# ARITHMETIC MACAULAYFICATIONS
# USING IDEALS OF DIMENSION ONE

IAN M. ABERBACH

## 1. Introduction

Let $R$ be a commutative Noetherian ring. For an ideal $I \subseteq R$ the Rees ring is $R[It] = R \oplus It \oplus I^2 t^2 \oplus \cdots$, where $t$ is an indeterminate. $R$ has an *arithmetic Macaulayfication* if there exists an $I \subseteq R$ such that $R[It]$ is Cohen-Macaulay. In this case $\mathrm{Proj}(R[It])$ is a Macaulayfication for $\mathrm{Spec}(R)$, however, the property that $R[It]$ is Cohen-Macaulay is significantly stronger than the condition that $\mathrm{Proj}(R[It])$ is a Cohen-Macaulay scheme.

Brodmann [Br1,3] and Faltings [F] have results on the existence of Macaulayfications. In essence, the argument involves blowing up an ideal generated by (part of) a system of parameters, where parameters kill certain local cohomology modules which obstruct the Cohen-Macaulay property.

Work of Brodmann [Br2] and Goto and Yamagishi [GY, 7.11] show the existence of an arithmetic Macaulayfication for local rings whose completions are equidimensional and Cohen-Macaulay on the punctured spectrum. The main result of [AHS] gives an explicit construction of an arithmetic Macaulayfication for rings of postive prime characteristic $p$ (see Sections 2 and 3 for information on tight closure and parameter test elements):

THEOREM 1.1 [AHS, THEOREM 4.1]. *Let $(R, m)$ be an excellent normal local domain of dimension $d$. Let $x_1, \ldots, x_d$ be any system of parameters such that each $x_i$ is a (parameter) test element. Then $R[Jt]$ is Cohen-Macaulay, where $J = ((x_1, \ldots, x_d)^{d-2})^*$.*

In particular, if $R$ is $F$-rational on the punctured spectrum, then such a $J$ may always be found. The fact that each $x_i$ is a test element is analogous to (in fact, stronger than) killing the local cohomology which obstructs the Cohen-Macaulay property.

We extend this result to the following:

THEOREM 4.1. *Suppose that $(R, m)$ is an excellent normal local domain of dimension $d \geq 3$. Let $x_1, \ldots, x_d$ be a system of parameters such that $x_1, \ldots, x_{d-1}$ are*

*parameter test elements. Set $J = ((x_1, \ldots, x_{d-1})^{d-2})^*$. Then $R[Jt]$ is a Macaulay-fication of $R$.*

Under mild conditions, the non-$F$-rational locus is closed. See [V]. As a corollary of Theorem 4.1, whenever the dimension of the non-$F$-rational locus of $R$ is 1 or less, then $R$ has an arithmetic Macaulayfication.

Kurano has independently obtained similar results. See [K].

I wish to thank the referee for helping to make this paper both more readable and more correct.

## 2. Tight closure and Rees ring results

We start with a short review of relevant tight closure results found in [HH1–2], [AHS].

Let $R$ be a Noetherian ring of positive prime characteristic $p$ and let $I \subseteq R$ be an ideal. For $I = (x_1, \ldots, x_n)$, let $I^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$. The element $x$ is in the tight closure of $I$, denoted by $I^*$, if there exists $c \in R$, but not in any minimal prime (this set is denoted by $R^0$), such that $cx^{p^e} \in I^{[p^e]}$ for all $p^e \gg 0$. The element $c$ depends on both $x$ and $I$. If $c$ works for all tight closure tests (for all ideals) then $c$ is called a test element. Test elements are often plentiful as shown by a result of [HH2].

PROPOSITION 2.1. *If $(R, m)$ is a reduced excellent local ring, $c \in R^0$, and $R_c$ is regular then some power of $c$ is a test element.*

If $I = (x_1, \ldots, x_k)$ has height $k$ then we say that $I$ is a *parameter ideal*. The element $c$ is a *parameter test element* if it works for all tight closure tests involving all parameter ideals. See Theorem 3.4 below.

One important category of results which we will apply here is colon capturing. Whenever $R$ is suitably nice (e.g., excellent local and equidimensional) then colon ideals involving ideals generated by polynomials in parameters lie in the tight closure of the "expected" answer (which comes from treating the parameters as a regular sequence). In particular we quote [AHS, Theorem 2.3] (see also [HH1–2]).

