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GROWTH OF THE BERGMAN KERNEL ON PLANAR REGIONS

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Statement of results

Let Ω be a bounded open set in the complex plane. The Bergman space $L^2_a(\Omega)$ is the Hilbert space of holomorphic functions on Ω that are square-integrable with respect to area measure A. Evaluation at each point λ of Ω is continuous, so there is a corresponding kernel function k_λ in $L^2_a(\Omega)$ such that

$$\int_\Omega f(z)\overline{k_\lambda(z)}\, dA(z) = f(\lambda)$$

for every function f in $L^2_a(\Omega)$. In this paper, we are interested in estimating the growth of $\|k_\lambda\|$ as λ tends to the boundary of Ω.

If Ω is smoothly bounded, $\|k_\lambda\|$ will grow like the reciprocal of the distance to the boundary; but if the boundary point is somehow "buried" deep inside Ω, the growth of $\|k_\lambda\|$ can be slower (to aid the reader's intuition: the phenomenon is not caused by cusps, which cause the complement to be too thick, but by little holes accumulating at some point). In [McCY] the authors found geometric conditions for $\|k_\lambda\|$ to remain bounded as a boundary point is approached (so that evaluation at this boundary point is a bounded point evaluation) for certain special domains (L-regions). Also, using results of Fernstrom and Polking [FP], necessary and sufficient conditions were found, in terms of Bessel capacity, for a boundary point of an arbitrary domain to be a bounded point evaluation. We extend this last result to estimating the growth of $\|k_\lambda\|$ when it does not remain bounded.

Let $\Lambda_{\Theta,\Gamma}$ be the sector in the left half-plane bounded by $y = x\tan\Theta$, $y = -x\tan\Theta$, and $x^2 + y^2 = \Gamma^2$. We shall always assume that 0 is in the boundary of Ω and is the point of interest, and that for some Θ in $(0, \frac{\pi}{2})$ and some $\Gamma > 0$,

$$\Lambda_{\Theta,\Gamma} \subset \Omega.$$

We shall look at the growth rate of $\|k_\lambda\|$ as λ tends to 0 along the negative real axis.

Let $G(x, y)$ be the Bessel kernel, which is most easily defined as the inverse Fourier transform of $(1 + x^2 + y^2)^{-\frac{1}{2}}$. For each set E in $\mathbb{C}$, the Bessel capacity is defined as

$$C(E) = \inf_f \int |f(x, y)|^2\, dx\, dy$$

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where $f \in L^2(\mathbb{R}^2)$, $f \geq 0$, and $\int G(t-x, s-y) f(t,s)\, dt\, ds \geq 1$ for $(x,y) \in E$. Let $A_k = \{z = x+iy \;:\; 2^{-k-1} \leq |z| < 2^{-k}\}$ and $A_k' = \{z \;:\; 2^{-k-2} \leq |z| \leq 2^{-k+1}\}$.

Our results are the following:

THEOREM 1. *Suppose*

$$\sum_{k=1}^{\infty} kC(A_k \setminus \Omega) < \infty.$$

Then there are constants F_1 *and* F_2 *so that, as* $\lambda \to 0^-$, k_λ *satisfies the growth condition*

$$\|k_\lambda\|^2 \leq F_1 \sum_{k=1}^{\infty} 2^{2\min(k, \log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega);$$

$$\|k_\lambda\|^2 \geq F_2 \sum_{k=1}^{\infty} 2^{2\min(k, \log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega).$$

COROLLARY 2. *Let* $0 < \alpha < 1$. *A necessary and sufficient condition for* $\|k_\lambda\|$ *to be* $O(|\lambda|^{-\alpha})$ *as* $\lambda \to 0^-$ *is that*

$$\limsup_{k \to \infty} 2^{2k(1-\alpha)} C(A_k \setminus \Omega) < \infty.$$

Our techniques also work for the Bergman space $L_a^p(\Omega)$ for $2 < p < \infty$. Letting $q = \frac{p}{p-1}$ be the conjugate index of p, define the q-capacity by

$$C_q(E) = \inf_f \int |f(x,y)|^q \, dx\, dy$$

where $f \in L^q(\mathbb{R}^2)$, $f \geq 0$, and $\int G(t-x, s-y) f(t,s)\, dt\, ds \geq 1$ for $(x,y) \in E$ (so our previous definition of capacity is C_2, though we shall continue to write it as C with no subscript).

