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ORTHOGONAL MARTINGALES UNDER DIFFERENTIAL SUBORDINATION AND APPLICATIONS TO RIESZ TRANSFORMS

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1. Introduction

Let $\mathbb{H}$ be a separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be a family of right-continuous sub-σ-fields of a probability space $\{\Omega, P, \mathcal{A}\}$ such that $\mathcal{F}_0$ contains all sets of probabilty zero. For two $\mathcal{F}$-adapted continuous-time $\mathbb{H}$-valued martingales $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$, let $[X, Y] = \{[X, Y]_t\}_{t\geq 0}$ be the quadratic covariation process between X and Y (see, for example, [DM]). Unless otherwise stated, we assume all the martingales in the paper are $\mathbb{H}$-valued where $\mathbb{H}$ is a separable Hilbert space, and have right-continuous paths with left-limits (r.c.l.l.). For notational simplicity, we use $[X] = \{[X]_t\}_{t\geq 0}$ to denote $[X, X]$. We say the martingale Y is *differentially subordinate to* the martingale X, if $[X]_t - [Y]_t$ is a nondecreasing and nonnegative function of t. The notion of differential subordination permits generalizations of various sharp martingale inequalities of Burkholder [Bur 1-4] from the discrete-time and diverse stochastic integral settings to the present more general setting (see [Wan]). For example, if X and Y are continuous-time martingales with Y being differentially subordinate to X, then Theorem 1 of [Wan] says

$$\|Y\|_p \leq (p^* - 1)\|X\|_p, \qquad \text{for } 1 < p < \infty \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$ and the inequality is strict if $p \neq 2$ and $0 < \|X\|_p < \infty$. It is also sharp since it is already sharp in the special cases considered in [Bur1]. For a martingale X, the norm $\|X\|_p$ is defined by

$$\|X\|_p = \sup_t \|X_t\|_p.$$

Because of the close relationship between martingales and harmonic analysis, new sharp inequalities under differential subordination for continuous-time martingales have very important applications in analysis. For example, in Bañuelos and Wang

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[BW] we showed that the Beurling-Ahlfors operator B, which acts on L^p functions in the complex plane $\mathbb{C}$, defined by

$$Bf(z) = -\frac{1}{\pi} p.v. \int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2}\, dm(\xi)$$

where $dm(\xi)$ is the Lebesgue measure on $\mathbb{C}$, has a representation as a martingale transform and that (1.1) implies

$$\|Bf\|_p \leq 4(p^* - 1)\|f\|_p \text{ for } 1 < p < \infty. \tag{1.2}$$

The inequality (1.2) gives the best known estimate for the L^p constant for the operator B and it gives hope that the well known conjecture of Iwaniec [Iwa], $\|B\|_p = p^* - 1$, is true. Iwaniec's conjecture is important because of its applications to quasi–conformal mappings, and partial differential equations; (see [Ast] [IM] and [IMNS]).

Another interesting application of martingale differential subordination in analysis is to the norms of the Riesz transforms $R_j\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $j = 1, \ldots, n$, defined by

$$R_j f(x) = c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)\, dy,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The L^p norms of the Riesz transforms are related to a variation of the inequality (1.1). We say two real valued martingales X and Y are orthogonal if the quadratic covariation process $[X, Y]_t$ is 0 for every t. For two $\mathbb{H}$-valued martingales $X = (X_1, X_2, \ldots, X_i, \ldots)$ and $Y = (Y_1, Y_2, \ldots, Y_i, \ldots)$, X and Y are orthogonal if for each $i, j \geq 1$, $[X_i, Y_j] = 0$. The following theorem was proved in [BW]. For $1 < p < \infty$, let

$$C_p = \cot\left(\frac{\pi}{2p^*}\right) \quad \text{and} \quad E_p = \sqrt{1 + C_p^2}.$$

THEOREM A. *Let X and Y be two $\mathbb{R}$-valued continuous-time orthogonal martingales with continuous paths. If Y is differentially subordinate to X, then for $1 < p < \infty$,*

$$\|Y\|_p \leq C_p \|X\|_p, \qquad \|\sqrt{X^2 + Y^2}\|_p \leq E_p \|X\|_p \tag{1.3}$$

and the inequalities are sharp. In addition, if $1 < p \leq 2$, X may be taken to be $\mathbb{H}$-valued; if $2 \leq p < \infty$, Y may be taken to be $\mathbb{H}$-valued.

Inequalities (1.3) are the martingale versions of Pichorides [Pic] and Essén [Ess] type inequalities for harmonic and conjugate harmonic functions. Using the inequalities (1.3) and the martingale representation of the Riesz transforms R_j [Bañ 1-2], Bañuelos and Wang ([BW]) prove that the norm of the Riesz transform $\|R_j\|_p = C_p$ for $1 < p < \infty$; a result first obtained by Iwaniec and Martin [IW] using the Calderón–Zygmund method of rotations.

