 - z\bar{\zeta}|^\alpha} \frac{1}{\left(\log \frac{1}{|1 - z\bar{\zeta}|}\right)^r}, \quad \text{where } r > 1.$$

We will first see that $g_z \in L^p(d\sigma)$ and

$$\|g_z\|_{L^p(d\sigma)} \leq \frac{C}{\left(\log \frac{1}{1 - |z|}\right)^{r-1/p}}.$$

Indeed, let $z_0 = z/|z|$ and $B_0 = B(z_0, 4(1 - |z|))$. Then

$$\begin{aligned} \int_S |g_z(\zeta)|^p d\sigma(\zeta) &= \int_{B_0} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n \left(\log \frac{1}{|1 - z\bar{\zeta}|}\right)^{rp}} \\ &\quad + \int_{B_0^c} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n \left(\log \frac{1}{|1 - z\bar{\zeta}|}\right)^{rp}} = \text{I} + \text{II}. \end{aligned}$$

In I, $|1 - z\bar{\zeta}| \approx 1 - |z|$. Hence

$$\text{I} \leq \frac{C}{\left(\log \frac{1}{1 - |z|}\right)^{rp}} \int_{B_0} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n} \leq C \frac{1}{\left(\log \frac{1}{1 - |z|}\right)^{rp-1}}.$$

In II, let

$$B_k = \{\zeta \in S; 4 \cdot 2^k(1 - |z|) \leq |1 - \zeta \bar{z}_0| \leq 4 \cdot 2^{k+1}(1 - |z|)\}.$$

Then

$$\begin{aligned} \text{II} &\simeq C \sum_{k < C \log \frac{1}{1-|z|}} \int_{B_{k+1} \setminus B_k} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n \left(\log \frac{1}{|1 - z\bar{\zeta}|} \right)^{rp}} \\ &\leq C \sum_{k < C \log \frac{1}{1-|z|}} \frac{1}{\left(\log \frac{1}{2^k(1-|z|)} \right)^{rp}} \simeq C \frac{1}{\left(\log \frac{1}{1-|z|} \right)^{rp-1}}. \end{aligned}$$

Next, define for $z \in B$, the holomorphic function defined by $f_z(\omega) = C_\alpha g_z(\omega)$. By Theorem 2.1,

$$f_z \in H_\alpha^p \quad \text{and} \quad \|f_z\|_{p,\alpha} \leq \frac{C}{\left(\log \frac{1}{1-|z|} \right)^{r-1/p}}.$$

Since $n - \alpha < 1$, $\text{Re} f_z(\omega) \simeq K_\alpha * g_z(\omega) \geq 0$ and

$$\text{Re} f_z(z) \geq C \int \frac{1}{|1 - z\bar{\zeta}|^n} \frac{1}{\left(\log \frac{1}{|1 - z\bar{\zeta}|} \right)^r} d\sigma(\zeta) \geq \frac{C}{\left(\log \frac{1}{1-|z|} \right)^{r-1}}$$

(where in last inequality we have used (1) for $p = 1$). Thus if we define

$$h_z(\omega) = \left(\log \frac{1}{1-|z|} \right)^{r-1/p} f_z(\omega),$$

by the above estimates we have

- (i) $\|h_z\|_{p,\alpha} \leq C$,
- (ii) $\text{Re} h_z \geq 0$ and

$$\text{Re} h_z(z) \geq C \left(\log \frac{1}{1-|z|} \right)^{1-1/p}.$$

Let $E \subset S$ be compact with $H^m(E) = 0$. For each $k \in \mathbb{N}$ let $\{B(\zeta_{jk}, \delta_{jk})\}_j$ be a disjoint family of non-isotropic balls with $E \subseteq \cup B(\zeta_{jk}, C_1 \delta_{jk})$ and $\sum_j \delta_{jk}^m < 1/2^k$, $C_1 > 0$ an absolute constant.

Define $z_{jk} \in B$ so that $\zeta_{jk} = z_{jk}/|z_{jk}|$ and $1 - |z_{jk}| = \varepsilon_{jk}$, where

$$\delta_{jk} = \frac{C}{\left(\log \frac{1}{\varepsilon_{jk}}\right)^{(p-1)\mu/n}}.$$

Let $F_k = \sum_j \delta_{jk}^m h_{jk}$ and $F = \sum_k F_k$, where $h_{jk} = h_{z_{jk}}$. By (i) and the election of δ_{jk} , $F \in H_\alpha^p$, and by (ii),

$$\begin{aligned} \operatorname{Re} F(z_{jk}) &\geq \operatorname{Re} F_k(z_{jk}) \geq C \delta_{jk}^m \left(\log \frac{1}{\varepsilon_{jk}}\right)^{1-1/p} \\ &\geq C \delta_{jk}^{m-n/\mu p} \rightarrow +\infty \quad \text{as } k \rightarrow +\infty. \end{aligned}$$

provided we choose $1 < p$ with $m - n/\mu p < 0$.