THEOREM 3. *Suppose* $2 < p < \infty$, *and*

$$\sum_{k \to \infty} 2^{k(2-q)} C_q(A_k \setminus \Omega) < \infty.$$

Then as $\lambda \to 0^-$, *the norm of the functional of evaluation at* λ *in the Bergman space* $L_a^p(\Omega)$ *is comparable to*

$$\left[\sum_{k=1}^{\infty} 2^{q\min(k, \log_2 \frac{1}{|\lambda|})} C_q(A_k \setminus \Omega)\right]^{\frac{1}{q}}.$$

We note that a disk Δ of radius δ has

$$C(\Delta) \sim \left[\log\left(\frac{1}{\delta}\right)\right]^{-1}$$

and, for $q < 2$,

$$C_q(\Delta) \sim \delta^{2-q},$$

so it is easy to construct examples of regions of the form $\mathbb{D}(0, 1) \setminus \cup_{n=1}^{\infty} \mathbb{D}(2^{-n}, r_n)$ that have kernels growing at a desired rate.

Proofs

We shall let F denote a generic constant, that may change from one line to the next. We shall let $K_\varepsilon = \{z : dist(z, K) < \varepsilon\}$.

LEMMA 4. *(a) For each Borel set E,*

$$C(E) = \inf_{E \subset U} C(U)$$

where U ranges over the open sets.
(b) For any Borel sets E_1 and E_2,

$$C(E_1 \cup E_2) \leq C(E_1) + C(E_2).$$

Proof. See [Me]. □

LEMMA 5. *Let K be a compact subset of $\mathbb{C}$. Suppose that*

$$\limsup\, kC(A_k \backslash K) = 0.$$

Then there exists a constant $M > 0$ such that, for each $\varepsilon > 0$ and each $k \geq 0$, there is a function $\psi_k \in C^\infty$ satisfying

(i) $\psi_k(z) = 1$ *for z in a neighborhood of $A'_k \backslash K_\varepsilon$,*
(ii) $\int_{|z| \leq 2^{-k+1}} |\bar{\partial} \psi_k|^2 dA \leq M \cdot C(A'_k \backslash K)$,
(iii) $\int_{|z| \leq 2^{-k+1}} |\psi_k|^2\, dA \leq M \cdot 2^{-2k} C(A'_k \backslash K)$.

Proof. The proof follows from the proof of Lemma 10 in [FP]. Their hypotheses are much stronger—namely that $\sum 2^{2k} C(A_k \backslash K) < \infty$—but their proof works in our special case provided just $kC(A_k \backslash K)$ tends to zero. To use their proof, we only have to show that there exists k_0 such that if $k \geq k_0$ then

$$\int_{\sqrt{t^2+s^2} > 2^{-k+2}} G(x-t, y-s) g_k(t, s)\; dt\; ds < \frac{1}{2} \tag{1}$$

for $x^2 + y^2 < 2^{-2k+3}$, where g_k is defined as in their proof. We have

$$\int_{\sqrt{t^2+s^2}>2^{-k+2}} G(x-t, y-s) g_k(t,s)\, dt\, ds \le \left\{ \int_{\sqrt{t^2 s^2}\ge 2^{-k}} G(t,s)^2\, dt\, ds \right\}^{\frac{1}{2}} \cdot \|g_k\|$$

$$\le F\left(\log \frac{1}{2^{-k}}\right)^{\frac{1}{2}} \cdot [C(A'_k \backslash K)]^{\frac{1}{2}} \quad (2)$$

where the second inequality follows from Lemma 4 in [FP] and the fact that $\| g_k \| \le 2[C(A'_k \backslash K)]^{\frac{1}{2}}$. As the expression in (2) tends to 0, we get (1) as desired. □

For a positive Borel measure μ, let

$$U^\mu(z) = \int |z-w|^{-1}\, d\mu(w)$$

and

$$c(E, V) = \sup_\mu \mu(E)$$

where the supremum is taken over all positive Borel measures with spt$\mu \subseteq E$ such that

$$\int_V |U^\mu(z)|^2\, dA \le 1.$$

(Throughout the paper, V will be a fixed bounded open set containing Ω or K. It is only necessary to introduce it as $|U^\mu|^2$ is not integrable in a neighborhood of infinity.)