In this paper we will generalize inequalities (1.3) to continuous-time martingales which may or may not have continuous paths. Moreover we will also show that the inequalities (1.3) are strict for the nontrivial cases. More precisely, we prove the following theorem:

THEOREM 1. *Let X and Y be two $\mathbb{R}$-valued continuous-time orthogonal martingales. If Y is differentially subordinate to X, then for $1 < p < \infty$,*

$$\|Y\|_p \le C_p\|X\|_p, \qquad \|\sqrt{X^2+Y^2}\|_p \le E_p\|X\|_p \tag{1.4}$$

and the inequalities are sharp. In addition, if $1 < p \le 2$, X may be take to be $\mathbb{H}$-valued; if $2 \le p < \infty$, Y may be taken to be $\mathbb{H}$-valued. Moreover, inequalities (1.4) are strict if $p \ne 2$ and $0 < \|X\|_p < \infty$.

Since the discrete-time martingales can be imbedded into continuous-time martingales, Theorem 1 covers general martingales (continuous-time or discrete-time) which are mutually orthogonal and have the correct subordination condition.

We note that the strictness is new even for the continuous path martingales considered in Theorem A. As a consequence, the strictness of the inequalities (1.4) gives some new information about Riesz transforms. In fact, using the argument similar to the one in Section 4 of [BW], we will show in Section 3 that the norms of Riesz transforms are not attainable.

The rest of the paper is organized as follows. In Section 2 we will prove Theorem 1. The proof is based on the techniques of Burkholder [Bur 1-4] and Wang [Wan] with the appropriate functions modified from those used by Pichorides and Essén. They are the same function used by Bañuelos and Wang [BW]. One needs to be a little more careful in treating the jumps in the general martingale case. In Section 3, we will give the applications of Theorem 1 to Riesz transforms.

2. Main result

We begin with some of the properties of martingale orthogonality and differential subordination. First we study orthogonality. Recall that two r.c.l.l. martingales $X = \{X_t\}_{t\ge0}$ and $Y = \{Y_t\}_{t\ge0}$ are orthogonal if the quadratic covariation process $[X, Y] = \{[X, Y]_t\}_{t\ge0}$ is identically 0. For any r.c.l.l. process Z, let $Z^c = \{Z^c_t\}_{t\ge0}$ be the continuous part of Z with $Z_0^c = 0$. For $t > 0$, let $\Delta_t Z = Z_t - Z_{t-}$ and define $\Delta_0 Z = Z_0$. Then, we have the decomposition

$$Z_t = Z_t^c + \sum_{0 \le s \le t} \Delta_s Z, \quad t \ge 0. \tag{2.1}$$

It is well known, see for example [DM] or [Pro], that the continuous part of quadratic covariation process $[X, Y]$ is the quadratic covariation process of the continuous part of X and Y; i.e.,

$$[X, Y]^c = [X^c, Y^c]. \tag{2.2}$$

It is also well known that the purely jump part of the quadratic covariation $[X, Y]$ is the product of the purely jumps of X and Y:

$$\Delta_t[X, Y] = \langle \Delta_t X, \Delta_t Y \rangle, \quad t \ge 0. \tag{2.3}$$

Applying (2.1) to $[X, Y]$, we have the following lemma.

LEMMA 1. *Two martingales X and Y are orthogonal if and only if*

$$[X^c, Y^c]_t = 0 \textit{ and } \langle \Delta_t X, \Delta_t Y \rangle = 0, \quad t \ge 0.$$

Now we go to martingale differential subordination. Applying the decomposition (2.1) to the difference of the quadratic variation processes $[X] - [Y]$ and using the relationships (2.2)-(2.3), we have the following lemma.

LEMMA 2. *The martingale Y is differentially subordinate to the martingale X if and only if* (1) *Y^c is differentially subordinate to X^c and* (2) $|\Delta_t Y| \le |\Delta_t X|$ *for* $t \ge 0$.

Combining Lemmas 1 and 2, we have:

COROLLARY 1. *Let X and Y be two orthogonal martingales. Then Y is differentially subordinate to X if and only if* (1) *Y^c is differentially subordinate to X^c*, (2) *X^c and Y^c are orthogonal*, (3) $\langle \Delta_t X, \Delta_t Y \rangle = 0$ *and* $|\Delta_t Y| \le |\Delta_t X|$ *for* $t \ge 0$.

We now state Theorem 1 again.

THEOREM 1. *Let X and Y be two $\mathbb{R}$-valued continuous-time orthogonal martingales. If Y is differentially subordinate to X, then for* $1 < p < \infty$,

$$\|Y\|_p \le C_p \|X\|_p, \qquad \|\sqrt{X^2 + Y^2}\|_p \le E_p \|X\|_p \tag{2.4}$$

and the inequalities are sharp. In addition, if $1 < p \le 2$, *X may be take to be $\mathbb{H}$-valued; if* $2 \le p < \infty$, *Y may be taken to be $\mathbb{H}$-valued. Moreover, inequalities* (2.4) *are strict if* $p \ne 2$ *and* $0 < \|X\|_p < \infty$.

