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TRANSFERENCE OF MULTILINEAR OPERATORS

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0. Introduction and statement of results

Fix an integer $k \geq 2$. Let G be an amenable group and $(M, d\mu)$ a measure space. For $0 \leq j \leq k$, let $0 < p_j \leq \infty$, and assume that $p_0 = p$ is given by

$$\frac{1}{p_0} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}.$$

Assume that for any $0 \leq j \leq k$ and any $u \in G$, R_u^j is a bounded map from the Banach space $L^{p_j}(M)$ into itself. We denote by $\|R_u^j\|_{\mathrm{op}}$ the operator norm of $R_u^j \colon L^{p_j}(M) \to L^{p_j}(M)$. We say that R_u^j is strongly continuous if for any sequence $u_n \to u$ in the topology of G, we have $\|R_{u_n}^j - R_u^j\|_{\mathrm{op}} \to 0$. We call the family $(R_u^0, R_u^1, \ldots, R_u^k)_{u \in G}$ a transference $(k+1)$-tuple if the following are true:

$$\text{For } 0 \leq j \leq k, \text{ the maps } u \to R_u^j \text{ are strongly continuous.} \tag{0.1}$$

$$\sup\{\|R_u^j\|_{\mathrm{op}},\ u \in G\} = C_j < \infty, \ \text{ for } 0 \leq j \leq k. \tag{0.2}$$

$$R_v^0 R_u^j f = R_{vu}^j f, \text{ for all } u, v \in G,\ 1 \leq j \leq k, \text{ and all } f \in \mathcal{D}, \tag{0.3}$$

where $\mathcal{D}$ is some dense subclass of all the spaces $L^{p_j}(M)$ and we are implicitly assuming that the domain of any R_u^0 includes the ranges of each of the R_v^j. [BPW] used transference couples ($k = 1$) to transfer boundedness properties of convolution and maximal operators. In this paper, we will use $(k+1)$-tuples to transfer boundedness properties of multilinear operators from amenable groups into measure spaces. The general maximal transference presented in [BPW] can be extended to the multilinear setting, but this will not concern us in this paper.

We need to make the additional assumption that each R_u^0 is multiplicative. More precisely, this means that $R_u^0(fg) = (R_u^0 f)(R_u^0 g)$ whenever f, g, and fg belong to $\mathcal{D}$. This property is clearly satisfied if the R_u^0's are given by actions on the points of M; i.e. for all $u \in G$ there exist maps $U_u \colon M \to M$, such that

$$(R_u^0 f)(x) = f(U_{u^{-1}}^0 x). \tag{0.4}$$

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In this paper we shall, in fact, assume that (0.4) holds. In many settings (0.4) is a consequence of being multiplicative. Moreover, the restriction given by (0.4) is used explicitly for all the families R^j in the proof of the weak-type transference announced in Theorem 2. Let λ be left Haar measure on G. It is well known that if G is amenable with respect to left Haar measure λ, it is also amenable with respect to right Haar measure ρ. The spaces $L^{p_j}(G)$ are defined with respect to left Haar measure λ. Consider the multilinear operator T on the group G defined by

$$(0.5)\quad T(g_1,\dots,g_k)(v)=\int_{G^k} K(u_1,\dots,u_k)g_1(u_1^{-1}v)\dots g_k(u_k^{-1}v)\,d\lambda(u_1)\dots d\lambda(u_k),$$

for g_j in some dense subspace of $L^{p_j}(G)$, where K is a kernel on G which may not be integrable. For $k=1$, T is a usual convolution operator but for $k\geq 2$ it isn't. We transfer the operator T to an operator $\tilde{T}$ defined by

$$(0.6)\quad \tilde{T}(f_1,\dots,f_k)(x)$$
$$=\int_{G^k} K(u_1,\dots,u_k)(R^1_{u_1}f_1)(x)\dots(R^k_{u_k}f_k)(x)\,d\lambda(u_1)\dots d\lambda(u_k),$$

for f_j in $\mathcal{D}$. We have the following:

