ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 4, Winter 1996

DEFORMATION CLASSES OF GRADED MODULES AND MAXIMAL BETTI NUMBERS

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1. Introduction

In this paper, I determine the deformation classes of finitely generated graded modules over a polynomial ring $S = k[x_1, \ldots, x_n]$, where k is an infinite field. Theorem 34 states that each deformation class is the set of modules with a given Hilbert function. Furthermore, I show in Theorem 31 that among all quotient modules with a fixed Hilbert function of a given finitely generated graded free module F, the quotient by the lexicographic submodule has the largest graded Betti numbers.

The deformation classes of subschemes of projective space were determined by Hartshorne in his thesis [Ha]. He proved that the Hilbert scheme, $\mathbf{Hilb}^{p(z)}(\mathbb{P}^{n-1})$, is linearly connected. That is, any two subschemes of $\mathbb{P}^{n-1}$ may be deformed to one another if and only if they have the same Hilbert polynomial; if they do, then the deformation may be realized as a sequence of deformations, each defined over $\mathbb{A}^1$. (All deformations in this paper are defined over $\mathbb{A}^1$.) Hartshorne's technique was to construct a deformation from $\mathcal{O}_V = \mathcal{O}_{\mathbb{P}}/\mathcal{I}_V$, the structure sheaf of a subscheme $V \subseteq \mathbb{P}^{n-1}$ with Hilbert polynomial $p(z)$, to $\mathcal{O}_{\mathbb{P}}/\mathcal{J}$, where $\mathcal{J}$ is the sheafification of a "Borel-fixed" ideal. Then, he constructed special families called "fans" which give a sequence of deformations between any two such $\mathcal{O}_{\mathbb{P}}/\mathcal{J}$ with Hilbert polynomial $p(z)$.

Reeves, in her thesis [Re1,2], refined Hartshorne's techniques in characteristic zero and showed that if d is the degree of $p(z)$, then there is a sequence of no more than $d+2$ deformations defined over $\mathbb{A}^1$ taking $\mathcal{O}_{\mathbb{P}}/\mathcal{I}_V$ to $\mathcal{O}_{\mathbb{P}}/\mathcal{L}$, where $\mathcal{L}$ is the sheafification of the unique "lexicographic ideal" L such that S/L has Hilbert polynomial $p(z)$, and has no submodule of finite length. This is the essential point in her theorem on the radius of the Hilbert scheme.

The main technique in this paper is a refinement of the technique that Reeves used in her thesis. Indeed, the operation that I call Φ in this paper is the essential operation in her argument. On the way to proving the two main theorems of this paper, I will show that Reeves' bound of $d + 2$ holds in positive characteristic, and also for deformations of quotient sheaves of a sum of line bundles $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-d_i)$. In particular, the quot scheme $\mathbf{Quot}^{p(z)}(\mathcal{E})$ is linearly connected for such an $\mathcal{E}$.

Lexicographic submodules of a free module F play a central role in this paper, and I will now describe them. Let $S = k[x_1, \ldots, x_n]$ where k is a field and let F be a

Received June 30, 1995
1991 Mathematics Subject Classification. Primary 13D02; Secondary 14D10.

graded free S-module of rank r. Fix a basis $e_1, \ldots, e_r$ of F where each e_i has degree d_i and $d_1 \leq \cdots \leq d_r$.

Definitions 1. A *monomial* of F is an element $x^\mu e_i$ where $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ is a monomial of S. The *lexicographic order on monomials of* S is the order in which $x^\mu > x^\nu$ if $\mu_s > \nu_s$ and $\mu_i = \nu_i$ for every $i < s$. The *lexicographic order on monomials of* F is the order in which $x^\mu e_i > x^\nu e_j$ if $i < j$, or $i = j$ and $x^\mu > x^\nu$. A *monomial subspace* of F_d, the vector space of homogeneous elements of F of degree d, is a subspace spanned by a set of monomials. The *lexicographic subspace* of F_d of dimension ℓ is the monomial subspace spanned by the first ℓ monomials of F_d in lexicographic order. A submodule L of F is a *lexicographic submodule* if it is graded and L_d is a lexicographic subspace of F_d for every d.

