ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 1, Spring 1996

CELL-LIKE MAPS AND ASPHERICAL COMPACTA

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A class of compacta on which cell-like maps cannot raise dimension was presented by Daverman [Da]. That class is expanded here, by establishing that all compact metric spaces contain compacta of codimension one on which cell-like maps do not raise dimension.

Classical results promise that cell-like maps defined on 1-dimensional compacta do not raise dimension. Dranishnikov [Dr] proved the existence of an infinite dimensional compactum whose integral cohomological dimension equals 3, from which it follows by the Edwards-Walsh construction [Wa] that there is a cell-like map on a 3-dimensional compactum with infinite-dimensional image. More recently, Dydak and Walsh [DW] confirmed that the same phenomenon could occur with 2-dimensional domain. Daverman [Da] introduced a notion of strongly hereditarily aspherical compacta, showed that cell-like maps on such compacta do not raise dimension, and provided examples in dimensions up to 5 with this asphericity property. Davis and Januszkiewicz [DJ] presented methods which give higher dimensional examples, by providing detailed elaborations of Gromov's useful idea [Gr] of hyperbolizing simplexes and polyhedra. A fortuitous consequence of the Cartan-Hadamard Theorem, for our purposes, is the fact that hyperbolization leads to asphericalization. One of our key results, a broad existence theorem, stems from techniques intimately related to this hyperbolization procedure. It is the following SHA Subset Theorem 3.1: Every compact metric space X contains a 0-dimensional F_σ-subset F such that all compact subsets of $X \setminus F$ are strongly hereditarily aspherical. As a striking consequence, every finite-dimensional, compact metric space X contains a 0-dimensional F_σ-subset F such that, for any cell-like map $p\colon X \to Y$ with infinite dimensional image, $p(F)$ is infinite-dimensional.

Corresponding to notions of hereditarily aspherical and strongly hereditarily aspherical compacta set forth in [Da], we introduce notions of hereditarily aspherical and strongly hereditarily aspherical maps. An issue still unresolved is whether the two types of compacta are distinct. Adding evidence for the suspicion that they are, we point out why the two types of maps are distinct. Among the highlights of §2 is Corollary 2.4, promising that a cell-like mapping which is strongly hereditarily aspherical over its image cannot raise dimension; near the end of the paper, in Theorem 3.7, we

Received December 28, 1993
1991 Mathematics Subject Classification. Primary 54F20, 54F45; Secondary 54B25, 55M10, 57P99.

produce a map from a 2-dimensional compactum onto an infinite dimensional space which is hereditarily aspherical, but obviously not strongly so, over its image.

The heart of this effort is the determination of high dimensional analogs of a grope. A grope G is an aspherical, acyclic 2-complex having a natural simple closed curve "boundary", ∂G, where inclusion $\partial G \to G$ induces an injection of fundamental groups. As originally described, G is a union of disks with handles, in each of which are identified finitely many handle curves, arranged so every handle curve is the boundary of some disk with handles at the next stage, and with the set of handle curves in any component of a given stage chosen to generate its fundamental group. These objects first appeared in papers of Štan'ko [Št], buried implicitly deep in his marvelous constructions for approximating embeddings of codimension 3 compacta in manifolds by homotopically tame embeddings. When he extended Štan'ko's results, Edwards [Ed] assigned them a more conspicuous role and later on, perhaps more significantly, made strong use of them in his unpublished initial work on the double suspension theorem. Cannon really brought these devices into the limelight in the later 1970s, first in his work with Ancel [AC] on the locally flat approximation theorem for codimension one embeddings, most resoundingly in his final solution of the double suspension theorem [C2], but also in a revelatory survey article exposing extensive connections to wildness and decomposition problems [C1]. Here Proposition 3.6 provides a model n-dimensional grope for every n, namely, an aspherical, acyclic n-complex $\alpha\sigma^n$ having a natural boundary, $\partial\alpha\sigma^n$, where not only does again inclusion $\partial\alpha\sigma^n \to \alpha\sigma^n$ induce an injection at the fundamental group level but also the same holds for inclusion $X^k \to \alpha\sigma^n$ of each k-stratum X^k above the k-skeleton of σ^n, and where the part of X^k above any k-simplex of σ^n is like a connected sum of copies of $\alpha\sigma^k$.

The authors gratefully acknowledge suggestions from Professor A. Koyama clarifying certain parts of this paper.

1. Definitions and notation

A compact subset X of a metric space S is said to have *Property k-UV in S* if for each neighborhood U of X in S there exists a neighborhood V of X, $V \subset U$, such that every map $\partial B^{k+1} \to V$ extends to a map $B^{k+1} \to U$. Recall that the space X itself is said to have *Property k-UV* if for each (some) embedding λ of X in an ANR Y, $\lambda(X)$ has Property k-UV in Y. Elementary features of ANRs make it plain that the aforementioned extension property is indeed invariant under embeddings in ANRs. Generally, X is said to have *Property UV^n* if it has Property k-UV for $0 \leq k \leq n$. Moreover, X is said to be *aspherical* if it has Property k-UV for all $k \geq 2$ and to be *hereditarily aspherical* if each compact subset $A \subset X$ is aspherical. Also, a map p: $X \to X'$ between compact metric spaces is said to be k-UV or UV^n if each point preimage has the property specified.

Throughout this paper p: $Q \to Q'$ will denote a surjective mapping between compact metric spaces, with Q the Hilbert cube, X' a compact subset of Q', and

$X = p^{-1}(X')$. Say that p is *strongly hereditarily aspherical over* X' if for each $\varepsilon > 0$, X' is covered by a collection of open sets $\{U_i\}$ of diameter less than ε such that every finite union of the form $\cup p^{-1}(U_{i(j)})$ is aspherical; say simply that p is *hereditarily aspherical over* X' if for each compact set $A \subset X'$ and each integer $k \geq 2$, $p^{-1}(A)$ has Property k-UV. One can easily check that the strong hereditary asphericity property implies the unmodified one and that $X \subset Q$ is (strongly) hereditarily aspherical as a space (the strong version being in the sense of [Da]) if and only if the identity map $Q \to Q$ is (strongly) hereditarily aspherical over X.