LEMMA 6. *Let E be a Borel set. Then*

$$C(E)^{\frac{1}{2}} \le c(E, V) \le FC(E)^{\frac{1}{2}}.$$

(For a discussion of why this is true, and for other equivalent notions of analytic 2-capacities, see [He].)

The next lemma is the key to proving the upper bound estimate.

LEMMA 7. *Let K be a compact subset of $\mathbb{C}$, $0 \in \partial K$, and $\Lambda_{\Theta,\Gamma} \subseteq K$ for some $0 < \Theta < \frac{\pi}{2}$ and $\Gamma > 0$. Suppose that*

$$\limsup_{k\to\infty} kC(A_k \backslash K) = 0.$$

Then there is a constant M such that for each rational function r with poles off K, and each λ in $\mathbb{R} \cap \Lambda_{\Theta,\Gamma}$,

$$|r(\lambda)| \le M \cdot \left(\int_K |r|^2\, dA\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} 2^{2\min(k, \log_2 \frac{1}{|\lambda|})} C(A_k \setminus K)\right)^{\frac{1}{2}}.$$

Proof. Without loss of generality, we can assume K is contained in the disk $\mathbb{D}(0, \frac{1}{4})$, and so $\Gamma < \frac{1}{4}$. It follows from Lemma 4 that

$$\limsup_{k\to\infty} kC(A'_k \backslash K) < \infty.$$

Construct a smooth function $\varphi \in C_o^\infty(\mathbb{R}^2)$ such that

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in \{z : |z| \geq 2\} \text{or } |z| \leq \frac{1}{4}\} \\ 1 & \text{if } z \in \{z : \frac{1}{2} \leq |z| \leq 1\} \cap (\Lambda_{\Theta, 1})^c \\ 0 & \text{if } z \in \Lambda_{\frac{\Theta}{2}, 1} \end{cases}$$

For each integer k set

$$\varphi_k(z) = \begin{cases} \varphi(2^k z) / \sum_{j=-\infty}^{\infty} \varphi(2^j z) & \text{if } \varphi(2^j z) \neq 0 \text{ for some } j \\ 0 & \text{else.} \end{cases}$$

Fix a rational function r with poles off K. Let g be a smooth function which is 1 in a neighborhood of K, zero in a neighborhood of the poles of r and zero off $\mathbb{D}(0, \frac{1}{4})$. Let λ be between $-\Gamma$ and 0. From Green's theorem,

$$r(\lambda) = \frac{1}{\pi} \int_{\mathbb{D}(0, \frac{1}{4})} \frac{1}{\lambda - z} \bar{\partial}\,(gr)\; dA(z).$$

Let ε be small enough for K_ε to be contained in the set where g is 1. Then $\bar{\partial}\,(gr) = 0$ on K_ε. Let ψ_k be as in Lemma 5; then $\sum \psi_k \varphi_k = 1$ near $\mathbb{D} \backslash K_\varepsilon$, because if z is in A_k, $\varphi_j(z)$ is non-zero only for $j = k-1, k, k+1$, and $\psi_j(z) = 1$ for these values of j. Therefore

$$\begin{aligned} r(\lambda) &= \frac{1}{\pi} \int \frac{1}{\lambda - z} \sum_{k=1}^{\infty} \psi_k \varphi_k \bar{\partial}\,(gr)\; dA \\ &= \frac{1}{\pi} \sum_k \int_{A_k \backslash \Lambda_{\frac{\Theta}{2}, \gamma}} \frac{1}{z - \lambda} \bar{\partial}\,(\psi_k \varphi_k) \cdot gr\; dA. \end{aligned}$$

Hence,

$$|r(\lambda)| \leq F \sum_k \int_{A_k \backslash \Lambda_{\Theta, \Gamma}} \frac{1}{|z - \lambda|} |\bar{\partial}\,(\psi_k \varphi_k)| |gr|\; dA. \tag{3}$$