THEOREM 1. *Let T be as in* (0.5), *where the R^j_u's satisfy* (0.1), (0.2), (0.3), *and* (0.4). *Assume that T is a bounded operator from $L^{p_1}(G)\times\cdots\times L^{p_k}(G)\to L^p(G)$ with bound N. Then $\tilde{T}$ can be extended to a bounded operator from $L^{p_1}(M)\times\cdots\times L^{p_k}(M)\to L^p(M)$ with bound no larger than $NC_0C_1\dots C_k$.*

We denote by $L^{p,\infty}(M)$ the space weak $L^p(M)$ with quasinorm

$$\|f\|_{L^{p,\infty}}=\sup_{\alpha>0}\alpha\,[\mu(\{x\in M\colon |f(x)|>\alpha\})]^{\frac{1}{p}}.$$

Let us now consider the case where all the R^j's are given by actions on points. That is, for all $1\leq j\leq k$ and for all $u\in G$, there exist maps $U^j_u\colon M\to M$ such that the representations R^j_u have the special form

$$(0.7)\qquad (R^j_uf)(x)=f(U^j_{u^{-1}}x).$$

In this case, we replace condition (0.3) by

$$(0.8)\quad U^j_{uv}f=U^j_uU^0_vf\qquad \text{for all } j=1,\dots,k, \text{ all } u,v\in G, \text{ and all } f\in\mathcal{D}.$$

We now have the following:

THEOREM 2. *Assume that the R^j_u's satisfy* (0.1), (0.2), (0.7), *and* (0.8). *Assume that T given by* (0.5) *extends to a bounded operator from $L^{p_1}(G)\times\cdots\times L^{p_k}(G)\to L^{p,\infty}(G)$ with norm N. Then $\tilde{T}$ can be extended to a bounded operator $L^{p_1}(M)\times\cdots\times L^{p_k}(M)\to L^{p,\infty}(M)$ with a bound no larger than $NC_0C_1\dots C_k$.*

Finally, observe that an immediate consequence of (0.3) is

$$R_v^0 R_{v^{-1}}^0 R_u^j = R_u^j \tag{0.9}$$

for all $u, v \in G$ and $1 \leq j \leq k$.

1. The proof of Theorem 1

We first assume that $L = \text{support}(K)$ is compact in all variables and that K is bounded in absolute value by some constant C_K on L. Once the required estimate is proved for such kernels K, with bounds independent of their support and their size, a density argument will give the conclusion for all kernels K.

The amenability of G is equivalent to Leptin's condition: given $\epsilon > 0$ and B a compact subset of G, there exists an open subset V of G, such that $\bar{B}$ is compact and

$$\lambda(B^{-1}V) \leq (1+\epsilon)\lambda(V). \tag{1.1}$$

For a given $\epsilon > 0$ and $L = \text{support}\,(K)$, fix such a V. Also fix $f_1, \dots, f_k \in \mathcal{D}$. The multiplicative property of R_v^0 and (0.9) imply

$$\begin{aligned} &\tilde{T}(f_1, \dots, f_k)(x) \\ &= \int_{G^k} K(u_1, \dots, u_k) R_v^0 \left[\prod_{j=1}^{k} (R_{v^{-1}u_j}^j f_j) \right] (x)\, d\lambda(u_1) \dots d\lambda(u_k) \end{aligned} \tag{1.2}$$

for all v in G. By the continuity of R_v^0, we can "move" R_v^0 outside the k-fold integral in (1.2). Since $\tilde{T}(f_1, \dots, f_k)$ is in $L^p(M)$, (with bounds that depend on K) and R_v^0 is bounded on $L^p(M)$ uniformly in $v \in G$, the following estimate holds for all v in G:

$$\begin{aligned} &\int_M |\tilde{T}(f_1, \dots, f_k)(x)|^p \, d\mu(x) \\ &\leq C_0^p \int_M \left| \int_{G^k} K(u_1, \dots, u_k) \prod_{j=1}^{k} \left(R_{v^{-1}u_j}^j f_j \right)(x)\, d\lambda(u_1) \dots d\lambda(u_k) \right|^p d\mu(x). \end{aligned} \tag{1.3}$$