THEOREM 2.2. *Let $(R, m)$ be an equidimensional excellent local ring. Let $x_1, \ldots, x_d$ be any system of parameters for $R$ and let $I$ and $J$ be any two ideals of the (polynomial) subring $A = (\mathbb{Z}/p\mathbb{Z})[x_1, \ldots, x_d]$ of $R$ generated by monomials in the variable. Then*

$$(IR)^* :_R JR \subseteq ((I :_A J)R)^*,$$

$$(IR)^* \cap (JR)^* \subseteq ((I \cap J)R)^*.$$

Section 3 contains more results when the parameters are assumed to be (parameter) test elements.

Let $(R, m)$ be a Noetherian ring of dimenesion $d$ and $I \subseteq R$ a proper ideal of $R$. Let $R[It]$ be the Rees ring and let $G = G(I) = R/I \oplus I/I^2 \oplus \cdots$ be the associated graded ring. If $r \in I^t - I^{t+1}$, let $\tilde{r} \in G$ be the form of degree $t$ obtained from $r$. Let $\tilde{m} \subseteq G$ be the unique homogeneous maximal ideal. By [TI, Theorem 1.1], the Rees ring is Cohen-Macaulay if and only if both

(1) $H^d_{\tilde{m}}(G)_n = 0$ for $n \geq 0$, and
(2) $H^i_{\tilde{m}}(G)_n = 0$ for $n \neq -1$ when $0 \leq i < d$.

Since $\tilde{m}$ is maximal, the module $H^d_{\tilde{m}}(G)$ is Artinian, so $a_d(G) = \sup\{n \mid H^d_{\tilde{m}}(G)_n \neq 0\}$ is finite. The integer $a_d(G)$ is called the $a$-invariant of $G$.

When $I$ is equimultiple (i.e., the analytic spread of $I$ is equal to $ht(I)$) and has a minimal reduction generated by $ht(I)$ elements (this is automatic when $R$ contains an infinite field) then we may pick $y_1, \ldots, y_d \in R$ such that $\tilde{y}_1, \ldots, \tilde{y}_d$ is a homogeneous sequence of parameters by choosing $(y_1, \ldots, y_s)$ a minimal reduction of $I$ and $y_{s+1}, \ldots, y_d$ a system of parameters for $R/I$ (where $s = ht(I)$). Note that $\deg \tilde{y}_i = 1$ for $1 \leq i \leq s$, while $\deg \tilde{y}_i = 0$ for $i > s$.

Whenever $\tilde{y}_1, \ldots, \tilde{y}_k$ is (part of) a system of homogeneous system of parameters for $G$, then $H^i_{(\tilde{y}_1, \ldots, \tilde{y}_k)}(G)$ is the cohomology of the (graded) Čech complex (obtained via a direct limit of Koszul cohomology)

$$0 \to G \to G_{\tilde{y}_1} \oplus \cdots \oplus G_{\tilde{y}_k} \to \cdots \to G_{\tilde{y}_1 \cdots \tilde{y}_k} \to 0.$$

*Remark* 2.3.    As in [AHS] we note that we may rewrite the conditions for Cohen-Macaulayness of $R[It]$ given in [TI, Theorem 1.1] as

(1) $H^d_{\tilde{m}}(G)_n = 0$ for all $n \geq 0$, and
(2) $H^{k-1}_{(\tilde{y}_{i_1}, \ldots, \tilde{y}_{i_k})}(G)_n = 0$ for $n \neq -1$ for all subsets $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$.

This equivalent form follows from an extension of [AHS, Lemma 4.3].

LEMMA 2.4.    *Let $G$ be an $\mathbb{N}$-graded ring, and let $z_1, \ldots, z_k$ be any homogeneous elements of $G$. Let $S$ be any subset of $\mathbb{Z}$ that is bounded above. Suppose that for each subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ of size $k$, the module $H^{k-1}_{(z_{i_1}, \ldots, z_{i_k})}(G)$ is zero in all degrees $t \notin S$. Then for all $i$, $H^i_n(G)$ is zero in degree $t \notin S$, where $n$ is any ideal of $G$ generated by more than $i$ of the elements $z_j$.*

*Proof.*    We will use the long exact sequence in cohomology

$$\cdots \to H^{i-1}_{(z_{i_1}, \ldots, z_{i_{k-1}})}(G_{z_{i_k}}) \to H^i_{(z_{i_1}, \ldots, z_{i_k})}(G) \to H^i_{(z_{i_1}, \ldots, z_{i_{k-1}})}(G) \to \cdots$$

All of the maps preserve degree.