Given a compact metric space X and another space H, as in [Dr2] we use the Kuratowski notation $X\tau H$ to mean that X has the *extension property with respect to* H, namely, for any closed subset A of X and map f: $A \to H$, f can be extended to a map F: $X \to H$.

Let L be a polyhedron with triangulation $\mathcal{L}$. An *asphericalization* of $(L, \mathcal{L})$ is a map g: $\alpha L \to L$ such that for every subcomplex L' of L (with respect to $\mathcal{L}$), $g^{-1}(L')$ is aspherical. Given a map f: $X \to L$, another map f': $X \to \alpha L$ is called an *$\mathcal{L}$-lifting to the asphericalization* g: $\alpha L \to L$ provided each pair $f(x), gf'(x)$, $x \in X$, resides in some simplex of $\mathcal{L}$.

A metric space S is said to be *uniformly locally i-connected*, written i-ULC, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each map of ∂B^{i+1} into an δ-subset of S extends to a map of B^{i+1} into an ε-subset of S; S is referred to as ULC^k if it is i-ULC for $0 \leq i \leq k$.

2. Hereditarily aspherical maps

This section presents some basic results about hereditarily aspherical and strongly hereditarily aspherical maps, confirms that cell-like maps of the latter type do not raise dimension, and provides an improved characterization of strongly hereditarily aspherical compacta.

LEMMA 2.1. *Suppose the* map p: $Q \to Q'$ *is strongly hereditarily aspherical over* $x \in Q'$. *Then* $p^{-1}x$ *is cell-like if and only if* $p^{-1}x$ *has Property* UV^1.

Proof. The forward implication is obvious. For the converse, given a neighborhood W of $p^{-1}x$, first apply strong hereditary asphericity to obtain an aspherical neighborhood U, $p^{-1}x \subset U \subset W$, next invoke UV^1 to find a smaller neighborhood V such that $\pi_1(V) \to \pi_1(U)$ is trivial, and finally name a closed neighborhood $D \subset V$ of $p^{-1}x$ with D homeomorphic to $P \times Q$ for some connected, finite polyhedron P. We confirm cell-likeness of $p^{-1}x$ by demonstrating the contractibility of D in $U \subset W$. Clearly $D = P \times Q$ deformation retracts to a copy of P in D. Moreover, the inclusion of that copy to P extends to a map of the cone cP on P, via path connectedness to extend from the 1-skeleton of cP into $D \subset V$, UV^1 features to extend from the 2-skeleton into U, and, finally, asphericity to extend over the successive skeleta. □

COROLLARY 2.2. *Suppose the map $p\colon Q \to Q'$ is strongly hereditarily aspherical over X'. Then $p \mid X$ is cell-like if and only if $p \mid X$ is UV^1.*

Already we can discern distinctions between the two notions of asphericity for maps. Let $p\colon Q \to Q' = Q/T$ be the quotient map, where T denotes Taylor's example [Ta] of a space with nontrivial shape admitting a cell-like map onto Q. By construction T has Property k-UV for all $k \geq 0$; in other words, P is hereditarily aspherical over the image of T. As T fails to be cell-like, Lemma 2.1 assures that p cannot be strongly hereditarily aspherical over the same point.

CONTROLLED LIFTING LEMMA 2.3. *Let $p\colon Q \to Q'$ be a cell-like mapping which is strongly hereditarily aspherical over X' and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for each finite polyhedron K and function $\eta\colon K^{(0)} \to X'$ with $\operatorname{dist}(\eta(v), \eta(v')) < \delta$ for all $v, v' \in K^{(0)}$ in a common simplex of K, there exists a map $\varphi\colon K \to Q$ such that $\operatorname{diam} p\varphi(\sigma) < \varepsilon$ for each simplex $\sigma \in K$ and $p\varphi(v) = \eta(v)$ for all $v \in K^{(0)}$. Furthermore, if K' is a subcomplex of K and $\varphi'\colon K' \to Q$ is a map such that $\operatorname{diam} p\varphi'(\sigma') < \varepsilon$ for each simplex $\sigma' \in K'$ and $p\varphi'(v) = \eta(v)$ for all $v \in K' \cap K^{(0)}$, then $\varphi\colon K \to Q$ can be obtained so $\varphi \mid K' = \varphi'$.*

Proof. Let $\mathfrak{J}_0$ be a finite cover of X' by open sets of diameter less than $\varepsilon/4$ such that for every finite union W of elements of $\mathfrak{J}_0$, $p^{-1}W$ is aspherical. Use cell-likeness of p to determine a finite cover $\mathfrak{J}_1$ of X' by connected open sets where to each $V \in \mathfrak{J}_1$ there corresponds some $U \in \mathfrak{J}_0$ such that $\operatorname{St}^2(V, \mathfrak{J}_1)$ is null-homotopic in U. For $A \subset Q'$ and $j \in \{0, 1\}$, here

$$\operatorname{St}(A, \mathfrak{J}_j) = \cup\{V \in \mathfrak{J}_j \mid V \cap A \neq \varnothing\}$$