Next, we average inequality (1.3) over V and we interchange the order of integration to the right hand side of the averaged inequality. We obtain

$$\begin{aligned} &\int_M |\tilde{T}(f_1, \dots, f_k)(x)|^p \, d\mu(x) \\ &\leq \frac{C_0^p}{\lambda(V)} \int_M \int_V \left| \int_{G^k} K(u_1, \dots, u_k) \prod_{j=1}^{k} \left(R_{v^{-1}u_j}^j f_j \right)(x)\, d\lambda(u_1) \dots d\lambda(u_k) \right|^p \\ &\quad d\lambda(v) d\mu(x). \end{aligned} \tag{1.4}$$

We denote by χ_A the characteristic function of the set A. Observe that we can replace $(R^j_{v^{-1}u_j} f_j)(x)$ by $h_j(u_j^{-1}v, x)$ in (1.4), where $h_j(w, x) = (R^j_{w^{-1}} f_j)(x)\chi_{L^{-1}V}(w^{-1})$. Clearly $h_j(\cdot, x) \in L^{p_j}(G)$ for all $x \in M$. By the boundedness of T from $L^{p_1}(G) \times \cdots \times L^{p_k}(G) \to L^p(G)$, we deduce the estimate

$$\int_M |\tilde{T}(f_1, \dots, f_k)(x)|^p \, d\mu(x) \le \frac{C_0^p N^p}{\lambda(V)} \int_M \prod_{j=1}^k \|h_j(\cdot, x)\|^p_{L^{p_j}(G)} d\mu(x). \tag{1.5}$$

At this point, we apply Hölder's inequality with exponents

$$1 = \frac{1}{p_1/p} + \cdots + \frac{1}{p_k/p}$$

to the right hand side of (1.5). We first assume that all $p_j < \infty$ for all j. We have

$$\begin{aligned} &\int_M |\tilde{T}(f_1, \dots, f_k)(x)|^p \, d\mu(x) \\ &\le \frac{C_0^p N^p}{\lambda(V)} \prod_{j=1}^k \left(\int_M \|(R^j f_j)(x)\chi_{L^{-1}V}\|^{p_j}_{L^{p_j}(G)} \, d\mu(x) \right)^{\frac{p}{p_j}}. \end{aligned} \tag{1.6}$$

Interchanging the order of integration in (1.6), we obtain

$$\begin{aligned} \int_M |\tilde{T}(f_1, \dots, f_k)(x)|^p \, d\mu(x) &\le \frac{C_0^p N^p}{\lambda(V)} \prod_{j=1}^k \left(\int_{L^{-1}V} \int_M |R^j_{u_j} f_j|^{p_j} \, d\mu \, du_j \right)^{\frac{p}{p_j}} \\ &\le N^p \frac{\prod_{j=0}^k C_j^p}{\lambda(V)} \prod_{j=1}^k \left(\int_{L^{-1}V} \int_M |f_j g|^{p_j} \, d\mu \, du_j \right)^{\frac{p}{p_j}} \\ &= N^p \frac{\prod_{j=0}^k C_j^p}{\lambda(V)} \prod_{j=1}^k \lambda(L^{-1}V)^{\frac{p}{p_j}} \prod_{j=1}^k \|f_j\|^p_{L^{p_j}} \\ &\le (1+\epsilon) N^p \prod_{j=0}^k C_j^p \prod_{j=1}^k \|f_j\|^p_{L^{p_j}}, \end{aligned} \tag{1.7}$$

by Leptin's condition (1.1). Since $\epsilon > 0$ was arbitrary, the required conclusion follows. If some of the p_j's, but not all of them, are equal to ∞, factor out all the L^∞ norms from the second integral in (1.5) and since $1/p$ is the sum of the remaining $1/p_j$'s, we can apply Hölder's inequality to these p_j's. The rest of the proof is the same. Finally if $p_j = \infty$ for all j, then the argument above can be easily adapted to this case. The proof of Theorem 1 is now complete.