Suppose that the lemma is false, and choose the smallest $k$ such that $n = (z_1, \ldots, z_k)$ (after possibly relabelling) and $H_n^i(G)_t \neq 0$ for some $i < k$, where $t \notin \mathcal{S}$. By assumption $H_n^{k-1}(G)_t = 0$, so we may assume that $i < k - 1$. By our choice of $k$, $H_{(z_{i_1}, \ldots, z_{i_{k-1}})}^{i-1}(G)_t = 0$ for all $t \notin \mathcal{S}$. Because $H_{(z_{i_1}, \ldots, z_{i_k})}^{i-1}(G_{z_k}) \cong H_{(z_{i_1}, \ldots, z_{i_k})}^{i-1}(G) \otimes G_{z_k}$ and $\mathcal{S}$ is bounded above, we have $H_{(z_{i_1}, \ldots, z_{i_{k-1}})}^{i-1}(G_{z_k}) = 0$ if $\deg z_k > 0$ and $H_{(z_{i_1}, \ldots, z_{i_{k-1}})}^{i-1}(G_{z_k})_t = 0$ for $t \notin \mathcal{S}$ if $\deg z_k = 0$. In either case it follows from the above long exact sequence that $H_n^i(G)_t = 0$ for $t \notin \mathcal{S}$. $\quad \square$

## 3. Results for ideals generated by parameter test elements

We will use a number of results proved in [AHS] for ideals generated by parameter test elements.

LEMMA 3.1. *Let $x_1, \ldots, x_d$ be parameters in an equidimensional reduced excellent local ring and let $I = (x_1, \ldots, x_d)$. Assume that each $x_i$ is a parameter test element. Then*

(1) $(I^N)^* = I^{N-1}I^*$ *for all $N \geq 1$,*
(2) $(x_1, \ldots, x_k)^* \cap I = (x_1, \ldots, x_k)$, *and*
(3) $(I^N)^* = (I^*)^N$.

*Proof.* Parts (1) and (2) are [AHS, Lemma 3.1]. Part (3), which is used in [AHS], follows from (1) since $(I^N)^* = I^{N-1}I^* \subseteq (I^*)^N$. The other containment holds for any ideal. $\quad \square$

Lemma 3.1(1) implies that $(x_1, \ldots, x_d)$ is a (minimal) reduction of $I^*$, so $I^*$ is equimultiple.

MAIN LEMMA 3.2. *Let $(x_1, \ldots, x_d)$ be parameters in an equidimensional excellent local ring and let $I = (x_1, \ldots, x_d)$. Assume that each $x_i$ is a parameter test element. Then for all integers $N$, $t \geq 1$ and any $k \leq d$,*

$$(\sharp) \qquad \left(I^N + (x_1^{t_1}, \ldots, x_k^{t_k})\right)^* = (I^N)^* + \sum x_{i_1}^{t_{i_1}-1} \cdots x_{i_j}^{t_{i_j}-1}(x_{i_1}, \ldots, x_{i_j})^*,$$

*where the sum ranges over all subsets $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, k\}$.*

We need a slight extension of the above lemma. By convention, $I^m = R$ for $m \leq 0$.

LEMMA 3.3. *Under the same hypotheses as above, if $z \in (I^N + (x_1^{t_1}, \ldots, x_k^{t_k}))^* \cap (I^A)^*$ where $A \leq N$ then the coefficient of $x_{i_1}^{t_{i_1}-1} \cdots x_{i_j}^{t_{i_j}-1}$ in equation $(\sharp)$ may be taken to be in $(x_{i_1}, \ldots, x_{i_j})^* \cap (I^{A-(t_1+\cdots+t_j)+j)})^*$.*

*Proof.* We repeat much of the proof given in [AHS], with the changes that lead to the stronger statement. The proof will be by induction on $\sum t_i$. Let $I_1 = (x_2, \ldots, x_d)$.