and $\operatorname{St}^2(A, \mathfrak{J}_j) = \operatorname{St}(\operatorname{St}(A, \mathfrak{J}_j), \mathfrak{J}_j)$. Choose $\delta > 0$ so that every pair of points in X' at distance less than δ apart both belong to a common element of $\mathfrak{J}_1$. Given K and η as prescribed, we define a map $\varphi\colon K \to Q$ in stages over successive skeleta. For $v \in K^{(0)}$ choose $x_v \in p^{-1}\eta(v) \subset X$ and then set $\varphi(v) = x_v$. For any 1-simplex e of K there is $V_e \in \mathfrak{J}_1$ with $\eta(\partial e) \subset V_e$; connectedness of point preimages implies connectedness of $p^{-1}(V_e)$ and ensures the existence of a map $\varphi\colon e \to p^{-1}(V_e)$ extending $\varphi \mid \partial e$. This provides a map $\varphi\colon K^{(1)} \to Q$, and the definition of $\mathfrak{J}_1$ allows an extension $\varphi\colon K^{(2)} \to Q$ such that to each 2-simplex $\Delta \in K$ corresponds $U_\Delta \in \mathfrak{J}_0$ with $\varphi(\Delta) \subset p^{-1}(U_\Delta)$. We use the asphericity of preimages of $\mathfrak{J}_0$ to extend $\varphi\colon K^{(2)} \to Q$ inductively over successive skeleta to $\varphi\colon K^{(k)} \to Q$ $(k > 2)$ where, specifically, for each k-simplex $\sigma \in K$, $\varphi(\sigma)$ lies in

$$p^{-1}(\cup\{U_\Delta \mid \Delta \text{ is a 2-simplex of } K \text{ in } \partial\sigma\}).$$

Of course, $K^{(k)} = K$ when k is large enough. That $\operatorname{diam} p\varphi(\sigma) < \varepsilon$ for $\sigma \in K$ follows because $p\varphi(\sigma) \subset \operatorname{St}^2(\eta(v), \mathfrak{J}_0)$ for every vertex $v \in \sigma$.

Only minor modifications are required to obtain the supplementary conclusion. □

Remarkable is the absence of any *a priori* dimension restriction on the polyhedron K arising in Lemma 2.3. Of course, for any fixed integer k, standard methods involving cell-like mappings (or even UV^k mappings), with no reference to strong asphericity, give rise to $\delta > 0$ satisfying the conclusion of 2.3 for k-complexes.

COROLLARY 2.4. *If $p\colon Q \to Q'$ is a cell-like mapping which is strongly hereditarily aspherical over X', then* $\dim X' \leq \dim X$.

Proof. According to unpublished work of Kozlowski [Ko] (see [DS, Theorem 2.1] for an explanation), it suffices to produce approximate right inverses to $p \mid X$, meaning that, given any $\varepsilon > 0$, one can find a map $\varphi'\colon X' \to Q$ such that $\operatorname{diam}\{p\varphi'(x'), x'\} < \varepsilon$ for all $x' \in X'$. Let $\mathfrak{I}$ be a finite cover of X' by open sets of diameter less than $\delta/2 < \varepsilon/2$, where $\delta > 0$ satisfies the conclusion of Lemma 2.3 for $\varepsilon/2$. Let $\mu\colon X' \to K$ be a barycentric map to K, the nerve of $\mathfrak{I}$. Name $\eta\colon K^{(0)} \to X'$ by simply picking $\eta(v)$ from the open set $V \in \mathfrak{I}$ corresponding to $v \in K^{(0)}$, and use Lemma 2.3 to get the promised map $\varphi\colon K \to Q$ such that $\operatorname{diam} p\varphi(\sigma) < \varepsilon/2$ and $p\varphi(v) = \eta(v)$ for $\sigma \in K$ and $v \in K^{(0)}$. Set $\varphi' = \varphi\eta$. Now to any $x' \in X'$ there corresponds $\eta(v)$ with $\{x', \eta(v)\} \subset V \in \mathfrak{I}$; equivalently, $\mu(x')$ lives in some $\sigma \in K$ with vertex v. Since $\{x', p\varphi'(x')\} \subset V \cup p\varphi(\sigma)$ and $V \cap p\varphi(\sigma) \neq \varnothing$, $\operatorname{diam}\{x', p\varphi'(x')\} \leq \operatorname{diam} V \cup p\varphi(\sigma) < \varepsilon$, as required. □

PROPOSITION 2.5. *If $v\colon Y \to Y'$ is a cell-like, strongly hereditarily aspherical, surjective map defined on a compact ANR Y, then Y' is a compact ANR.*

Outline of a proof. Embed Y in Q and extend φ to a cell-like map $p\colon Q \to Q'$ which is 1-1 on $Q \setminus Y$. Specify $\varepsilon > 0$. Find a closed neighborhood D of Y in Q, retraction $R\colon D \to Y$ and $\gamma > 0$ such that $\operatorname{diam} pR(A) < \varepsilon/3$ whenever $A \subset D$ and $\operatorname{diam} A < \gamma$. Restrict D so R moves points less than γ. Find $\delta > 0$ corresponding to $p(Y)$ and $\varepsilon/3$ from Lemma 2.3. Restricting further, if necessary, choose D as $P \times Q$ for some finite polyhedron P where $\operatorname{diam}(z \times Q) < \delta$ for $z \in P$; then the image under pR of an obvious deformation Ψ_t of $P \times Q$ to $P \times *$ has small image in Y'. As in the proof of Corollary 2.4, obtain a map $\varphi\colon Y' \to Q$ with $p\varphi$ close to the identity on Y'. Control φ so $\varphi(Y') \subset D$ and $pR\varphi$ is close to the identity. Apply Lemma 2.3 to get a homotopy between $R \mid P \times *$ and $R\varphi pR \mid P \times *$, tracks of which have small diameter under p. All this combines to give a homotopy in Y between the identity and $R\varphi pR$ having small image under p. According to [Ko, Theorem 5], Y' is an ANR.

THEOREM 2.6. *If X is LC^1 and p is cell-like and hereditarily aspherical over X', then* $\dim X' \leq \dim X$.

This follows via the argument of Ščepin detailed in [Da, Theorem 8].