2. The proof of Theorem 2

We now suitably modify the proof of Theorem 1 to obtain Theorem 2. This modification is precisely the one used in Theorem (2.6) in [CW]. In this reference there is a discussion that motivates the arguments given which is certainly applicable here.

Let ϵ, L and V be as before. We first assume that all of the p_j's are finite. Fix $\alpha > 0$. Let $A_\alpha = \{x \in M\colon |\tilde{T}(f_1, \dots, f_k)(x)| > \alpha\}$ and for $v \in V$, let $B_\alpha(v) = \{x \in M\colon |\tilde{T}(f_1, \dots, f_k)(U_v^0 x)| > \alpha\}$. It is easy to check that $B_\alpha = R^0_{v^{-1}}[A_\alpha]$. By the boundedness of R^0 on $L^p(M)$ we obtain

$$(\mu(A_\alpha))^{\frac{1}{p}} \leq C_0(\mu(B_\alpha(v)))^{\frac{1}{p}}\ , \tag{2.1}$$

for all $v \in V$. Averaging the p^{th} power of (2.1) over V, we obtain

$$\begin{aligned} \mu(A_\alpha) &\leq \frac{C_0^p}{\lambda(V)} \int_V \mu(B_\alpha(v))d\lambda(v) = \frac{C_0^p}{\lambda(V)} \int_V \int_M \chi_{B_\alpha(v)}(x)\, d\mu(x)\, d\lambda(v) \\ &= \frac{C_0^p}{\lambda(V)} \int_M \int_V \chi_{D_\alpha(x)}(v)\, d\lambda(v)\, d\mu(x) = \frac{C_0^p}{\lambda(V)} \int_M \lambda(D_\alpha(x))\, d\mu(x), \end{aligned} \tag{2.2}$$

where $D_\alpha(x) = \{v \in V\colon x \in B_\alpha(v)\}$. By property (0.6) we have $D_\alpha(x) = \{v \in V\colon \int_{G^k} K(u_1, \dots, u_k) f_1(U_{u_1^{-1}v}x) \dots f_k(U_{u_k^{-1}v}x)\, d\lambda(u_1) \dots d\lambda(u_k) > \alpha\}$. We can now replace $f_j(U_{u_j^{-1}v}x)$ by $h_j(u_j^{-1}v, x)$, where $h_j(w, x) = f_j(U_w x)\chi_{L^{-1}V}(w)$. Clearly $h_j(\cdot, x) \in L^{p_j}(G)$ for all $x \in M$. The assumed weak type estimate for T gives

$$\lambda(D_\alpha(x)) \leq \frac{N^p}{\alpha^p} \prod_{j=1}^k \|h_j(\cdot, x)\|^p_{L^{p_j}(G)}\ . \tag{2.3}$$

Using (2.2) and (2.3), we obtain

$$\begin{aligned} \mu(A_\alpha) &\leq \frac{C_0^p N^p}{\lambda(V)\alpha^p} \int_M \prod_{j=1}^k \|h_j(\cdot, x)\|^p_{L_{p_j}(G)}\, d\mu(x) \\ &\leq \frac{C_0^p N^p}{\lambda(V)\alpha^p} \prod_{j=1}^k \left(\int_M \|h_j(\cdot, x)\|^{p_j}_{L_{p_j}(G)}\, d\mu(x) \right)^{\frac{p}{p_j}}\ , \end{aligned} \tag{2.4}$$