The proof of the theorem is based on Burkholder's technique of constructing special functions and applying Itô's formula. We give the following general result on orthogonal martingales under differential subordination. The proof is a modification of Proposition 1 of Wang [Wan], it should also be compared with Proposition 1.2 of [BW] where martingales have continuous-paths. Some notation is useful. Let $\langle\cdot,\cdot\rangle$ denote the inner product of $\mathbb{H}$. It is sufficient to consider just the Hilbert space l^2. Let $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ to be the first and second order derivatives of $f(x,y)$ defined on $\mathbb{H}\times\mathbb{H}$ to $\mathbb{R}$. We restrict ourselves to considering a function f satisfying

$$f((0,x),(0,y)) = f(x,y) \tag{2.5}$$

for $x=(x_1,x_2,x_3,\ldots)$, $y=(y_1,y_2,y_3,\ldots)\in\mathbb{H}$.

PROPOSITION 1. *Let $f(x,y)$ be a continuous function on $\mathbb{H}\times\mathbb{H}$, bounded on bounded sets; C^1 on $\mathbb{H}\times\mathbb{H}\setminus(\{|x|=0\}\cup\{|y|=0\})$, whose first order derivative is bounded on bounded sets not containing 0, the origin of $\mathbb{H}\times\mathbb{H}$, and satisfying*

$$f(x+h,y+k)-f(x,y)-\langle f_x(x,y),h\rangle-\langle f_y(x,y),k\rangle\le 0 \tag{2.6}$$

when $|x||y|\neq 0$, $|k|\le|h|$ and $\langle h,k\rangle=0$; C^2 on S_i, $i\ge 1$, where S_i is a sequence of open connected sets, such that the union of the closure of S_i is $\mathbb{H}\times\mathbb{H}$. Suppose for each $i\ge 1$, there exists a nonnegative measurable function $c_i(x,y)$ defined on S_i such that for $(x,y)\in S_i$ with $|x||y|\neq 0$,

$$\langle hf_{xx}(x,y),h\rangle+\langle kf_{yy}(x,y),k\rangle\le -c_i(x,y)(|h|^2-|k|^2) \tag{2.7}$$

for all $h,k\in\mathbb{H}$. Assume further that for each $n\ge 1$, there exists a function M_n which is nondecreasing in n such that

$$\sup c_i(x,y)\le M_n<\infty \tag{2.8}$$

where the supremum is taken over all $(x,y)\in S_i$ such that $1/n^2\le|x|^2+|y|^2\le n^2$ and all $i\ge 1$. Let X and Y be $\mathbb{H}$ valued orthogonal martingales such that Y is differentially subordinate to X. Then for $0\le s\le t$,

$$E(f(X_t,Y_t)|\mathcal{F}_s)\le f(X_s,Y_s)\quad a.e. \tag{2.9}$$

provided f is nonnegative or $\sup_t|f(X_t,Y_t)|$ is integrable.

Proof of Proposition 1. It is enough to prove (2.9) for $s=0$ and $\mathcal{F}_0=\{\emptyset,\Omega\}$, the trivial σ-field. Let $T_n=\inf\{t\ge 0\colon |X_t|^2+|Y_t|^2+[X]_t+[Y]_t\ge n^2\}$. Then $X_{t\wedge T_n-}$, $Y_{t\wedge T_n-}$ are bounded by n for all $t\ge 0$, and $\lim_{n\to\infty}T_n=\infty$. Here and elsewhere in the paper, we define $X_{0-}=0$ and $Y_{0-}=0$. We will show that for any $n\ge 0$,

$$Ef(X_{t\wedge T_n},Y_{t\wedge T_n})\le f(X_0,Y_0). \tag{2.10}$$

This implies (2.9) by either Fatou's lemma or the Lebesgue dominated convergence theorem. Fix $n \geq 0$. Let $X_t = X_{t\wedge T_n}$, $Y_t = X_{t\wedge T_n}$ to simplify notation.

Given $\epsilon > 0$ and $0 < a < 1/2$. Choose $n_1 \geq n$ such that $1/(n_1+2) \leq a$. By enlarging the dimension of $\mathbb{H}$ if necessary, we may assume $\overline{X}_t = (a, X_t)$ and $\overline{Y}_t = (a, Y_t) \in \mathbb{H}$. Choose $m_{a,\epsilon}$ such that $E\sum_{i\geq m_{a,\epsilon}}([\overline{X}_i]_{\infty-} + [\overline{Y}_i]_{\infty-}) \leq \epsilon/M_{n_1+2}$. This is possible since

$$\begin{aligned} E\sum_{i\geq 1}([\overline{X}_i]_{\infty-} + [\overline{Y}_i]_{\infty-}) &= 2a^2 + E\sum_{i\geq 1}([X_i]_{\infty-} + [Y_i]_{\infty-}) \\ &\leq 2a^2 + n^2. \end{aligned}$$

For $m > 1$, we let

$$\overline{X}^m = (a, X_1, \ldots, X_{m-1}, 0, \ldots), \quad \overline{Y}^m = (a, Y_1, \ldots, Y_{m-1}, 0, \ldots),$$

where

$$X = (X_1, X_2, \ldots), \quad Y = (Y_1, Y_2, \ldots).$$

Now let $m > m_{a,\epsilon}$ and let g be a C^∞ nonnegative function on $\mathbb{R}^m \times \mathbb{R}^m$ such that g has support inside the unit ball and assume

$$\int_{R^m}\int_{R^m} g(x, y)\,dx\,dy = 1.$$

Such g exists. In fact, we can choose g such that g is a radial function: $g(x, y) = g(|x|^2 + |y|^2)$.