The point of this paper is that the LC^1 hypothesis of Theorem 2.6 is necessary and that Corollary 2.4 cannot be improved by deleting the word "strongly". The example described in Theorem 3.7 nails this down.

LEMMA 2.7. *Suppose S is a ULC^k metric space and $X \subset S$ has Property k-UV (in the absolute sense). Then X has Property k-UV in S.*

Proof. Let λ: $X \to A$ be an embedding in an ANR A. Then λ extends to a map g: $Z \to A$, for some neighborhood Z of X in S. Fix a neighborhood U of X. We spell out calculations needed to determine the smaller neighborhood V of X. To that end, let ρ_A and ρ_S denote distances in A and S, respectively.

Find $\delta_{k+1} > 0$ such that U contains the δ_{k+1}-neighborhood of X in S. Since S is i-ULC for all $i \leq k$, there exists $\delta_k > 0$ such that every map from ∂I^{k+1} into a $2\delta_k$-subset of S extends to a map from I^{k+1} into a δ_{k+1}-subset of S. Recursively, for $0 < m \leq k$ find $\delta_{k-m} > 0$ such that every map from ∂I^{k-m+1} into a $2\delta_{k-m}$-subset of S extends to a map from I^{k-m+1} into a δ_{k-m+1}-subset of S. Then determine $\eta > 0$ such that $\operatorname{diam} \lambda^{-1}(S) < \delta_0/2$ for each subset $S \subset \lambda(X)$ satisfying $\operatorname{diam} S < \eta$. Let W denote the $(\eta/3)$-neighborhood of $\lambda(X)$ in A. Find a smaller neighborhood W' of $\lambda(X)$ there such that every map $\partial B^k \to W'$ extends to a map $B^k \to W$. Let $V^* = Z \cap g^{-1}(W') \supset X$, and restrict V^* so there exists $\gamma \in (0, \delta_0)$ with $\operatorname{diam} g(C) < \eta$ whenever $C \subset V^*$ and $\operatorname{diam} C < \gamma$. Finally, let V be the intersection of V^* with the γ-neighborhood of X.

Now consider a map f: $\partial B^{k+1} \to V$. By hypothesis, gf extends to a map F: $B^{k+1} \to W \subset A$. Name a triangulation T of B^{k+1} with mesh so small that, for each $\sigma \in T$, $\operatorname{diam} F(\sigma) < \eta/3$ and, when $\sigma \subset \partial B^k$, $\operatorname{diam} f(\sigma) < \delta_0$. Given any vertex $v \in T$ not in ∂B^{k+1}, choose a point $v' \in \lambda(X)$ such that $\operatorname{dist}(v', F(v)) < \eta/3$. Define F': $\partial B^k \cup T^{(0)} \to S$ as $F'(v) = \lambda^{-1}(v')$ for vertices $v \in T^{(0)}$ and $F'(z) = f(z)$ for $z \in \partial B^{k+1}$. Check that for each 1-simplex e of T not in ∂B^{k+1}, $\operatorname{diam} F'(\partial e) < 2\delta_0$ (the tricky case occurs when $\partial e = \{v, w\}$ with $v \in \operatorname{Int} B^{k+1}$, $w \in \partial B^{k+1}$: here choose $x \in X$ so $\rho_S(x, f(w) = F'(w)) < \gamma < \delta_0$ and note that $\rho_A(g(x) = \lambda(x), gf(w)) < \eta$, which implies $\rho_A(\lambda(x), F(v)) < \eta$ and $\rho_S(x, F'(v)) < \delta_0$). Apply the predetermined 1-ULC features to obtain an extension F': $e \to S$ into a δ_1-subset of S. Under the assumption that F': $\partial B^{k+1} \cup T^{(i-1)} \to S$ is an extension of f with $\operatorname{diam} F'(\sigma) < \delta_{i-1}$ for each $(i-1)$-simplex $\sigma \in T$, the i-ULC arrangements give rise to an extension F': $\partial B^{k+1} \cup T^{(i)} \to S$ such that $\operatorname{diam} F'(\sigma) < \delta_i$ for each i-simplex $\sigma \in T$. When $i = k+1$, this provides F': $B^{k+1} \to U$ extending f on ∂B^{k+1}, as desired. □

COROLLARY 2.8. *Suppose p: $Q \to Q'$ is a cell-like, surjective map, and $X' \subset Q'$ has Property k-UV (in the absolute sense). Then X' has Property k-UV in Q' and X has Property k-UV.*

Proof. Here Q' is known to be ULC^k for all k, so X' has Property k-UV by Lemma 2.7. That X also has Property k-UV follows because, due to cell-likeness,

p induces isomorphisms $\pi_i(p^{-1}(U)) \to \pi_i(U)$ for all connected open subsets U of Q'. □

COROLLARY 2.9. *Suppose p: $Q \to Q'$ is a cell-like surjective mapping and $X' \subset Q'$ is a hereditarily aspherical compactum. Then p is hereditarily aspherical over X'.*

THEOREM 2.10. *For any compact metric space X, the following statements are equivalent.*

(1) *X is strongly hereditarily aspherical.*

(2) *For every map f: $X \to L$ to a polyhedron L and every triangulation $\mathcal{L}$ of L, f admits an $\mathcal{L}$-lifting f': $X \to \alpha L$ to some asphericalization g: $\alpha L \to L$ of $(L, \mathcal{L})$.*

(3) *X can be expressed as an inverse limit of a system $\{L_i, g_i^{i+1}\}$, where g_i^{i+1}: $L_{i+1} \to L_i$ is an asphericalization of L_i with respect to some triangulation $\mathcal{L}_i$ and where mesh $g_i^{i+1}(\mathcal{L}_{i+k}) \to 0$ as $k \to \infty$.*

Proof. Assume (1), and consider a map f of X to a polyhedron L endowed with triangulation $\mathcal{L}$. Let $\mathfrak{I}$ be the cover of L by open stars of vertices of $\mathcal{L}$, and then let $\mathfrak{I}'$ be an open cover of X refining $f^{-1}(\mathfrak{I})$ such that every finite union of elements of $\mathfrak{I}'$ is aspherical. Let αL be the nerve of $\mathfrak{I}'$ and f': $X \to \alpha L$ a barycentric map. Define g: $\alpha L \to L$ on vertices of αL by choosing $g(v)$ to be a vertex w of L such that $f(V_v) \subset \mathrm{St}(g(v), L)$ and extend g linearly over the simplexes of αL. It follows that f' is an $\mathcal{L}$-lifting and, thus, (2) holds.