Note that in the definition of lexicographic order, I do not compare the degrees of monomials as one would with degree-lexicographic order.

PROPOSITION 2 (Macaulay [Ma], Hulett [Hu1,3]). *Let N be a graded submodule of F. Then there is a lexicographic submodule L of F such that* $\dim L_d = \dim N_d$ *for every d.*

Macaulay proved Proposition 2 in the case that $F = S$ so that L is a lexicographic ideal. Hulett proved the theorem in the form stated above.

Macaulay also proved that among all homogeneous ideals with the same Hilbert function, that is with the same dimension in every degree, the lexicographic ideal has the largest number of minimal generators of each degree. It is not difficult to compute the minimal generators for the lexicographic ideal for a given Hilbert function, so it is easy to bound the number of generators that an ideal requires in each degree if we know its Hilbert function. Bigatti and Hulett proved a remarkable generalization of this theorem when k has characteristic zero. If M is a finitely generated graded S-module with minimal graded free resolution

$$0 \to F_n \to \cdots \to F_0 \to M \to 0,$$

then $\beta_{ij}(M)$ is the number of degree j minimal generators of F_i. These numbers are the *graded Betti numbers* of M. They are well defined; in fact $\beta_{ij}(M) = \dim \operatorname{Tor}_i^S(M, S/\mathfrak{m})_j$ where $\mathfrak{m} = (x_1, \ldots, x_n)$. See [EvGr].

THEOREM 3 (Bigatti [Bi], Hulett [Hu1-3]). *If k has characteristic zero, N is a graded submodule of F and L is the lexicographic submodule of F such that F/N and F/L have the same Hilbert function, then $\beta_{ij}(F/N) \leq \beta_{ij}(F/L)$ for every i and j.*

Bigatti and Hulett independently proved this theorem for $F = S$ and Hulett later proved this theorem in the form above. The theorem is a generalization of Macaulay's because the number of minimal generators of N of degree j is $\beta_{1j}(F/N)$.

The free resolution of S/L was explicitly described by Eliahou and Kervaire [ElKe] and, since F/L is a direct sum of cyclic modules $(S/L_{(i)})e_i$ where $L_{(i)}$ is a lexicographic ideal, their description easily extends to a description of the free resolution of F/L. Using their result it is easy to write a combinatorial formula for $\beta_{ij}(F/L)$ in terms of the generators of L. So we have sharp bounds for the graded Betti numbers of a module in terms of its Hilbert function and the degrees of its generators, and these bounds are not very difficult to compute.

But it is essential to Hulett's and Bigatti's arguments that the characteristic of k is zero. In this paper I give a new argument which works if k is any field, even a finite field. I also prove the analogous statement for regular local rings in Corollary 33.

I am grateful to E. G. Evans, A. Galligo, H. Hulett and A. Reeves for helpful conversations related to the contents of this paper. I am especially grateful to D. Eisenbud, who was my thesis advisor during much of the work leading to this paper.

I dedicate this paper to William Pardue on the occasion of his sixtieth birthday.

2. Notation

I use the following notations and conventions in this paper. Except where otherwise specified, k is an infinite field. The characteristic of k is p, which may be zero. $S = k[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ is the graded maximal ideal of S. F is a free graded S-module with a fixed basis $e_1, \ldots, e_r$ such that e_i is homogeneous of degree d_i and $d_1 \leq \cdots \leq d_r$. All S-modules are graded and finitely generated; if M is an S-module then M_d is the vector space of homogeneous elements of M of degree d. Any reference to a submodule of F applies to ideals as well, since F may be S.

Define the *codimension* codim N of a proper submodule N of F to be the height of the annihilator of F/N. In particular, the codimension of an ideal is its height. Define the codimension of F to be n. If I is an ideal, then $Z(I)$ is the zero locus of I in $\mathbb{P}_k^{n-1}$. The codimension of $\emptyset$ in $\mathbb{P}_k^{n-1}$ is n. With these conventions, the codimension of $Z(I)$ in $\mathbb{P}_k^{n-1}$ is equal to the codimension of I, even if $I = S$.