Let l be positive integer such that $1/l < a$ and $1/l \leq \sqrt{2}a - 1/(n_1+2)$. For $x, y \in \mathbb{R}^m$, define

$$f^l(x, y) = \int_{R^m}\int_{R^m} f(x - u/l, y - v/l) g(u, v)\,du\,dv.$$

In the above, we use the notation $f(x, y) = f((x, 0, \ldots), (y, 0, \ldots))$ for $x, y \in \mathbb{R}^m$.

By the dominated convergence theorem, since f is continuous and bounded on bounded sets,

$$Ef(X_{t-}, Y_{t-}) = \lim_{a\to 0}\lim_{m\to\infty}\lim_{l\to\infty} Ef^l(\overline{X}^m_{t-}, \overline{Y}^m_{t-}). \tag{2.11}$$

Because f is continuous and is C^1 on $\mathbb{H}\times\mathbb{H}\setminus(\{|x| = 0\}\cup\{|y| = 0\})$, integration by parts shows that if $|x| \geq a$ and $|y| \geq a$, then

$$\begin{aligned} f^l_{xx}(x, y) &= \int_{R^m}\int_{R^m} f_{xx}(x - u/l, y - v/l) g(u, v)\,du\,dv \\ f^l_{yy}(x, y) &= \int_{R^m}\int_{R^m} f_{yy}(x - u/l, y - v/l) g(u, v)\,du\,dv. \end{aligned}$$

Therefore, by (2.7), when $|x| \ge a$, $|y| \ge a$ and $h, k \in \mathbb{H}$, we have

$$\langle h f^l_{xx}(x,y), h\rangle + \langle k f^l_{yy}(x,y), k\rangle \le -c(x,y)(|h|^2 - |k|^2)$$

where

$$c(x,y) = \sum_i \iint_{R_i} c_i(x - u/l, y - v/l) g(u,v)\,du\,dv,$$

and $R_i = \{(u,v) \colon (x,y) - (u,v)/l \in S_i\}$.

Let $\{h^j_i\}$, $\{k^j_i\}$ be two triangular sequences $\in \mathbb{R}^m$ such that

$$\sup_j \sum_i |k^j_i|^2 < \infty, \quad \text{and} \quad \sup_j \sum_i |h^j_i|^2 < \infty.$$

Then, for $|x| \ge a$, $|y| \ge a$,

$$\begin{aligned} &\lim_{j\to\infty} \sum_i (\langle h^j_i f^l_{xx}(x,y), h^j_i\rangle + \langle k^j_i f^l_{yy}(x,y), k^j_i\rangle) \\ &\qquad\qquad \le -c(x,y) \lim_{j\to\infty} \sum_i (|h^j_i|^2 - |k^j_i|^2). \end{aligned} \tag{2.12}$$

By differential subordination and Corollary 1, this implies

$$\begin{aligned} &\sum_{i=1}^m \sum_{j=1}^m \int_{0+}^{t-} (f^l_{x_i x_j}(\overline{X}^m_{s-}, \overline{Y}^m_{s-})\, d[\overline{X}^{cm}_i, \overline{X}^{cm}_j]_s \\ &\qquad\qquad + f^l_{y_i y_j}(\overline{X}^m_{s-}, \overline{Y}^m_{s-}) d[\overline{Y}^{cm}_i, \overline{Y}^{cm}_j]_s) \\ &\qquad \le -\sum_{i=1}^m \int_0^{t-} c(\overline{X}^m_{s-}, \overline{Y}^m_{s-})\, d([\overline{X}^{cm}_i]_s - [\overline{Y}^{cm}_i]_s) \\ &\qquad \le M_{n_1+2} \sum_{i \ge m} ([\overline{X}^c_i]_{t-} + [\overline{Y}^c_i]_{t-}), \end{aligned} \tag{2.13}$$

since $0 \le c(\overline{X}^m_{s-}, \overline{Y}^m_{s-}) \le M_{n_1+2}$ because $\sqrt{2}a \le \sqrt{|\overline{X}^m_{s-}|^2 + |\overline{Y}^m_{s-}|^2} \le n+1$.