Next assume (2). Not only will we determine the system $\{L_i, g_i^{i+1}\}$ involving asphericalizations g_i^{i+1}: $L_{i+1} \to L_i$, we will simultaneously construct $1/i$ maps ν_i: $X \to L_i$ such that $g_{i-1}^{i}\nu_i$ is a $\mathcal{L}_{i-1}$-lifting of ν_{i-1}. To get started, take μ_1: $X \to N_1$ to be a barycentric map to the nerve of some finite open cover of X, impose a triangulation $\mathcal{P}_1$ on N_1 of mesh less than 1, and then apply (2) to obtain a lifting ν_1: $X \to L_1 = \alpha N_1$ to some asphericalization G_1: $L_1 \to N_1$ of $(N_1, \mathcal{P}_1)$. Assume inductively that all this data $(L_j, \mathcal{L}_j, \nu_j, g_j^{j+1})$, with the exception of g_k^{k+1}, has been constructed for $j = 1, \ldots, k$. By [Hu, Theorem 8.1] there exists a $1/(k+1)$-map μ_{k+1}: $X \to N_{k+1}$ to the nerve N_{k+1} of some finite open cover of X, as well as a map φ_{k+1}: $N_{k+1} \to L_k$ such that $\varphi_{k+1}\mu_{k+1}$ is $\mathcal{L}_k$-close to ν_k. Specify a triangulation $\mathcal{P}_{k+1}$ of N_{k+1} with mesh so small that $\operatorname{diam}(\mu_{k+1})^{-1}(\sigma) < 1/(k+1)$ and $\operatorname{diam} g_i^k\varphi_{k+1}(\sigma) < 1/k$ for each $\sigma \in \mathcal{P}_{k+1}$ and $1 \leq i \leq k$. Apply (2) to obtain an asphericalization G_{k+1}: $L_{k+1} = \alpha N_{k+1} \to N_{k+1}$. Set $g_k^{k+1} = \varphi_{k+1}G_{k+1}$ and name a triangulation $\mathcal{L}_{k+1}$ of L_{k+1} such that G_{k+1} sends cells of $\mathcal{L}_{k+1}$ into cells of $\mathcal{P}_{k+1}$. The sequence $\{L_i, g_i^{i+1}\}$ iteratively constructed in this way fulfills the small mesh features of (3), and the other surrounding conditions ensure that the inverse limit of this sequence is homeomorphic to X. Hence (2) implies (3).

To complete the cycle, the proof that (3) implies (1) can be found in [Da, Proposition 1]. □

3. Strongly hereditarily aspherical subsets

The proof of the following occupies the bulk of this section.

SHA SUBSET THEOREM 3.1. *Each compact metric space X contains a 0-dimensional G_δ-subset W (alternatively, a 0-dimensional F_σ-subset) such that compacta in $X\setminus W$ are strongly hereditarily aspherical.*

For applications of 3.1 it is useful to note that $\dim(X\setminus W) \geq \dim(X) - 1$ [HW, p. 28] and that, by the Sum Theorem For Dimension n [HW, p. 30], $X\setminus W$ contains a compact subset C for which $\dim(C) = \dim(X\setminus W)$.

COROLLARY 3.2. *For $n = 2, 3, \ldots$ there exists an n-dimensional strongly hereditarily aspherical compactum in S^{n+1}; furthermore, there exists a strongly hereditarily aspherical compactum embedded as a separating subset of the Hilbert cube.*

Corollary 3.2 answers Question 1 of [Da]. Before turning to the arguments, we present an example loosely related to the 0-dimensional F_σ-subset of S^n promised in Theorem 3.1.

Example. A Cantor set in S^n with aspherical complement. Consider the Blankinship construction [Bk] of a Cantor set C in $T^{n-2} \times \mathbb{R}^2$ (here T^{n-2} denotes the $(n-2)$-dimensional torus). The crucial observation is that $(T^{n-2} \times \mathbb{R}^2)\setminus C$ is aspherical (to see why, check how Blankinship presents this space essentially as a union of chambers of the form $M \times T^{n-3}$, where M is a solid torus from which is deleted the interiors of four solid tori, linked as in the related 3-dimensional Antoine's necklace construction; since M is known to be aspherical and, for each component S of ∂M, S is aspherical and $\pi_1(S) \to \pi_1(M)$ is an injection, asphericity of $(T^{n-2} \times \mathbb{R}^2)\setminus C$ follows from [Wh, Theorem 5]. Pass to the universal cover $\theta\colon \mathbb{R}^n = \mathbb{R}^{n-2} \times \mathbb{R}^2 \to T^{n-2} \times \mathbb{R}^2$. Clearly $\mathbb{R}^n\setminus\theta^{-1}(C)$ is also aspherical. Define the desired Cantor set in $S^n = \mathbb{R}^n \cup \{\infty\}$ as $\theta^{-1}(C) \cup \{\infty\}$.