The case that each $t_i = 1$ follows from ($\sharp$), since if $z \in (I^N + (x_1, \ldots, x_k))^* \cap (I^A)^* \subseteq ((I^N)^* + (x_1, \ldots, x_k)^*) \cap (I^A)^*$ and we write $z = u + v$ with $u \in (I^N)^*$ and $v \in (x_1, \ldots, x_k)^*$ then $v \in (x_1, \ldots, x_k)^* \cap (I^A)^*$.

We may now assume that $t_1 \geq 2$ (after possibly reordering the parameters) and that the stronger statement holds for $\sum t_i$ smaller. By Theorem 2.2,

$$(I^N + (x_1^{t_1}, \ldots, x_k^{t_k}))^* \cap (I^A)^* \subseteq (I^N + x_1^{t_1} I^{A-t_1} + \cdots + x_k^{t_k} I^{A-t_k})^*.$$

Suppose that $z \in (I^N + (x_1^{t_1}, \ldots, x_k^{t_k}))^* \cap (I^A)^*$. Because $x_1$ is a test element and the above inclusion holds,

$$x_1 z \in I^N + x_1^{t_1} I^{A-t_1} + \cdots + x_k^{t_k} I^{A-t_k}.$$

This can be rewritten as

$$x_1(z - u - x_2^{t_2} u_2 - \cdots - x_k^{t_k} u_k - x_1^{t_1-1} v) \in I_1^N + x_2^{t_2} I_1^{A-t_2} + \cdots + x_k^{t_k} I_1^{A-t_k},$$

where $u \in I^{N-1}$, $u_j \in I^{A-t_j-1}$ and $v \in I^{A-t_1}$. By Theorem 2.2,

$$z - u - x_2^{t_2} u_2 - \cdots - x_k^{t_k} u_k - x_1^{t_1-1} v \in ((I_1^N + (x_2^{t_2}, \ldots, x_k^{t_k}): x_1)) \cap (I_1^A: x_1)$$

$$\subseteq (I_1^N + (x_2^{t_2}, \ldots, x_k^{t_k}))^* \cap (I_1^A)^*$$

Elements of the right hand side may be expanded out in the desired manner by the induction hypothesis.

Thus we may assume that $z = u + x_2^{t_2} u_2 + \cdots + x_k^{t_k} u_k + x_1^{t_1-1} v$ for $u = \sum_\alpha u_\alpha m_\alpha$, where $m_\alpha$ is a monomial of degree $N - 1$ in $x_1, \ldots, x_d$. By altering $v$ and the $u_j$ for $2 \leq j \leq k$, we may assume without loss of generality that in any monomial the exponent of $x_i$ is strictly less than $t_i$ and the exponent of $x_1$ is strictly less than $t_1 - 1$. It then follows from Theorem 2.2 and the fact that $u \in (I_1^N + (x_1^{t_1-1}, x_2^{t_2}, \ldots, x_k^{t_k}))^*$ that each $u_\alpha \in (x_1, \ldots, x_d)^*$. Hence $u \in (I^N)^*$. Thus we may assume that

$$z = x_2^{t_2} u_2 + \cdots + x_k^{t_k} u_k + x_1^{t_1-1} v \in (I^N + (x_1^{t_1}, \ldots, x_d^{t_d}))^* \cap (I^A)^*,$$

where $u_j \in I^{A-t_j-1}$ and $v \in I^{A-t_1}$. If we can show that each $u_j$ is actually in $(I^{A-t_j})^*$ then we will be able to eliminate that term, since $x_j^{t_j} u_j \in x_j^{t_j-1}(x_j)^*(I^{A-t_j})^*$, is one of the terms in the desired large sum.