Applying Itô's formula to $f^l(\overline{X}^m, \overline{Y}^m)$, we have

$$f^l(\overline{X}^m_{t-}, \overline{Y}^m_{t-}) = f^l(\overline{X}^m_0, \overline{Y}^m_0) + I_1 + I_2 + I_3 + I_4, \tag{2.14}$$

where

$$I_1 = \int_{0+}^{t-} \langle f^l_x(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), d\overline{X}^m_s\rangle + \int_{0+}^{t-} \langle f^l_y(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), d\overline{Y}^m_s\rangle,$$

$$
\begin{aligned}
2I_2 &= \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{0+}^{t-} f^l_{x_i x_j}(\overline{X}^m_{s-}, \overline{Y}^m_{s-})\, d[\overline{X}^{cm}_i, \overline{X}^{cm}_j]_s \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{0+}^{t-} f^l_{y_i y_j}(\overline{X}^m_{s-}, \overline{Y}^m_{s-})\, d[\overline{Y}^{cm}_i, \overline{Y}^{cm}_j]_s, \\
I_3 &= \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{0+}^{t-} f^l_{x_i y_j}(\overline{X}^m_{s-}, \overline{Y}^m_{s-})\, d[\overline{X}^{cm}_i, \overline{Y}^{cm}_j]_s, \\
I_4 &= \sum_{0<s<t} (f^l(\overline{X}^m_s, \overline{Y}^m_s) - f^l(\overline{X}^m_{s-}, \overline{Y}^m_{s-}) \\
&\quad - \sum_{0<s<t} (\langle f^l_x(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), \Delta_s \overline{X}^m\rangle + \langle f^l_y(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), \Delta_s \overline{Y}^m\rangle).
\end{aligned}
$$

By Corollary 1, orthogonality implies $I_3 = 0$. Inequality (2.13) implies that

$$2I_2 \le M_{n_1+2} \sum_{i\ge m}([\overline{X}^c_i]_{t-} + [\overline{Y}^c_i]_{t-}).$$

Since

$$
\begin{aligned}
I_1 &= \int_{0+}^{t} \langle f^l_x(\overline{X}^m_{s-}, \overline{Y}^m_{s-}),\, d\overline{X}^m_s\rangle + \int_{0+}^{t} \langle f^l_y(\overline{X}^m_{s-}, \overline{Y}^m_{s-}),\, d\overline{Y}^m_s\rangle \\
&\quad - \langle f^l_x(\overline{X}^m_{t-}, \overline{Y}^m_{t-}), \Delta_t \overline{X}^m\rangle - \langle f^l_y(\overline{X}^m_{t-}, \overline{Y}^m_{t-}), \Delta_t \overline{Y}^m\rangle,
\end{aligned}
$$

it follows from (2.14) and the martingale property that

(2.15)

$$
\begin{aligned}
Ef^l(\overline{X}^m_t, \overline{Y}^m_t) &\le f^l(\overline{X}^m_0, \overline{Y}^m_0) + \epsilon + E\sum_{0<s\le t} \Delta_s f^l(\overline{X}^m, \overline{Y}^m) \\
&\quad - E\sum_{0<s\le t} (\langle f^l_x(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), \Delta_s \overline{X}^m\rangle + \langle f^l_y(\overline{X}^m_{s-}, \overline{Y}^m_{s-}), \Delta_s \overline{Y}^m\rangle),
\end{aligned}
$$

by the choice of ϵ.

Let $l \to \infty$, then $m \to \infty$, and then $\epsilon \to 0$. By the dominated convergence theorem, we have

$$
\begin{aligned}
Ef(\overline{X}_t, \overline{Y}_t) &\le f(\overline{X}_0, \overline{Y}_0) + E\sum_{0<s\le t} \Delta_s f(\overline{X}, \overline{Y}) \\
&\quad - E\sum_{0<s<t} (\langle f_x(\overline{X}_{s-}, \overline{Y}_{s-}), \Delta_s \overline{X}\rangle + \langle f_y(\overline{X}_{s-}, \overline{Y}_{s-}), \Delta_s \overline{Y}\rangle), \\
&\le f(\overline{X}_0, \overline{Y}_0)
\end{aligned}
$$

because of (2.6) and Corollary 1.

Letting $a \to 0$ proves (2.10).

Remark. If $\mathbb{H}\times\mathbb{H}$ is replaced by $\mathbb{R}\times\mathbb{H}$ and $\mathbb{H}\times\mathbb{R}$, $\mathbb{H}\times\mathbb{H}\setminus(\{|x|=0\}\cup\{|y|=0\})$ is replaced by $\mathbb{R}\times\mathbb{H}\setminus\mathbb{R}\times\{0\}$ and $\mathbb{H}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}$, and $|x||y|\neq 0$ is replaced by $|y|\neq 0$ and $|x|\neq 0$ respectively in the statement of Proposition 1, the above proof will be still valid. In fact, we need only to replace $\sqrt{2}a$ by a; replace $\overline{X}^m$ by X, $\mathbb{R}^m\times\mathbb{R}^m$ by $\mathbb{R}\times\mathbb{R}^m$, $|x|\geq a$, $|y|\geq a$ by $|y|\geq a$ in the case of $\mathbb{R}\times\mathbb{H}$, and replace $\overline{Y}^m$ by Y, $\mathbb{R}^m\times\mathbb{R}^m$ by $\mathbb{R}^m\times\mathbb{R}$, $|x|\geq a$, $|y|\geq a$ by $|x|\geq a$ in the case of $\mathbb{H}\times\mathbb{R}$. This leads to the corresponding lemmas which will be used in the proof of Theorem 1 next.