LEMMA 3.3. *Suppose $\{H_i\}$ is a countable collection of CW-complexes with each $S^0 * H_i$ contractible. Then each compact metric space X contains a 0-dimensional, G_δ-subset W such that $(X\setminus W)\tau H_i$ for all i.*

Proof. Contractibility of $S^0 * H_i$ makes it an absolute retract, so $X\tau(S^0 * H_i)$. The proof of Corollary 3 in [Dr2] gives a G_δ-subset W of X such that $W\tau S^0$ (in other words, W is 0-dimensional) and $Y\tau H_i$ for each compact $Y \subset W$. By Proposition 2.3 of [Dr2], $(X\setminus W)\tau H_i$. This establishes the lemma for one such space; for a countable family, use the countable wedge. □

LEMMA 3.4. *Suppose X is a compact metric space, L is a polyhedron with triangulation $\mathcal{L}$, and $g\colon K \to L$ is a map such that $X\tau g^{-1}(\sigma)$ for all $\sigma \in \mathcal{L}$. Then each map $f\colon X \to L$ admits a $\mathcal{L}$-lifting $f'\colon X \to K$.*

Proof. This is an easy inductive application of the extension property —assuming f' has been defined on $f^{-1}(L^{(k-1)})$ so that $f'(f^{-1}(\gamma)) \subset g^{-1}(\gamma)$ for all $\gamma \in L^{(k-1)}$, one invokes $X \tau g^{-1}(\sigma)$ for k-simplexes $\sigma \in \mathcal{L}$ to extend f' so that $f'(f^{-1}(\sigma)) \subset g^{-1}(\sigma)$. □

Next we review the *Gromov Construction* [Gr], exploited extensively in [DJ, §4]. The data consists of a space P equipped with a reflection r: $P \to P$ (r^2 = identity), where P is expressed as a union of closed subsets P_0 and P_1, $A = P_0 \cap P_1$, $r \mid A =$ identity, and $r(P_0 \backslash A) = P_1 \backslash A$. We call P_0 (or P_1) a *fundamental domain* of r. Define $\Omega(P, r)$ as $P \times [0, 1]/\sim$, where $\sim$ is the relation with nontrivial classes $x \times \{0\} = x \times \{1\}$ for $x \in P_1$. There is a natural inclusion j: $P \to \Omega(P, r)$ such that $j(x_i \in P_i)$ is either ($i = 0$) the image of $x_0 \times \{0\}$ or ($i = 1$) the image of $r(x_1) \times \{1\}$.

LEMMA 3.5. *Suppose $\zeta \in H_*(P)$ satisfies $r_*(\zeta) = -\zeta$, where r: $P \to P$ is the reflection, as above, and suppose $H_*(A) \to H_*(P)$ is trivial. Then the inclusion j: $P \to \Omega(P, r)$ satisfies $j_*(\zeta) = 0$.*

Proof. Name a k-cycle $z = c_0 + c_1$ representing ζ, where c_i is supported in P_i. By hypothesis, ζ is also carried by $-r_\#(c_0 + c_1)$. Let d be a $(k + 1)$-cycle such that

$$c_0 + c_1 = -r_\#(c_0) - r_\#(c_1) + \partial d,$$

and express d as $d_0 + d_1$ with d_i supported in P_i. Hence there exists a k-chain e carried by A such that

$$(*) \qquad c_0 = -r_\#(c_1) + \partial d_0 + e.$$

The image of $z \times I$ in $\Omega(P,r)$ satisfies

$$\partial(z \times I) = c_0 \times 1 + c_1 \times 1 - c_0 \times 0 - c_1 \times 0 = c_0 \times 1 - c_0 \times 0$$

due to the identification of $P_1 \times 0$ with $P_1 \times 1$. As a result,

$$\begin{aligned} j_\#(z) &= j_\#(c_0 + c_1) = c_0 \times 0 + r_\#(c_1) \times 1 \\ &= c_0 \times 0 - c_0 \times 1 + \partial d_0 \times 1 + e \times 1 \\ &= -\partial(z \times I) + \partial d_0 \times 1 + e \times 1. \end{aligned}$$

The preceding equality reveals that $e \times 1$ is a k-cycle in $j(A)$ and, as such, it is nullhomologous in $j(P) \subset \Omega(P, r)$. Hence, $j_*(\zeta) = j_*([z]) = 0$. □

ADDENDUM TO 3.5. *Let p: $P \to P_0$ denote the obvious retraction with $p = p \cdot r$. The conclusion of Lemma 3.5 holds with no assumption of triviality for $H_*(A) \to H_*(P)$ provided order $(p_*(\zeta)) \neq 2$.*

Proof. Here $p_*(\zeta) = p_* \cdot r_*(\zeta) = -p_*(\zeta)$, yielding $p_*(\zeta) = 0$. Consequently, in the notation of 3.5,

$$p_{\#}(c_0 + c_1) = c_0 + r_{\#}(c_1) = \partial d_0,$$

promising that one can take $e = 0$ in equation $(*)$, from which the conclusion quickly follows, just as before. □

This leads to the promised generalization of a grope. Recall that a grope G is an aspherical, acyclic 2-complex having a natural simple closed curve "boundary", ∂G, where inclusion $\partial G \to G$ induces an injection of fundamental groups. See the description given by Cannon [C1, Supplement 13]. Actually, it is possible to obtain a finite 2-complex possessing exactly the same stated features [DV].

Another device used in [DJ] is put to work here as well. In order to define $\alpha\sigma^{n+1}$, we will presume $\psi_n: \alpha\sigma_n \to \sigma_n$ as already defined. Given a nondegenerate simplicial map $\pi: S \to \sigma^n$ we then will use $\alpha\sigma^n \Delta S$ to denote the fiber product of $\alpha\sigma^n$ and S over σ^n, also known as the *Williams functor* [Wi] of $(\alpha\sigma_n, \psi_n)$ and S over σ^n (explicitly, $\alpha\sigma^n \Delta S$ is the set of all $\langle x, s\rangle \in \alpha\sigma^n \times S$ such that $\psi_n(x) = \pi(s)$).