Let $-2^{-N} \leq \lambda < -2^{-N-1}$ and $z \in A_k \backslash \Lambda_{\Theta, \Gamma}$. If $k \geq N$, then

$$\frac{1}{|z - \lambda|} \leq F \cdot \frac{1}{2^{-N}};$$

if $k < N$, then

$$\frac{1}{|z-\lambda|} \leq F \cdot \frac{1}{2^{-k}}.$$

Therefore (3) gives

$$\begin{aligned}
|r(\lambda)| &\leq F \sum_{k=1}^{N-1} 2^k \left(\int_{A_k} |\bar{\partial}(\psi_k \varphi_k)|^2 \, dA \right)^{\frac{1}{2}} \left(\int_{A_k} |gr|^2 \, dA \right)^{\frac{1}{2}} \\
&\quad + F \sum_{k=N}^{\infty} 2^N \left(\int_{A_k} |\bar{\partial}(\psi_k \varphi_k)|^2 \, dA \right)^{\frac{1}{2}} \left(\int_{A_k} |gr|^2 \, dA \right)^{\frac{1}{2}} \\
&\leq F \|gr\|_{L^2_a(\mathbb{D})} \left(\sum_{k=1}^{N-1} 2^{2k} \int_{A_k} |\bar{\partial}(\psi_k \varphi_k)|^2 \, dA + \sum_{k=N}^{\infty} 2^{2N} \int_{A_k} |\bar{\partial}(\psi_k \varphi_k)|^2 \, dA \right)^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 5 and the fact that $|\bar{\partial}\varphi_k| \leq F \cdot 2^k$, we have

$$\int_{A_k} |\bar{\partial}(\psi_k \varphi_k)|^2 \, dA \leq F \cdot C(A_k' \backslash K).$$

Hence,

$$|r(\lambda)| \leq F \|gr\|_{L^2_a(\mathbb{D})} \left(\sum_{k=1}^{N-1} 2^{2k} C(A_k' \backslash K) + \sum_{k=N}^{\infty} 2^{2N} C(A_k' \backslash K) \right)^{\frac{1}{2}}.$$

Since g is arbitrary, subject to being 1 on K, we conclude that

$$|r(\lambda)| \leq M \left(\sum_{k=1}^{\infty} 2^{2 \min(k, \log_2 \frac{1}{|\lambda|})} C(A_k \setminus K) \right)^{\frac{1}{2}} \|r\|_{L^2(K,A)}. \qquad \square$$

Proof of Theorem 1. (Upper Bound) Suppose that

$$\limsup_{k \to \infty} kC(A_k \backslash \Omega) = 0,$$

so by Lemma 4 (b),

$$\limsup_{k \to \infty} kC(\bar{A}_k \backslash \Omega) = 0.$$

Using Lemma 4 (a), we can find open subsets U_k such that

$$\bar{A}_k \backslash \Omega \subset U_k$$

and

$$C(U_k) < 2C(A_k \backslash \Omega).$$

Let $K_k = \bar{A}_k \setminus U_k$, and $K = \cup_{k=1}^{\infty} K_k \cup \{0\}$. Then K is a compact subset,

$$\limsup_{k\to\infty} kC(A_k \backslash K) = 0,$$

and

$$K \cap A_k \subseteq \Omega \cap A_k,$$

so

$$K\backslash\{0\} \subseteq \Omega.$$

Let $f \in L_a^2(\Omega)$ be analytic in a neighborhood of zero. By Runge's theorem, there exists a sequence of rational functions r_n converging to f uniformly on K; so by Lemma 7 we have

$$\begin{aligned} |f(\lambda)| &\leq M\left(\sum_{k=1}^{\infty} 2^{2\min(k,\log_2 \frac{1}{|\lambda|})} C(A_k \setminus K)\right)^{\frac{1}{2}} \left(\int_K |f|^2\, dA\right)^{\frac{1}{2}} \\ &\leq M\left(\sum_{k=1}^{\infty} 2^{2\min(k,\log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega)\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2\, dA\right)^{\frac{1}{2}}. \end{aligned} \tag{4}$$

Since such functions f are dense in $L_a^2(\Omega)$ (see [Al]), (4) holds for all f in $L_a^2(\Omega)$. Therefore, for $\lambda \in \mathbb{R} \cap \Lambda_{\Theta,\Gamma}$,

$$\|k_\lambda\| = \sup_{\|f\|_{L_a^2(\Omega)}=1} |f(\lambda)| \leq M\left(\sum_{k=1}^{\infty} 2^{2\min(k,\log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega)\right)^{\frac{1}{2}}.$$