Suppose that we have succeeded in reducing to the case that $z = x_j^{t_j} u_j + \cdots + x_k^{t_k} u_k + x_1^{t_1-1} v$ for some $2 \leq j \leq k$. Write $u_j = \sum u_\alpha m_\alpha$ where $m_\alpha$ is a monomial of degree $A - t_j - 1$ in $x_1, \ldots, x_d$. By adjusting $u_{j+1}, \ldots, u_k$ and $v$ we may assume that in any monomial the exponent of $x_i$ is strictly less than $t_i$ (for $j + 1 \leq i \leq k$) and the exponent of $x_1$ is strictly less than $t_1 - 1$. Since $x_j^{t_j} u_j \in (I^A + (x_{j+1}^{t_{j+1}}, \ldots, x_k^{t_k}, x_1^{t_1-1}))^*$, by Theorem 2.2 we have $u_j \in (I^{A-t_j} + (x_{j+1}^{t_{j+1}}, \ldots, x_k^{t_k}, x_1^{t_1-1}))^*$. The only way that

a sum of monomials of degree $A - t_j - 1$ in $x_1, \ldots, x_d$ (where in any monomial the exponent of $x_i$ is strictly less than $t_i$ and the exponent of $x_1$ is strictly less than $t_1 - 1$) can end up in $(I^{A-t_j} + (x_{j+1}^{t_{j+1}}, \ldots, x_k^{t_k}, x_1^{t_1-1}))^*$ is for the coefficients to be in $I^*$. The point here is that Theorem 2.2 allows us to treat the $x_i$'s as if they were variables (and hence a regular sequence) up to tight closure.

At this point we have reduced to the case that $z = x_1^{t_1-1} v$ where $v \in I^{A-t_1}$. Applying Theorem 2.2 again yields

$$v \in (I^{N-t_1+1} + (x_1, x_2^{t_2}, \ldots, x_d^{t_d}))^* \cap (I^{A-t_1+1})^*,$$

which may be expanded using the induction hypothesis in order to conclude that $x_1^{t_1-1} v$ is in the desired ideal.   $\square$

The statement of Lemma 3.3 is actually stronger than what is needed to prove Theorem 4.1. The following weaker result is given in [AHS]:

LEMMA 3.4.   *Let $x_1, \ldots, x_d$ be parameters in an equidimensional excellent local ring $R$ and let $I$ denote the ideal they generate. Assume that each $x_i$ is a parameter test element. Then for any integers $N$, $t_i \geq 1$, $k \leq d$ and for any integer $a$ with $\sum_{1 \leq i \leq k} (t_i - 1) + 2 \leq a \leq N$,*

$$(I^N + (x_1^{t_1}, \ldots, x_k^{t_k}))^* \cap (I^a)^* = (I^N)^* + \sum_{1 \leq i \leq k} x_i^{t_i} (I^{a-t_i})^*.$$

The proof of this uses the following result [AHS, Corollary 3.3]:

LEMMA 3.5.   *Let $x_1, \ldots, x_d$ be parameters in an equidimensional excellent local ring $R$ and let $I$ denote the ideal they generate. Assume that each $x_i$ is a parameter test element. For arbitrary integers $N$ and $t$, with $N \geq t \geq 1$,*

$$(I^N)^* \cap (x_{i_1}^t, \ldots, x_{i_j}^t) = (I^{N-t})^* (x_{i_1}^t, \ldots, x_{i_j}^t),$$

*where $\{i_1, \ldots, i_j\}$ is any subset of $\{1, \ldots, d\}$.*

Ideals of parameter test elements which fulfill the hypotheses of Lemmas 3.2–5 are abundant by virtue of the following results (see [AHS, §5]).

THEOREM 3.6.   *Let $(R, m)$ be a reduced, equidimensional, excellent local ring. For every $c \in R^0$ such that $R_c$ is F-rational, there is a power of $c$ which is a test element for all $m$-primary ideals $I$ such that $R/I$ has finite phantom projective dimension.*

It follows from this (see [AHS, Corollary 5.2]) that such a $c$ has a power which is a test element for all ideals generated by monomials in parameters. A ring is called F-rational if every ideal generated by parameters is tightly closed. All regular rings and direct summands of regular rings actually satisfy the stronger condition that every ideal is tightly closed.

## 4. The main theorem

THEOREM 4.1.   *Suppose that $(R, m)$ is an excellent normal local domain of dimension $d \geq 3$. Let $x_1, \ldots, x_d$ be a system of parameters such that $x_1, \ldots, x_{d-1}$ are parameter test elements. Set $J = ((x_1, \ldots, x_{d-1})^{d-2})^*$. Then $R[Jt]$ is an arithmetic Macaulayfication of $R$.*

*Proof.*   The proof of this theorem is very similar to the proof of Theorem 4.1 in [AHS]. We have followed the general outline of [AHS], but have made the necessary modifications.