Next take $\zeta \in E$ and let $k \in \mathbf{N}$ be fixed. There exists $j \in \mathbf{N}$ so that $\zeta \in B(\zeta_{jk}, C_1 \delta_{jk})$ and since $\varepsilon_{jk} < \delta_{jk}$,

$$\begin{aligned} |1 - z_{jk} \bar{\zeta}| &\leq 2\left((1 - |z_{jk}|) + |1 - \zeta_{jk} \bar{\zeta}|\right) \leq 2(\varepsilon_{jk} + C_1 \delta_{jk}) \\ &\leq \frac{C}{\left(\log \frac{1}{\varepsilon_{jk}}\right)^{(p-1)\mu/n}} = \frac{C}{\left(\log \frac{1}{1 - |z_{jk}|}\right)^{(p-1)\mu/n}} \end{aligned}$$

we have $z_{jk} \in \mathcal{E}_\mu(\zeta)$. Hence $P_\mu F(\zeta) = +\infty$ for every $\zeta \in E$. ■

Finally, for $p = 2$ we can give one more example.

PROPOSITION 3.2. *Let $E \subset S$ be a compact set with $\operatorname{diam}(E) < 1$ so that $H^m(E) = 0$, $m < n$. Then for every $\mu > 1$ with $(n + m)/2 < n/\mu$, there is $f \in H_\alpha^2$, $\alpha = n/2$, so that*

$$E = \left\{ \zeta \in S; \nexists \lim f(z), z \rightarrow \zeta, z \in \mathcal{E}_\mu(\zeta) \right\},$$

where

$$\mathcal{E}_\mu(\zeta) = \left\{ z; |1 - z \bar{\zeta}| < \frac{C}{\left(\log \frac{1}{1 - |z|}\right)^{\mu/n}} \right\}.$$

The proof is based in the following lemma.

LEMMA 3.2. *Let $m < n, \mu > 1$ be as in Proposition 3.2 and let $\eta < 1$. There exists $\delta < 0$ so that for any finite disjoint collection $\{B(\zeta_j, \delta_j)\}$ of pairwise disjoint non-isotropic balls, with $\delta_j < \delta, \zeta_j \in S$ and $|1 - \zeta_j \bar{\zeta}_j| < \eta$ for every j, k , then there exists $F \in H^\infty(B)$ satisfying:*

- (i) $\operatorname{Re} F(z) \geq 0$ for z closed enough to $\cup\{\zeta_j\}$, $\operatorname{Re} F(z_j) \geq C$ if $z_j/|z_j| = \zeta_j, 1 - |z_j| = \varepsilon_j$ with $\varepsilon_j = e^{-(c/\delta_j)^{n/\mu}}$.
- (ii) $\|F\|_{2, \alpha}^2 \leq C \sum_j \delta_j^m, \alpha = n/2$.

Proof of Lemma 3.2. If h is the holomorphic function on D given by

$$h(w) = \frac{1}{w} \log \frac{1}{1-w},$$

for each $z \in B$ we define the holomorphic function on B given by

$$f_z(\omega) = h(\bar{z}\omega) = \frac{1}{\bar{z}\omega} \log \frac{1}{1-\bar{z}\omega}.$$

Taking real parts we obtain

$$\operatorname{Re} f_z(\omega) = \operatorname{Re} \frac{1}{\bar{z}\omega} \log \frac{1}{|1-\bar{z}\omega|} - \operatorname{Im} \frac{1}{\bar{z}\omega} \operatorname{Arg} \frac{1}{1-\bar{z}\omega}.$$

Since

$$\operatorname{Im} \frac{1}{\bar{z}\omega} \operatorname{Arg} \frac{1}{1-\bar{z}\omega} \leq 0$$

for each z, ω in B , in order to see that $\operatorname{Re} f_z(\omega) \geq 0$, we just need to prove that the product of real parts is positive. If z and ω are chosen sufficiently close to $\cup\{\zeta_j\}$ this is deduced from the hypothesis on the ζ_j 's.

Defining

$$F(z) = \sum_j \left(\log \frac{1}{\varepsilon_j} \right)^{-1} f_{z_j}(z), \quad z \in B,$$

we deduce (i) from the above, and from the fact that $f_z(z) \geq C \log 1/(1-|z|)$. In order to prove (ii), we will compute the norm by duality.