(Lower Bound) By Lemma 6, we can choose positive measures μ_k carried by $A_k \setminus \Omega$ with

$$\int_\Omega |U_k^{\mu}(z)|^2\, dA \leq 1,$$

and $\|\mu_k\|$ comparable to $C(A_k \setminus \Omega)^{\frac{1}{2}}$. Moreover, we can assume that each of the measures μ_k has support entirely within one of the three sectors $\{z : -\pi + \frac{\Theta}{2} \leq \arg(z) \leq -\frac{\Theta}{2}\}$, $\{z : -\frac{\Theta}{2} \leq \arg(z) \leq \frac{\Theta}{2}\}$, $\{z : \frac{\Theta}{2} \leq \arg(z) \leq \pi - \frac{\Theta}{2}\}$.

Let

$$\gamma_k = 2^{\min(k,\log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega)^{\frac{1}{2}}.$$

For some conjugacy class modulo 3, the sum of γ_k^2 for k in this conjugacy class is comparable to $\sum_{k=1}^{\infty} \gamma_k^2$. We shall choose only those μ_k for k in this conjugacy class, and set the other μ_k's to zero (the point of this is to ensure the supports are well separated).

Let

$$f_k(z) = \int \frac{d\mu_k(w)}{z-w}$$

be the Cauchy transform of μ_k. Let us fix some λ between $-\Gamma$ and 0, and let N be the closest integer to $\log_2 \frac{1}{|\lambda|}$. By the way we have chosen the measures, $|f_k(\lambda)|$ is comparable to $2^k \|\mu_k\|$ for $k < N$ and to $2^N \|\mu_k\|$ for $k > N$; For $k = N$, we can only say that it dominates $2^N \|\mu_k\|$. Thus, for all k, we have

$$|f_k(\lambda)| \geq F\ \gamma_k. \tag{5}$$

We also have $\int_\Omega |f_k|^2 \, dA \leq 1$, and the estimate

$$|f_n(z)| \leq \|\mu_n\| \, [\, \mathrm{dist}(z, A_n)]^{-1}$$

gives, for $|n-k| \geq 2$,

$$\int_{A_k} |f_n|^2 \, dA \leq FC(A_n \setminus \Omega) 2^{2\min(n,k)-2k}. \tag{6}$$

Let $f = \sum \alpha_n f_n$, where we shall choose the α's later. We have

$$\begin{aligned}
\int_\Omega |f|^2 \, dA &= \sum_{k=1}^{\infty} \int_{A_k \cap \Omega} \left| \sum_{|n-k|\geq 2} \alpha_n f_n + \alpha_k f_k \right|^2 dA \\
&\leq 2 \left[\sum_{k=1}^{\infty} \int_{A_k \cap \Omega} |\alpha_k f_k|^2 \, dA + \sum_{k=1}^{\infty} \int_{A_k \cap \Omega} \left| \sum_{|n-k|\geq 2} \alpha_n f_n \right|^2 dA \right].
\end{aligned}$$

Now, by the inequalities of Minkowski and Cauchy-Schwartz,

$$\begin{aligned}
\int_{A_k \cap \Omega} \left| \sum_{|n-k|\geq 2} \alpha_n f_n \right|^2 dA &\leq \left[\sum_{|n-k|\geq 2} |\alpha_n| \left(\int_{A_k \cap \Omega} |f_n|^2 \, dA \right)^{\frac{1}{2}} \right]^2 \\
&\leq \left(\sum_{|n-k|\geq 2} |\alpha_n|^2 \right) \left(\sum_{|n-k|\geq 2} \int_{A_k \cap \Omega} |f_n|^2 \, dA \right). \qquad (7)
\end{aligned}$$

Using this and inequality (6), we get

$$\int_\Omega |f|^2 \, dA \leq F \left(\sum |\alpha_n|^2 \right) \left(1 + \sum_{n,k:|n-k|\geq 2} C(A_n \setminus \Omega \right) 2^{2\min(n,k)-2k}).$$