Proof of Theorem 1. We first consider the case $1<p\leq 2$. By Minkowski's inequality, it is enough to show

$$\|\sqrt{X^2+Y^2}\|_p\leq E_p\|X\|_p. \tag{2.16}$$

For $x\in\mathbb{H}$ and $y\in\mathbb{R}$ and let

$$\begin{aligned} V_1(x,y) &= R^p-\sec^p\left(\frac{\pi}{2p}\right)|x|^p,\\ U_1(x,y) &= -\tan\left(\tfrac{\pi}{2p}\right)R^p\cos p\theta, \end{aligned}$$

where $|x|=R\cos\theta$, $y=R\sin\theta$, $-\dfrac{\pi}{2}\leq\theta\leq\dfrac{\pi}{2}$. It is clear that U_1 is continuous on $\mathbb{H}\times\mathbb{R}$ and C^1 on $\mathbb{H}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}$. In [BW], it is shown that U_1 satisfies the following properties:

(a) $V_1(x,y)\leq U_1(x,y)$ with equality holds only if $R=\sec(\frac{\pi}{2p})|x|$;
(b) For $x,h\in\mathbb{H}$, $|x|\neq 0$, $y,k\in\mathbb{R}$, there exists a nonnegative measurable function $c_1(x,y)$ which is bounded on $R\geq r$ for every $r>0$ such that

$$\langle hU_{1xx}(x,y),h\rangle+U_{1yy}(x,y)k^2\leq -c_1(x,y)(|h|^2-k^2).$$

In fact, strict inequality in (a) was not stated explicitly in Section 2 of [BW]. However, since $F(\theta)$ defined there attains its maximum only at $\frac{\pi}{2p}$, strictness follows easily.

We will show (b) implies the following:

(c) For $x,h\in\mathbb{H}$, $y,k\in\mathbb{R}$ with $|x|\neq 0$, $|k|\leq|h|$ and $hk=0$,

$$U_1(x+h,y+k)-U_1(x,y)-\langle U_{1x}(x,y),h\rangle-U_{1y}(x,y)k\leq 0. \tag{2.17}$$

In fact, it is enough to consider the case $k=0$. Let $a>0$ and $G(t)=U_1((a,x+th),y)-U_1((a,x),y)-t\langle U_{1x}((a,x),y),h\rangle$. Then the mean value theorem and (b) imply that $G(1)-G(0)\leq 0$. Let $a\to 0$ to show (2.17).

Applying Proposition 1 and the remark to U_1 and using properties (b)-(c), we have

$$EU_1(X_t, Y_t) \leq EU_1(X_0, Y_0) \tag{2.18}$$

for any $t \geq 0$. Note $U_1(X_0, Y_0) \leq 0$ since $|Y_0| \leq |X_0|$. Combining property (a) with (2.18), inequality (2.16) follows.

Next we consider the case $2 < p < \infty$. Again, by Minkowski's inequality, it is enough to show

$$\|Y\|_p \leq C_p \|X\|_p. \tag{2.19}$$

For $x \in \mathbb{R}$ and $y \in \mathbb{H}$, set

$$V_2(x, y) = |y|^p - \cot^p\left(\frac{\pi}{2p}\right)|x|^p,$$

and

$$U_2(x, y) = \begin{cases} V_2(x, y), & \text{if } 0 \leq \theta \leq \dfrac{\pi}{2q}, \pi - \dfrac{\pi}{2q} \leq \theta \leq \pi \\ \cot\dfrac{\pi}{2p} R^p \cos p\left(\dfrac{\pi}{2} - \theta\right), & \text{if } \dfrac{\pi}{2q} \leq \theta \leq \pi - \dfrac{\pi}{2q}, \end{cases}$$

where in this case, $x = R\cos\theta$, $|y| = R\sin\theta$, $0 \leq \theta \leq \pi$, and $q = p/(p-1)$. A simple calculation shows that U_2 is continuous on $\mathbb{R} \times \mathbb{H}$ and C^1 when $|y| \neq 0$.

Section 2 of [BW] shows U_2 satisfies the following properties:

(a′) $V_2(x, y) \leq U_2(x, y)$ with equality only if $|y| = \cot(\frac{\pi}{2p})|x|$.