The referee has pointed out that the next result was proved by Maunder [Ma], who actually produced a finite complex $\alpha\sigma^n$. We include an argument for completeness.

PROPOSITION 3.6. *Let σ^n denote the n-simplex. There exists a map $\psi_n: \alpha\sigma^n \to \sigma^n$ such that*:

(1) *$\alpha\sigma^n$ is an aspherical n-dimensional complex*;
(2) *ψ_n is an asphericalization*;
(3) $H_*(\alpha\sigma^n) = 0$;
(4) *$\pi_1((\psi_n)^{-1}(K) \to \pi_1(\alpha\sigma^n)$ is 1-1, for each connected subcomplex K of $\partial\sigma^n$*;
(5) *$(\psi_n)^{-1}(\gamma^i)$ is homeomorphic to $\alpha\sigma^i \Delta\beta^r(\sigma^i)$ for each i-simplex γ^i in $\partial\sigma^n$ and for some choice of r.*

Proof. The $n = 1$ case is trivial: $\alpha\sigma^1 = \sigma^1$ and $\psi_1 =$ identity.

The $n = 2$ case takes a bit more work. Let $\alpha\sigma^2$ be a grope and $\psi: \alpha\sigma^2 \to \sigma^2$ a map collapsing to a point the complement of an annular neighborhood of the grope's boundary, $\partial\alpha\sigma^2$, coordinatized so $\psi_2 \mid \partial\alpha\sigma^2$ acts like inclusion.

The remaining cases involve intrinsically infinite complexes, which we describe recursively. Suppose ψ_i and $\alpha\sigma^i$ have been defined for $i \leq n$. Let S denote the first barycentric subdivision of the standard triangulation of $\partial\sigma^{n+1}$, endowed with the standard nondegenerate simplicial mapping $\pi: S \to \sigma^n$ sending the barycenter of any j-simplex to the vertex of σ^n labelled j. Apply the Williams functor to obtain a complex $P = \alpha\sigma^n \Delta S$ and map $\psi_{n+1}: P \to \partial\sigma^{n+1}$ (projection to the second coordinate of $P = \alpha\sigma^n \Delta S$ in $\alpha\sigma^n \times S$) such that $(\psi_{n+1})^{-1}(\gamma^i) = \alpha\sigma^i \Delta\beta^r(\sigma^i)$ for some choice of r and for each i-simplex γ^i in S. Now a simplicial reflection r': $\partial\sigma^{n+1} \to \partial\sigma^{n+1}$ determined by tranposing a pair of vertices gives rise to a reflection r: $P \to P$, by functoriality of the Williams construction. A routine Mayer-Vietoris argument confirms that ψ_{n+1}: $(\psi_{n+1})^{-1}(K) \to K$ induces an isomorphism of homology groups, for every subcomplex K of $\partial\sigma^{n+1}$; consequently, r_*: $H_n(P) \to H_n(P)$ amounts to multiplication by -1, as the same certainly holds for r'_*: $H_n(\partial\sigma^{n+1}) \to H_n(\partial\sigma^{n+1})$. Perform the Gromov construction $\Omega(P,r)$. Not only does inclusion j: $P \to \Omega(P,r)$ induce the trivial homomorphism on n^{th} homology, by Lemma 3.5, but also other straightforward homology calculations show that

$$H_*(\Omega(P,r)) \cong H_1(\Omega(P,r)) + H_n(\Omega(P,r))$$

with $H_1(\Omega(P,r))$ carried by the image C_1 of $a \times [0,1]$, where $r(a) = a$, and $H_n(\Omega(P,r))$ by the image D_1 of $P \times 0$ in $\Omega(P,r)$. To $\Omega(P,r)$ attach a (2-dimensional) grope G_1 with $\partial G_1 = C_1$ and attach another copy of $\Omega(P,r)$ to D_1, specifically equating $j(P) \subset \Omega(P,r)$ with D_1. Repeat the procedure in the new copy and iterate, producing in this way countably indexed collections $\{G_i\}$ and $\{\Omega(P,r)_i\}$ of gropes and Gromov constructions such that G_{i+1} and $\Omega(P,r)_{i+1}$ kill the homology of $\Omega(P,r)_i$. Then the union $\alpha\sigma^{n+1}$ of both collections is an acyclic, aspherical $(n+1)$-complex; asphericity follows exactly as in [DJ, Lemma (1.h2)]. Moreover, statement (4) holds because, for each connected subcomplex K of $\partial\sigma^{n+1}$, $\pi_1((\psi_{n+1})^{-1}(K) \to \pi_1(P)$ is 1-1 by [DJ, Proposition 1h.1] and $\pi_1(P) \to \pi_1(\alpha\sigma^{n+1})$ is 1-1 by [DJ, Proposition 4c.2]. Define ψ_{n+1}: $\alpha\sigma^{n+1} \to \sigma^{n+1}$ as an arbitrary extension of $\psi_{n+1} \mid (P = \alpha\sigma^n \Delta S)$ with $\psi_{n+1}(\alpha\sigma^{n+1} \backslash P) \subset \operatorname{Int}\sigma^{n+1}$. □

Proof of Theorem 3.1. Let $\{H_i\}$ be an enumeration of all Williams complexes of the form $\alpha\sigma^n \Delta \beta^r(\sigma^n)$, for all integers $r \geq 0$ and $n > 0$, where $\beta^r(\sigma^n)$ denotes the r^{th} barycentric subdivision of σ^n and $\alpha\sigma^n$ the complex promised in Proposition 3.6 for n. By the Hurewicz Theorem $S^0 * H_i$ is contractible. Hence, Lemma 3.3 gives a 0-dimensional G_δ-subset W of X with $(X\backslash W)\tau H_i$ for every i. We claim that this makes every compact subset Y of $X\backslash W$ strongly hereditarily aspherical. To that end, consider a finite, n-complex L triangulated by $\mathcal{L}$ and map f: $Y \to L$. Let π: $\beta(L) \to \sigma^n$ be the standard simplicial mapping defined as in 3.6. Use the Williams functor to get an asphericalization g: $\alpha L \to L$ of $(L, \mathcal{L})$ such that, for each n-simplex $\gamma^n \in L$, $g^{-1}(\gamma^n)$ equals some H_i. Then invoke Lemma 3.4 to get an $\mathcal{L}$-lifting f': $Y \to \alpha L$ of f. Theorem 2.10 certifies that Y is strongly hereditarily aspherical.