We note that for all $n$, $(J^n)^* = J^n$. This follows from Lemma 3.1(3).

Let $G = G(J)$. We would like to show that the two conditions given in Remark 2.3 hold for $\mathcal{S} = \{-1\}$. Let $\tilde{m}$ be the homogeneous maximal ideal of $G$.

Let $y_i = x_i^{d-2}$ for $1 \leq i \leq d$. Then $\deg \tilde{y}_i = 1$ for $1 \leq i < d$ and $\deg \tilde{y}_d = 0$. The ideal $(\tilde{y}_1, \ldots, \tilde{y}_d)$ is primary to $\tilde{m}$, so we may compute $H_{\tilde{m}}^i(G)$ via the limit of Koszul cohomology on $\tilde{y}_1, \ldots, \tilde{y}_d$.

We first observe that each $\tilde{y}_i$ is a nonzerodivisor. This can be checked on homogeneous elements, so assume that $\tilde{z}$ has degree $a$ and $\tilde{y}_i \tilde{z} = 0$. Set $t = \deg \tilde{y}_i$ (so either $t = 1$ or $t = 0$). Then $y_i z \in J^{a+t+1}$, so by colon capturing we have $z \in J^{a+t+1} : y_i \subseteq (J^{a+1})^* = J^{a+1}$, contradicting the assumption that $\deg \tilde{z} = a$.

We next check that condition (2) of Remark 2.3 holds for $k = 2$. Choose any pair $\tilde{y}_{i_1}$, $\tilde{y}_{i_2}$, and by abuse of notation, relabel the elements so that we are considering $\tilde{y}_1$, $\tilde{y}_2$, where $\deg \tilde{y}_1 = 1$ and $\deg \tilde{y}_2 = m \in \{0, 1\}$. To see that $H_{(\tilde{y}_1, \tilde{y}_2)}^1(G) = 0$ we need only show that $\tilde{y}_1, \tilde{y}_2$ is $G$-regular. This condition can be checked using homogeneous coefficients. Suppose that $\tilde{z}\tilde{y}_2 = \tilde{w}\tilde{y}_1$ where $a = \deg \tilde{w} = \deg \tilde{z} + 1 - m$. Then $zy_2 - wy_1 \in J^{a+2}$, hence $z \in (J^{a+2} + y_1R) : y_2 \subseteq (J^{a+2-m} + y_1R)^* \subseteq J^{a+2-m} + x_1^{d-3}(x_1)^* = J^{a+2-m} + (y_1)$, using Lemma 2.2, the main lemma, and the fact that principal ideals are tightly closed in a normal domain. Without loss of generality, $z = y_1u$ where $u \in J^{\deg \tilde{z}} : y_1 = J^{a-1+m} : y_1 \subseteq J^{a+m-2}$ by colon capturing. Hence $\tilde{z} \in (\tilde{y})$. Therefore $\tilde{y}_1, \tilde{y}_2$ is $G$-regular.

We now show that the $a$-invariant of $G$ is negative. Let $\tilde{y} = \tilde{y}_1 \cdots \tilde{y}_d$. Any element $\eta \in H_{\tilde{m}}^d(G)$ can be represented as $\eta = [\tilde{z}/\tilde{y}^t]$ where $\deg \eta = a - (d-1)t$ if $\deg \tilde{z} = a$. Note that $t$ can be assumed to be as large as we desire since we can multiply and divide by $\tilde{y}^{t'}$ for any $t' \geq 0$. We need to show that if $a \geq (d-1)t$ then $\eta = 0$. We will use the fact that for any ideal $B = (r_1, \ldots, r_k)$ and any integer $n$, $B^{(n-1)k+1} = (r_1^n, \ldots, r_k^n)B^{(n-1)(k-1)}$.