where we applied Hölder's inequality as before. By Fubini's Theorem and the boundedness of the maps R^j on $L^{p_j}(M)$, we obtain the following bound for (2.4)

$$N^p \frac{\prod_{j=0}^k C_j^p}{\lambda(V)\alpha^p} \prod_{j=1}^k \left(\int_{L^{-1}V} \int_M |f_j|^{p_j} \, d\mu \, du_j \right)^{\frac{p}{p_j}} \tag{2.5}$$

$$= N^p \frac{\prod_{j=0}^k C_j^p}{\lambda(V)\alpha^p} \prod_{j=1}^k \lambda(L^{-1}V)^{\frac{p}{p_j}} \prod_{j=1}^k \|f_j\|_{L^{p_j}(M)}^p$$

$$\leq (1+\epsilon) \frac{N^p}{\alpha^p} \prod_{j=1}^k C_j^p \prod_{j=0}^k \|f_j\|_{L^{p_j}(M)}^p ,$$

where we used Leptin's condition (1.1) in the last inequality above. Since $\epsilon > 0$ was arbitrary, (2.4) and (2.5) imply the required weak type inequality. The removal of the restriction on the support and the size of K is standard. Finally, the case where some or all of the p_j's are infinite is treated as in the previous section.

3. Remarks and applications

We begin by observing that the kernels $K(u_1, \ldots, u_k)$ of the previous sections can depend on l variables only, say $u_1, \ldots, u_l$, while the remaining $k - l$ variables can be linear functions of the first l variables. Let us consider the case where $u_{l+1}, \ldots, u_k$ are related to the variable u_l by the relation $\frac{u_l}{b_l} = \frac{u_{l+1}}{b_{l+1}} = \cdots = \frac{u_k}{b_k}$, where $b_l, \ldots, b_k$ are nonzero real numbers. More precisely, let

$$K = K_0(u_1, \ldots, u_l)\delta_{\frac{u_l}{b_l} = \frac{u_{l+1}}{b_{l+1}} = \cdots = \frac{u_k}{b_k}}, \tag{3.1}$$

where $1 \leq l < k$, δ is the Dirac distribution, and K_0 is a function of l variables. For this kernel K, the k-fold integral (0.5) defining T reduces to an l-fold integral. Assuming first that K_0 is compactly supported and bounded, the proofs of Theorems 1 and 2 apply as before with minor modifications. Then a density argument will give the conclusion for general K_0.

We are now going to give some applications of our theorems. Let $G = \mathbb{Z}$ with counting measure, $M = \mathbb{R}$ with Lebesgue measure, and $K(n_1, \ldots, n_k)$ be a complex-valued function on $\mathbb{Z}^k$, or a distribution of the type (3.1). For $1 \leq j \leq k$, let a_j be multipliers for $L^{p_j}(\mathbb{R})$ and define the operators R_u^j acting on $L^{p_j}(\mathbb{R})$ as follows:

$$(R_u^0 f)(x) = f(x-u) = (\hat{f}(\xi)e^{2\pi i u\xi})^\vee, \qquad (R_u^j f)(x) = (\hat{f}(\xi)a_j(\xi)e^{2\pi i u\xi})^\vee,$$

where we are using the definition $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx$. It is easy to see that the family $(R_u^0, \ldots, R_u^k)_{u\in\mathbb{Z}}$ satisfies (0.1)-(0.4), and thus it is a transference $(k+1)$-tuple as the ones we considered. Assume that the operator

$$T(g_1, \ldots, g_k)(n) = \sum_{(m_1,\ldots,m_k)\in\mathbb{Z}^k} K(m_1, \ldots, m_k) g_1(n-m_1)\ldots g_k(n-m_k) \tag{3.2}$$