If we define the polynomial in R of degree n given by

$$Q = Q(R) = (R + (n-2)\operatorname{Id}) \dots (R + \operatorname{Id}) R^2,$$

it is then immediate to check that

$$Qf_z(\omega) = \frac{C(n)}{(1 - \bar{z}\omega)^n}.$$

Associated to Q we define the operators \tilde{Q}_α and $\tilde{Q}_{-\alpha}$, $\alpha = n/2$, by

$$\begin{aligned}\tilde{Q}_\alpha P_k &= ((k + (n - 1)) \dots (k + 2)(k + 1)^2)^{1/2} P_k, \\ \tilde{Q}_{-\alpha} P_k &= ((k + (n - 1)) \dots (k + 2)(k + 1)^2)^{-1/2} P_k,\end{aligned}$$

where P_k is a homogeneous polynomial of degree k . Then, if g is a holomorphic function in a neighbourhood of \bar{B} ,

$$\begin{aligned}\left| \int_S \overline{\tilde{Q}_\alpha F(\omega)} g(\omega) d\sigma(\omega) \right| &= \left| \int_S \overline{QF(\omega)} \cdot \tilde{Q}_{-\alpha} g(\omega) d\sigma(\omega) \right| \\ &= C(n) \left| \sum_j \int_S \left(\log \frac{1}{\varepsilon_j} \right)^{-1} \right. \\ &\quad \left. \times \frac{1}{(1 - z_j \bar{\omega})^n} \tilde{Q}_{-\alpha} g(\omega) d\sigma(\omega) \right|.\end{aligned}$$

Using Cauchy's integral formula and Schwarz's inequality, the last expression is bounded by

$$C \left(\sum_j \left(\frac{1}{\log \frac{1}{\varepsilon_j}} \right)^\lambda \right)^{1/2} \left(\sum_j \left(\log \frac{1}{\varepsilon_j} \right)^{\lambda-2} |\tilde{Q}_\alpha g(z_j)|^2 \right)^{1/2}, \quad (2)$$

where $\lambda < 1$ is to be chosen.

Since $\log 1/\varepsilon_j = \delta_j^{-n/\mu}$, if we choose λ so that $\lambda n/\mu = m$ we have (2) bounded by

$$C \left(\sum_j \delta_j^m \right)^{1/2} \cdot \left(\sum_j \delta_j^{(2n/\mu)-m} \cdot |\tilde{Q}_{-\alpha} g(z_j)|^2 \right)^{1/2} \quad (3)$$

Next, for each $\zeta \in B(\zeta_j, c\delta_j)$, $z_j \in \mathcal{E}_\mu(\zeta)$. Taking infimums, we deduce that

$$|\tilde{Q}_{-\alpha} g(z_j)| \leq \inf_{\zeta \in B(\zeta_j, \delta_j)} P_\mu \tilde{Q}_{-\alpha} g(\zeta).$$

Since $2n/\mu - m > n$ and the balls are disjoint, we get

$$\begin{aligned} & C \left(\sum_j \delta_j^m \right)^{1/2} \left(\int_S |P_\mu \tilde{Q}_{-\alpha} g(\zeta)|^2 d\sigma(\zeta) \right)^{1/2} \\ & \leq C \left(\sum_j \delta_j^m \right)^{1/2} \|R^\alpha \tilde{Q}_{-\alpha} g\|_{L^2(d\sigma)} \simeq C \left(\sum_j \delta_j^m \right)^{1/2} \|g\|_{L^2(d\sigma)}, \end{aligned}$$

where in the first estimate we have applied Theorem 3.1, and the last estimate is deduced from the fact that $R^\alpha \tilde{Q}_{-\alpha}$ is a multiplier in $L^2(d\sigma)$.

Finally, we deduce from the above that

$$\|\tilde{Q}_\alpha F\|_2 \leq C \sum \delta_j^m,$$

and since $\|\tilde{Q}_\alpha F\|_2 \simeq \|R^\alpha F\|_2$, we have proved (ii) ■

Proof of Proposition 3.2. With fixed $k \in \mathbb{N}$, let $\{B(\zeta_{jk}, \delta_{jk})\}$, be a family of non-isotropic disjoint balls satisfying $E \subset \cup_j B(\zeta_{jk}, c_1 \delta_{jk})$ and let $m_k \nearrow +\infty$ so that $m_k (\sum_j \delta_{jk}^m)^{1/2} \leq 1/2^k$. Let z_{jk} and F_k be as in Lemma 3.2, and define $F = \sum_k m_k F_k$. Then, (ii) of Lemma 3.2, together with the election of the m_k 's give that $F \in H_\alpha^2$.