For the alternative form locating a 0-dimensional F_σ-subset, simply use [Ol] to obtain a G_δ-subset Z of X containing $X\backslash W$ such that $Z\tau H_i$ for all i. Then $X\backslash Z$ is the required F_σ-set for, just as shown above, each compact subset Y of Z is strongly hereditarily aspherical.

THEOREM 3.7. *There exist a 2-dimensional compactum $F \subset Q$ and cell-like map $p\colon Q \to Q'$ such that*
$F' = p(F)$ and $F = p^{-1}(F')$,
p is hereditarily aspherical over F',
$\dim F' = \infty$.

Proof. Consider a Dydak-Walsh cell-like map $p\colon X \to X'$ where $\dim X = 2$ and $\dim X' = \infty$. Embed X in Q and extend p to a cell-like map $p\colon Q \to Q'$ which is 1-1 on $Q\backslash X$. Apply SHA Subset Theorem 3.1 to find a 0-dimensional $Z \subset X'$ such that $X'\backslash Z$ is a countable union of strongly hereditarily aspherical compact subsets T_i. Corollary 2.9 implies p is hereditarily aspherical over T_i for each i. Moreover, there exists an index k with $\dim T_k = \infty$, as otherwise we would have $\dim T_i \leq 2$ for all i and therefore $\dim X' < \infty$, since X' would be expressed as a union of the 0-dimensional space Z and its 2-dimensional complement. □

COROLLARY 3.8. *There exists a cell-like map $p\colon Q \to Q'$ and a 2-dimensional subset $F = p^{-1}p(F)$ of Q such that p is hereditarily aspherical but not strongly hereditarily aspherical over $p(F)$.*

The compactum F' of Theorem 3.7 reveals that covering dimension and integral cohomological dimension do not coincide for strongly hereditarily aspherical spaces, answering Question 3 of [Da] in the negative.

At one time we thought perhaps every 2-dimensional compact metric space which is k-UV for all $k \geq 2$ could be expressed as an inverse limit of aspherical 2-dimensional polyhedra. However, the proof of Karimov's result [Ka] shows this is false. Still open is:

GENERALIZED WHITEHEAD CONJECTURE. *If X is a 2-dimensional compactum having Property 2-UV, then every subcompactum has Property 2-UV.*

When X is a 2-complex, the conjecture above is equivalent to the classical Whitehead Conjecture, unsolved at this juncture. The truth of the Generalized Whitehead Conjecture would imply that the example promised by Corollary 3.8 is actually hereditarily aspherical and would provide a negative answer to Question 2 of [Da]. Furthermore, if that example could be expressed as an inverse limit of aspherical 2-complexes, then the classical Whitehead Conjecture would be false.

Example. A cell-like subset of an n-manifold that does not embed in $\mathbb{R}^n$. McMillan [Mc] has described an arc α in an n-manifold, no neighborhood of which embeds in $\mathbb{R}^n$. A minor adaption of the source of α gives an example of the desired sort. The source is obtained as a cell-like image of a wedge of the suspensions of two continua X^+, X^- disjointly embedded in S^3. For $n = 4$ McMillan uses a 4-manifold arising as the interior of (B^4 plus two attached 2-handles), where the individual 2-handles are attached to $S^3 = \partial B^4$ so as to cover X^+ and X^- separately. Each suspension point corresponds to the center of either B^4 or one of the 2-handles. Expanding McMillan's wedge, one can simply thicken each of the three points corresponding to suspension points to 4-cells, in the natural way so each 4-cell meets the wedge in a cone. The union C of McMillan's example and the three 4-cells is the desired cell-like set. If C were embedded in $\mathbb{R}^4$, one could trim back the 4-cells slightly and tube together the two not containing the wedge point to produce an embedding λ of $(X^+ \cup X^-) \times [0, 1]$ in $S^3 \times [0, 1]$ satisfying

$$\lambda((X^+ \cup X^-) \times [0, 1]) \cap S^3 \times \{i\} = \lambda((X^+ \cup X^-) \times \{i\}), i \in \{0, 1\},$$

with $\lambda((X^+ \cup X^-) \times \{0\})$ embedded in $S^3 \times \{0\}$ just like McMillan's example and with the two components of $\lambda((X^+ \cup X^-) \times \{1\})$ separated in $S^3 \times \{1\}$ by some 2-sphere, an arrangement which the argument of [Mc] shows cannot occur.

Other questions

1. If $\dim X = 2$ and X' has rational cohomological dimension at most 1, is p hereditarily aspherical over X'?

2. Is the compactum F of Theorem 3.7 itself hereditarily aspherical?

3. If p: $F \to F'$ is cell-like, where $\dim F = 2$, F' is hereditarily aspherical and fibers are 1-dimensional, is F hereditarily aspherical? Can $\dim F' > \dim F$? Same questions if F' has rational cohomological dimension 1.

4. Can every acyclic 2-dimensional compactum be embedded in $\mathbb{R}^4$? What if acyclic and aspherical? What about cell-like sets? What about cell-like subsets of 4-manifolds?

5. Can proper, cell-like maps on 4-complexes raise dimension? What about on 4-manifolds? On 3-complexes? The best result available concerning 3-complexes, which appears in [KRW], gives a negative answer provided all point preimages are 1-dimensional.

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