maps $L^{p_1}(\mathbb{Z}) \times \cdots \times L^{p_k}(\mathbb{Z})$ into $L^p(\mathbb{Z})$. Then Theorem 1 implies that the transferred operator

$$\tilde{T}(f_1, \dots, f_k)(x) = \sum_{(m_1,\dots,m_k)\in\mathbb{Z}^k} K(m_1, \dots, m_k)(R^1_{m_1} f_1)(x) \dots (R^k_{m_k} f_k)(x),$$

maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R})$ into $L^p(\mathbb{R})$. In particular, if the multipliers $m_j(\xi)$ have the special form $e^{2\pi i d_j \xi}$ for some d_j real constants, and T maps $L^{p_1}(\mathbb{Z}) \times \cdots \times L^{p_k}(\mathbb{Z})$ into $L^{p,\infty}(\mathbb{Z})$, then by Theorem 2, $\tilde{T}$ maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R})$ into $L^{p,\infty}(\mathbb{R})$. An interesting situation arises when the kernel K is the distribution

$$K(n_1, \dots, n_k) = \frac{1}{n_1} \delta_{\frac{n_1}{b_1} = \frac{n_2}{b_2} = \dots = \frac{n_k}{b_k}}, \tag{3.3}$$

where b_j are nonzero and pairwise distinct numbers, and the notation in (3.3) means that all the variables $n_1, \dots, n_k$ have collapsed to being multiples of the single variable n_1. For $p \geq 1$, it is a difficult open question whether the operator T in (3.2) maps $L^{p_1}(\mathbb{Z}) \times \cdots \times L^{p_k}(\mathbb{Z})$ into $L^p(\mathbb{Z})$. Replacing $\frac{1}{n_1}$ by $\frac{1}{n_1^\epsilon}$ or by $\frac{1}{n_1(\log n_1)^{1+\epsilon}}$ in (3.3) for some $\epsilon > 0$, we obtain examples of multilinear operators for which we know that the operator T in (3.2) is bounded.

Next, we turn to an application regarding fractional integrals. Let $G = \mathbb{R}^1$ and $M = \mathbb{R}^n$, both with usual Lebesgue measure. For $g_1, \dots g_k$ functions on $\mathbb{R}^1$, and $0 < \alpha < 1$ let

$$I_\alpha(g_1, \dots g_k)(x) = \int_{-\infty}^{+\infty} g_1(x - \theta_1 t) \dots g_k(x - \theta_k t)\, |t|^{\alpha - 1}\, dt,$$

where $\theta_1, \dots, \theta_k$ are fixed nonzero and pairwise distinct numbers. Let $p_1, \dots, p_k > 1$, and assume that their harmonic sum p satisfies $\frac{1}{1+\alpha} \leq p < \frac{1}{\alpha}$. By Theorem 1 in [G], I_α maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R})$ into $L^q(\mathbb{R})$, where $\frac{1}{q} + \alpha = \frac{1}{p}$. Fix a unit vector $\omega \in S^{n-1}$. Using the maps $R^0_u =$ Identity, $(R^j_u f)(x) = f(x - u\theta_j \omega)$ for all $u \in \mathbb{R}$ and $0 \leq j \leq k$, we have that the transferred operator

$$\tilde{I}_{\alpha,\omega}(f_1, \dots f_k)(x) = \int_{-\infty}^{+\infty} g_1(x - \theta_1 t\omega) \dots g_k(x - \theta_k t\omega)\, |t|^{\alpha - 1}\, dt,$$

maps $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_k}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ when $\frac{1}{q} + \alpha = \frac{1}{p}$. Here we are using the fact that the kernel of I_α has the special form $K(u_1, \dots, u_k) = |\frac{u_1}{\theta_1}|^{\alpha-1} \delta_{\frac{u_1}{\theta_1} = \frac{u_2}{\theta_2} = \dots = \frac{u_k}{\theta_k}}$. Compare this result with Theorem 1 in [G] in dimension n.

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