On the other hand, by (i) of last lemma,

$$\operatorname{Re} F(z_{jk}) \geq \operatorname{Re} m_k F_k(z_{jk}) \geq C m_k.$$

Proceeding in the same way as in Proposition 3.1, we deduce from the above, that for each $\zeta \in E$ and for each $k \in \mathbb{N}$, there exists $z_{jk} \in \mathcal{E}_\mu(\zeta)$, with $\operatorname{Re} F(z_{jk}) \geq C m_k$. Hence $P_\mu F(\zeta) = +\infty$ for each $\zeta \in E$.

Finally, from the definition of the E_k 's we deduce easily that if $\omega \in K \subset \bar{B} \setminus E$, K compact, there exists $C = C(K) > 0$ with $|F_k(\omega)| \leq C \sum_j \delta_{jk}^{n/\mu}$, and since $m < n/\mu$, the series defining F converges uniformly over the compact sets of $\bar{B} \setminus E$. ■

Remark 3.2. The same methods used in Proposition 3.2 actually show the following statement. Suppose F is an s -set ($s \leq n$); i.e., suppose there exists a positive measure ν on S and $c_1, c_2 > 0$ so that

$$c_1 \delta^s \leq \nu(B(\zeta, \delta)) \leq c_2 \delta^s$$

for each $\zeta \in F$, $\delta > 0$. Assume $E \subset F$ is compact with $\operatorname{diam} E < 1$ and $H^m(E) = 0$, $m < s$. Then E is an exceptional set (in the previous sense) with respect to \mathcal{E}_μ for each $\mu > 1$ so that

$$\frac{s+m}{2} < \frac{n}{\mu} < s.$$

Added in proof. We have recently proved that the estimate in Theorem 3.1 is true if $\nu(B(\zeta, \delta)) = O(\delta^{n/\mu})$ and that the condition $H^{n/\mu}(E) = 0$ in fact characterizes the exceptional sets in Corollary 3.1, provided $1 < p \leq 2$.

REFERENCES

- [A] P. AHERN, *Exceptional sets for holomorphic Sobolev functions*, Michigan Math. J. **35** (1988), 29–41.
- [A-B] P. AHERN and J. BRUNA, *On holomorphic functions in the ball with absolutely continuous boundary values*, Duke Math. J. **58** (1988), 129–141.
- [A-Ca] P. AHERN and C. CASCANTE, *Exceptional sets for Poisson integrals of potential on the unit sphere in \mathbb{C}^n , $p \leq 1$* . Pacific J. Math. **153** (1992), 1–13.
- [A-Co] P. AHERN and W. COHN, *Exceptional sets for Hardy-Sobolev functions, $p > 1$* , Indiana Math. J. **38** (1989) 417–451.
- [A-N] P. AHERN and A. NAGEL, *Strong L^p estimates for maximal functions with respect to singular measures; with applications to exceptional sets*. Duke Math. J. **53** (1986), 359–393.
- [Ci-Do-Su] P. CIFUENTES, J.R. DORRONSORO and J. SUEIRO, *Boundary tangential convergence on spaces of homogeneous type*, Preprint 1989.
- [Co] W. COHN, *Non isotropic Hausdorff measure and exceptional sets for holomorphic Sobolev functions*, Illinois Math. J. **33** (1989), 673–690.
- [Do] J.R. DORRONSORO, *Poisson integrals of regular functions*, Trans. Amer. Math. Soc. **297** (1986), 669–685.
- [Du] P. DUREN, *Theory of H^p spaces*, Academic Press, San Diego, 1970.
- [Gr] I. GRAHAM, *An H^p space theorem for the radial derivative of holomorphic functions on the unit ball in \mathbb{C}^n* , preprint.
- [Kr] S. KRANTZ, *Analysis on the Heisenberg group and estimates for functions in Hardy classes of several complex variables*, Math. Ann. **244** (1979), 243–262.
- [N-R-S] A. NAGEL, W. RUDIN and J. SHAPIRO, *Tangential boundary behavior of functions in Dirichlet-type spaces*. Ann. of Math. **116** (1982), 331–360.
- [N-S] A. NAGEL and E.M. STEIN, *On certain maximal functions and approach regions*. Adv. in Math. **54** (1984), 83–106.
- [Ru] W. RUDIN, *Function theory in the unit ball of \mathbb{C}^n* , Springer-Verlag, New York, 1980.
- [Su] J. SUEIRO, *Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces*. Math. Ann. **286** (1990), 661–678.
- [T-E] F.G. TRICOMI and A. ERDELYI, *The asymptotic expansion of a ratio of Gamma functions*. Pacific J. Math. **1** (1951), 133–142.
- [Tr] N.S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*. J. Math. Mech. **17** (1967), 473–483.

UNIVERSITAT DE BARCELONA
BARCELONA, SPAIN