But this last factor is dominated by

$$\sum_{n=1}^{\infty} nC(A_n \setminus \Omega)$$

which is finite by hypothesis. So we finally get

$$\int_\Omega |f|^2 dA \le F\left(\sum |\alpha_n|^2\right). \tag{8}$$

Now choose α_k so that $|\alpha_k| = \gamma_k$, and so that $\alpha_k f_k(\lambda) \ge 0$. By inequalities (5) and (8), we have

$$\frac{|f(\lambda)|^2}{\int_\Omega |f|^2 dA} \ge F\frac{(\sum |\alpha_k|\gamma_k)^2}{\sum |\alpha_k|^2} = F\sum_{k=1}^{\infty} \gamma_k^2.$$

So $(\sum_{k=1}^{\infty} \gamma_k^2)^{\frac{1}{2}}$ is a lower bound for k_λ, as desired. □

Proof of Corollary 2. (Sufficiency) If

$$\limsup_{k\to\infty} 2^{2k(1-\alpha)} C(A_k \setminus \Omega) < \infty$$

then the hypotheses of Theorem 1 are satisfied. Using the estimate on $\|k_\lambda\|$ from Theorem 1, together with the hypothesis that $C(A_k \setminus \Omega) \le F 2^{-2k(1-\alpha)}$, one gets that $|\lambda|^\alpha \|k_\lambda\|$ stays bounded.

(Necessity) Let f_k be as in the proof of Theorem 1. Then

$$\frac{|f_k(\lambda)|}{\|f_k\|} \ge F\ 2^{\min(k, \log_2 \frac{1}{|\lambda|})} C(A_k \setminus \Omega)^{\frac{1}{2}}.$$

Letting $\lambda = -2^{-k}$, we get

$$|\lambda|^\alpha \|k_\lambda\| \ge F\ 2^{k(1-\alpha)} C(A_k \setminus \Omega)^{\frac{1}{2}},$$

and the right-hand side must remain bounded, as desired. □

Proof of Theorem 3. This proceeds like the proof of Theorem 1, with appropriate modifications to Lemmata 5, 6 and 7 (in the proof of Lemma 5 the hypothesis that

$$\lim_{k\to\infty} 2^{k(2-q)} C_q(A_k \setminus \Omega) = 0$$

is used, and this is a sufficient condition for the upper bound to hold).

For the lower bound, there is one extra difficulty in getting the appropriate analogue of (7)—if one follows routinely, one gets $\sum |\alpha|^q$ instead of $\sum |\alpha|^p$. To get around this, note that (6) becomes

$$\int_{A_k} |f_n|^p dA \le F\ C_q^{\frac{p}{q}}(A_n \setminus \Omega) 2^{p\min(n,k)-2k}. \tag{6'}$$

Then (7) becomes

$$\int_{A_k\cap\Omega} \left| \sum_{|n-k|\ge 2} \alpha_n f_n \right|^p dA \le \left(\sum_{|n-k|\ge 2} |\alpha_n|^p \right) \left(\sum_{|n-k|\ge 2} \left[\int_{A_k\cap\Omega} |f_n|^p dA \right]^{\frac{q}{p}} \right)^{\frac{p}{q}}.$$

Using (6′), we get

$$\sum_{k=1}^{\infty} \int_{A_k \cap \Omega} \left| \sum_{|n-k|\geq 2} \alpha_n f_n \right|^p dA \leq \sum_{k=1}^{\infty} \left(\sum_{|n-k|\geq 2} |\alpha_n|^p \right)$$

$$\times \left(\sum_{|n-k|\geq 2} C_q(A_n \Omega) 2^{q \min(n,k) - \frac{2q}{p} k} \right)^{\frac{p}{q}}$$

$$\leq \left(\sum_{n=1}^{\infty} |\alpha_n|^p \right) \sum_{k=1}^{\infty} \left(\sum_{|n-k|\geq 2} C_q(A_n \Omega) 2^{q \min(n,k) - \frac{2q}{p} k} \right).$$

Now interchanging the order of summation and using the hypothesis of the theorem, (and the estimate $\int_{A_n \cap \Omega} |\alpha_n f_n|^p \, dA \leq |\alpha_n|^p$) we finally get

$$\int_{\Omega} |f|^p dA \leq F \left(\sum |\alpha_n|^p \right).$$

Choosing $|\alpha_k| = \gamma_k^{q-1}$, where $\gamma_k = 2^{\min(k, \log_2 \frac{1}{|\lambda|})} C_q^{\frac{1}{q}}(A_k \setminus \Omega)$, yields the desired lower bound. □

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