(b′) For $x, h \in \mathbb{R}$, $y, k \in \mathbb{H}$, $|y| \neq 0$, and $|y| \neq \cot(\frac{\pi}{2p})|x|$, there exists a non-negative measurable function $c_2(x, y)$ which is bounded on $R \leq r$ for every $r > 0$, such that

$$U_{2xx}(x, y)h^2 + \langle kU_{2yy}(x, y), k\rangle \leq -c_2(x, y)(|h|^2 - k^2).$$

Again, we will show that (b′) implies the following:

(c′) For $x, h \in \mathbb{R}$, $y, k \in \mathbb{H}$ with $|y| \neq 0$, $|k| \leq |h|$ and $hk = 0$,

$$U_2(x + h, y + k) - U_2(x, y) - U_{2x}(x, y)h - \langle U_{2y}(x, y), k\rangle \leq 0. \tag{2.20}$$

It is enough to consider the case $|k| = 0 < |h|$. In this case, inequality (2.20) becomes

$$U_2(x + h, y) - U_2(x, y) - U_{2x}(x, y)h \leq 0. \tag{2.21}$$

Fix x, y, h. Let l be an integers and define

$$U_2^l(z) = \int_R U_2(z + u/l, y)g(u)\, du$$

for some C^∞ function g which has support in $[-1, 1]$. Similar to the arguments at the beginning of the proof of Proposition 1, U_2^l is C^2 and satisfies

$$U_{2zz}^l(z) \le 0 \tag{2.22}$$

for all $z \in R$ because of (b′). This implies

$$U_2^l(x+h) - U_2^l(x) - U_{2x}(x)h \le 0.$$

Letting $l \to \infty$ gives (2.21).

Inequality (2.19) then follows from (a′)-(c′) and Proposition 1.

The inequalities (2.4) are sharp since they are already sharp when X and Y have continuous-paths as stated in Theorem A. It only remains to prove the strictness. The argument is similar to that given in [Bur 4] and [Wan].

Assume $p \neq 2$ and $0 < \|X\|_p < \infty$. We prove that inequality (2.16) is strict. The others are similar.

For V_1 and U_1 defined above, let $u(t) = EU_1(X_t, Y_t)$ and $v(t) = EV_1(X_t, Y_t)$. The above argument shows that $v(t) \le u(t)$. By the martingale convergence theorem, both X and Y have limits X_∞ and Y_∞ at $t = \infty$, respectively. Using Doob's maximal function inequality, we see that v and u are r.c.l.l. and have limits at infinity. Thus, strictness follows if we can show $v(\infty) < 0$.

If $E|X_0|^p \neq 0$, then $u(0) < 0$ since $|Y_0| \le |X_0|$. Thus $v(t) \le u(t) \le u(0) < 0$ for any $t \ge 0$. Therefore, without loss of generality, we may assume that $X_0 = Y_0 = 0$ and $\|X_t\|_p > 0$ for all $t > 0$.

It is enough to show $P(|Y_\infty| = \tan(\frac{\pi}{2p})|X_\infty|) < 1$ since by (a), this implies that $v(\infty) < u(\infty) \le 0$.

Suppose $|Y_\infty| = \tan(\frac{\pi}{2p})|X_\infty|$ almost surely. Let $U_t = U_1(X_t, Y_t)$. Then $U_\infty = 0$. Since by Proposition 1, $\{U_t, t \ge 0\}$ is a uniformly integrable supermartingale starting from 0, this implies $P(U_t = 0, \text{ for all } t \ge 0) = 1$. Therefore, $|Y| = \tan(\frac{\pi}{2p})|X|$.

Let $T_n = \inf\{t > 0, |X_t| + |Y_t| \ge n\}$. Then $|X_{T_n -}| \le n$, $|Y_{T_n -}| \le n$. Moreover, $(X_- \cdot X)^{T_n}$, $(Y_- \cdot Y)^{T_n}$ are martingales. By the definition of the quadratic variation, for any $t > 0$ we have

$$\begin{aligned} 0 &= |Y_{T_n \wedge t}|^2 - \tan\left(\frac{\pi}{2p}\right)\Bigg)^2 |X_{T_n \wedge t}|^2 \\ &= 2\left((Y_- \cdot Y)_{T_n \wedge t} - \tan\left(\frac{\pi}{2p}\right)^2 (X_- \cdot X)_{T_n \wedge t}\right) \\ &\quad + \left([Y, Y]_{T_n \wedge t} - \tan\left(\frac{\pi}{2p}\right)^2 [X, X]_{T_n \wedge t}\right) \\ &= 2J_1 + J_2. \end{aligned}$$

Observe that J_1 is a martingale, so $EJ_1 = 0$, and that EJ_2 is negative unless we have $E[X, X]_{T_n \wedge t} = 0$ because $\tan(\frac{\pi}{2p}) > 1$. Taking expectation of both sides, we must have $E[X,X]_{T_n\wedge t}=0$. Therefore, $E|X_t|^2=0$. This contradicts the fact that $E|X_t|^p>0$ for $t>0$ which we assumed at the beginning. This completes the proof.

3. Applications to Riesz transforms

The relationship between martingales, Riesz transforms, and harmonic functions have been studied extensively in the past. See, for example, [GV], [Var], and [Bañ 1-2]. Recently, Bañuelos and Wang [BW], using Theorem A, gave a proof that the Riesz transforms have norms C_p. This result was first obtained by Iwaniec and Martin [IW]. The technique used by Bañuelos and Wang also leads to other important and interesting results regarding Riesz transforms, the Beurling-Ahlfors operator, and harmonic functions. We will show here, using that technique, that Theorem 1 gives more information about the Riesz transforms. In particular, we show that the norms of the Riesz transforms are not attainable.