Say $a = t(d-1) + s$ where $t \geq 1$ and $s \geq 0$ (i.e., $\deg \eta \geq 0$). Let $I = (x_1, \ldots, x_{d-1})$. Suppose that $z \in J^a$. Then

$$
\begin{aligned}
z \in {} & (I^{a(d-2)})^* = I^{a(d-2)-1}I^* \qquad \text{by Lemma 3.1(1)} \\
= {} & I^{(t(d-1)+s)(d-2)-1}I^* = I^{(d-1)(t(d-2)-1)+(s+1)(d-2)}I^* \\
= {} & (x_1^{t(d-2)}, \ldots, x_{d-1}^{t(d-2)})I^{(d-2)(t(d-2)+s)-1}I^* \qquad \text{by the above fact} \\
= {} & (y_1^t, \ldots, y_{d-1}^t)J^{t(d-2)+s}.
\end{aligned}
$$

From this we may conclude that $\tilde{z} \in (\tilde{y}_1, \ldots, \tilde{y}_{d-1})$, thus $\tilde{z}/\tilde{y}^t \in Im(G_{\tilde{y}/\tilde{y}_d} \oplus \cdots \oplus G_{\tilde{y}/\tilde{y}_1})$, so that $\eta = 0$.

We now need to show that $H^{k-1}_{(\tilde{y}_{i_1}, \ldots, \tilde{y}_{i_k})}(G)$ vanishes in degree $\neq -1$ for $3 \leq k \leq d$ in order to fulfill the part of condition (2) of Remark 4.3 that we have not yet shown. Choose $k$ minimal if some cohomology module fails to vanish in degree $\neq -1$. Again abusing notation, renumber so that we call the elements $\{\tilde{y}_1, \ldots, \tilde{y}_k\}$, where $\deg \tilde{y}_j = 1$ except possibly $\tilde{y}_k$, which may have degree 0. Let $\tilde{y} = \tilde{y}_1 \cdots \tilde{y}_k$.

Let $\eta = [\frac{\tilde{y}^t \tilde{r}_1}{\tilde{y}^t}, \ldots, \frac{\tilde{y}^t \tilde{r}_k}{\tilde{y}^t}]$ represent a cohomology class of degree $b \neq -1$, which, if nonzero, is written with the fewest number of nonzero entries.

Since $\eta$ is a cohomology class, and $\tilde{y}$ is a nonzerodivisor, we know that $\tilde{y}_1^t \tilde{r}_1 + \cdots + \tilde{y}_k^t \tilde{r}_k = 0$. Let $k' = \max\{j | \tilde{r}_j \neq 0\}$. If we can show that $\tilde{r}_{k'} \in (\tilde{y}_1^t, \ldots, \tilde{y}_{k'-1}^t)$ then $\eta$ is equivalent to a cohomology class that can be written with a 0 in the $k'$ spot and we will be done.

Let $a = \deg \tilde{r}_1$ and let $c = \deg \tilde{r}_{k'}$. Note that $c = a$ if $\deg \tilde{y}_{k'} = 1$ and $c = a + t$ if $\deg \tilde{y}_{k'} = 0$, but either way, $c = b + (k-1)t$ (recall that $b = \deg \eta$).

Back in $R$, we have $y_1^t r_1 + \cdots + y_{k'}^t r_{k'} \in J^{a+t+1}$ so using Lemmas 2.2 and 2.4 (and letting $I = (x_1, \ldots, x_{d-1})$),

$$r_{k'} \in \left[ (J^{a+t+1} + (y_1^t, \ldots, y_{k'-1}^t) : y_{k'}^t \right] \cap J^c$$

$$\subseteq (J^{c+1} + (y_1^t, \ldots, y_{k'-1}^t))^* \cap J^c$$

$$\subseteq (I^{(d-2)(c+1)})^* + \sum (x_{i_1} \cdots x_{i_j})^{(d-2)t-1}(x_{i_1}, \ldots, x_{i_j})^*$$

where the sum is over $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, k'\}$ and each coefficient is in $(x_{i_1}, \ldots, x_{i_j})^* \cap (I^{(d-2)c-(d-2)jt+j})^*$.

Since $c(d-2) - (d-2)jt + j = (d-2)[b + (k-j-1)t] + j$ and we may assume that $t$ is as large as we like, we conclude the following about the summands:

(1) If $j < k - 1$ then $(d-2)[b + (k-j-1)t] + j > 1$ (by taking $t$ large enough) so that the coefficient is in $(x_{i_1}, \ldots, x_{i_j})^* \cap (I^2)^* \subseteq (x_{i_1}, \ldots, x_{i_j})^* \cap I \subseteq (x_{i_1}, \ldots, x_{i_j})$ by Lemma 3.1(1) and (2).

(2) If $j = k - 1$ and $b < -1$ then $(x_1 \cdots x_{k-1})^{(d-2)t-1} \in J^{c+1}$ since then $(k-1)[(d-2)t-1] = (k-1)(d-2)t - (k-1) \geq (d-2)(k-1)t + b(d-2)$ ($= (c+1)(d-2)$).

(3) If $j = k - 1$ and $b \geq 0$ then $c(d-2) - (k-1)(d-2)t + k - 1 = (b + (k-1)t)(d-2) - (k-1)(d-2)t + k - 1 = (d-2)b + k - 1 \geq 2$, since $b \geq 0$ and $k \geq 3$. Hence the coefficient of $(x_1 \cdots x_{k-1})^{t(d-2)-1}$ is in $(x_1, \ldots, x_{k-1})^*(I^2)^* \subseteq (x_1, \ldots, x_{k-1})^* \cap I \subseteq (x_1, \ldots, x_{k-1})$ by Lemma 3.1(1) and (2).

In all cases we have shown that $r_k \in (J^{c+1} + (y_1^t, \ldots, y_{k-1}^t)) \cap J^c$. By Lemma 3.5, in $G$, $\tilde{r}_k \in (\tilde{y}_1^t, \ldots, \tilde{y}_{k-1}^t)$, proving the theorem.   $\square$

COROLLARY 4.2.    *Let $(R, m)$ be an excellent normal domain of dimension $d \geq 3$. If the non-$F$-rational locus of $R$ is closed and has dimension $\leq 1$ then $R$ has an arithmetic Macaulayfication.*

*Proof.*    Suppose that $A \subseteq R$ defines the non-$F$-rational locus. Then $\dim(R/A) \leq 1$, so $ht(A) \geq d - 1$ since $R$ is a catenary domain. Thus we can pick $d - 1$ parameters in $A$ and taking high enough powers we obtain $x_1, \ldots, x_{d-1}$ to be parameter test elements by Theorem 3.6. Now apply Theorem 4.1.    □

COROLLARY 4.3.    *Let $(R, m)$ be an excellent normal domain of dimension 3. Then $R$ has an arithmetic Macaulayfication.*

*Proof.*    The non-regular locus of $R$ is closed and has height at least 2, since $R$ is normal. Thus by Proposition 2.1 we may pick $x_1, x_2$ which are test elements and are part of a system of parameters.    □

*Remark* 4.4.    When the defining ideal of the non-F-rational locus has dimension two, the methods used here and in [AHS] no longer work. For instance, when $\dim R = 4$ we can choose $J = (x_1, x_2)^*$, where $x_1$ and $x_2$ are parameter test elements and consider $R[Jt]$ and $G(J)$. When considering $H_m^3(G)$ we will be considering the condition that $r_1 x_1^t + \cdots + r_4 x_4^t \in J^N$, however, the Main Lemma will no longer be applicable to analyze $r_4$.

REFERENCES

[AHS]   I. M. Aberbach, C. Huneke and K. E. Smith, *A tight closure approach to arithmetic Macaulayfication*, Illinois J. Math., **40** (1996).
[Br1]   M. Brodmann, *A macaulayfication of unmixed domains*, J. Algebra **44** (1977), 221–234 .
[Br2]   _____, *Local cohomology of certain Rees and Form rings II*, J. Algebra **86** (1984), 457–493.
[Br3]   _____, *A few remarks on "Macaulayfication" of sheaves*, Commutative Algebra, International Conference, Vechta, Vechtaer Universitätsschriften, 1994, pp. 29–32.
[F]     G. Faltings *Über Macaulayfizierung*, Math. Ann. **238** (1978), 175–192 .
[GY]    S. Goto and K. Yamagishi, *The theory of unconditioned strong d-sequences and modules of finite local cohomology*, preprint.
[HH1]   M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.
[HH2]   _____, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.
[K]     K. Kurano, *Macaulayfication using tight closure*, preprint.
[TI]    N. V. Trung and S. Ikeda, *When is the Rees algebra Cohen-Macaulay?*, Comm. Alg. **17** (1989), 2893–2922.
[V]     J. Velez, *Openness of the F-rational locus and smooth base change*, J. Algebra **172** (1995), 425–453.

UNIVERSITY OF MISSOURI
    COLUMBIA, MISSOURI