THEOREM 2. *Let R_j, $j=1,\dots,n$ be the j-th Riesz transform in $\mathbb{R}^n$ and $1<p<\infty$. Then*

$$\|R_jf\|_p\le C_p\|f\|_p \qquad \text{and} \qquad \|\sqrt{(R_jf)^2+f^2}\|_p\le E_p\|f\|_p. \tag{3.1}$$

The above inequalities are sharp and strict if $p\neq 2$ and $0<\|f\|_p<\infty$.

Note that except for the strictness, Theorem 2 is Theorem 3 of [BW].

Proof. For the sake of completeness, we briefly describe the relationship between martingales and Riesz transforms. We refer the reader to [BW] for details.

Let $B_t=(X_t,Y_t)=(X_t^1,\dots,X_t^n,Y_t)$ be the background radiation process defined by Gundy and Varopoulos [GV] in the upper half space $\mathbb{R}^{n+1}_+$ of $\mathbb{R}^{n+1}$. The process $B=\{B_t\}_{-\infty\le t\le 0}$ starts at time $t=-\infty$ and has initial distribution of Lebesgue measure on $\mathbb{R}^n\times\{\infty\}$. It terminates on the boundary $\mathbb{R}^n\times\{0\}$ at time $t=0$. We may think of B as "Brownian motion" and observe that the usual rules of stochastic integration and potential theory apply. See Varopoulos [Var] for details.

Given any function $f(x)\in L^p(R^n)$, the space of L^p function in $\mathbb{R}^n$, define $U_f(x,y)$ to be the Poisson extension of f in $\mathbb{R}^{n+1}$. Let

$$\begin{aligned}\nabla U_f &=\left(\frac{\partial U_f}{\partial x_1},\dots,\frac{\partial U_f}{\partial x_n},\frac{\partial U_f}{\partial y}\right)^T,\\ dB_t &=(dX_t^1,\dots,dX_t^n,dY_t)^T\end{aligned}$$

where for $v = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$, v^T denote the transform of x. For $j = 1, \dots, n$, let $A_j = (a^j_{lm})$ be an $(n+1) \times (n+1)$ matrix such that

$$a^j_{lm} = \begin{cases} 1, & l = n+1, m = j+1 \\ -1, & l = j+1, m = n+1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus for every $v \in \mathbb{R}^{n+1}$,

$$\langle A_j v, A_j v\rangle \le \langle v, v\rangle \qquad \text{and} \qquad \langle A_j v, v\rangle = 0. \tag{3.2}$$

For any $(n+1) \times (n+1)$ matrix A and $f \in L^p(\mathbb{R}^n)$, let

$$A * f = \{A * f_t\}_{-\infty \le t \le 0} = \left\{ \int_{-\infty}^{t} A\nabla U_f(X_s, Y_s) \cdot dB_s \right\}_{-\infty \le t \le 0}$$

denote the martingale transform B by $A\nabla U_f$. Then using basic properties of conditional expectation and stochastic integrals, the fact that the Green's function for the background radiation process is $2y$, the Cauchy-Riemann equation, and the Littlewood-Paley identity, it follows that (see [BW])

$$\begin{aligned} R_j f(x) &= E(A_j * f_0 | B_0 = (x, 0)), \\ f(x) &= E(I * f_0 | B_0 = (x, 0)), \end{aligned} \tag{3.3}$$

where I is the identity matrix.

Since

$$\begin{aligned} [A_j * f]_t &= \int_{-\infty}^{t} \langle A_j \nabla U_f(B_s), A_j \nabla U_f(B_s)\rangle \, ds, \\ [I * f]_t &= \int_{-\infty}^{t} \langle \nabla U_f(B_s), \nabla U_f(B_s)\rangle \, ds, \\ [A_j * f, I * f]_t &= \int_{-\infty}^{t} \langle A_j \nabla U_f(B_s), \nabla U_f(B_s)\rangle \, ds, \end{aligned}$$

Relationship (3.2) implies that $A * f$ is differentially subordinate and orthogonal to $I * f$. Thus it follows from Theorem 1 that when $1 < p < \infty$, the following inequalities hold and are strict if $p \ne 2$ and $0 < \|f\|_p < \infty$:

$$\|A_j * f\|_p \le C_p \|I * f\|_p \qquad \text{and} \qquad \|\sqrt{A_j * f^2 + I * f^2}\|_p \le E_p \|I * f\|_p.$$

By (3.3) and the fact that conditional expectation is a contraction operator, we have

$$\|R_j f\| \le C_p \|f\|_p \qquad \text{and} \qquad \|\sqrt{(R_j f)^2 + f^2}\|_p \le E_p \|f\|_p$$

and the inequalities are strict if $p \ne 2$ and $0 < \|f\|_p < \infty$. This completes the proof.

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