The *Hilbert function* of a module M is the function from $\mathbb{Z}$ to $\mathbb{N}$ defined by $h_M(d) = \dim M_d$. For sufficiently large d, $h_M(d)$ agrees with a polynomial $p_M(d)$, which is the *Hilbert polynomial* of M. If N is a submodule of F, then the largest submodule P of F containing N such that F/N and F/P have the same Hilbert polynomial is the *saturation* of N, which is

$$N:_F \mathfrak{m}^\infty = \left\{ f \in F\colon \mathfrak{m}^s f \subseteq N \text{ for some } s \geq 0 \right\}.$$

The Hilbert polynomial of F/N has degree equal to $n - 1 - \operatorname{codim}(N)$, where we take the polynomial 0 to have degree -1.

A *monomial submodule* of F is a submodule generated by monomials. If $m = x^\mu e_i$ is a monomial of F and x^ν is a monomial of S, then the notation $x^\nu | m$ means that x^ν

divides x^μ. In this case, I write $\frac{1}{x^\nu}m$ for $\frac{x^\mu}{x^\nu}e_i$. The notation $x_j^\ell \| m$ means that $x_j^\ell | m$ and $x_j^{\ell+1} \not| m$. The lexicographic order on monomials of S and F is defined above. If $f \in F$ is the sum of $\alpha x^\mu e_i$, with $\alpha \in k^*$, and a k-linear combination of monomials which come later than $x^\mu e_i$ in lexicographic order, then $\alpha x^\mu e_i$ is called the *initial term* of f and is denoted by $\mathbf{in}(f)$. The initial term of 0 is 0. If N is a submodule of F, then the *submodule of initial terms* of N is the submodule of F generated by the initial terms of elements of N; it is denoted by $\mathbf{in}(N)$. If V is a subspace of F_d, then define the subspace of initial terms of V to be the subspace of F_d consisting of initial terms of elements of V; it is denoted by $\mathbf{in}(V)$. It is not hard to see that V and $\mathbf{in}(V)$ have the same dimension. Also, $\mathbf{in}(N_d) = \mathbf{in}(N)_d$, so N and $\mathbf{in}(N)$ have the same Hilbert function. The *generic submodule of initial terms* of N is $\mathbf{in}(\gamma(N))$ where $\gamma \in GL(n) \rtimes GL(F)$ is a generic element as in Proposition 4 below; this submodule of F is denoted by $\mathbf{Gin}(N)$. For more information on the theory of submodules of initial terms, see Chapter 15 of [Ei]; the treatment there is for ideals, but is easily extended to the case of submodules of free modules.

Starting in Section 6, I will often be working with submodules of F of the form $I_{(1)}e_1 \oplus \cdots \oplus I_{(r)}e_r$ where the last several $I_{(i)}$ are 0. The notation $I_{(1)}e_1 \oplus \cdots \oplus I_{(q)}e_q$ will be used for such a submodule where $I_{(q)} \neq 0$ and $I_{(i)} = 0$ for $i > q$.

3. Borel-fixed submodules

In this section I summarize the characteristic free theory of Borel-fixed submodules. The characteristic zero theory of Borel-fixed ideals is essential to the proofs of Reeves' Theorem on the radius of the Hilbert scheme, and to Theorem 3 above; it has also had wider applications to algebraic geometry, commutative algebra, and combinatorics. See, for example, [Co], [Go] and [Ka1,2]. Because of the lack of a theory in positive characteristic, nearly all of these applications have been restricted to characteristic zero.

Proposition 4 below shows that Borel-fixed submodules arise naturally as $\mathbf{in}(N)$ where N is a submodule of F in "generic coordinates." N is in generic coordinates after we make a general change of variables in S and a general change of homogeneous basis for F. More explicitly, the group $GL(n)$ of $n \times n$ invertible matrices with entries in k is also the group of graded k-algebra automorphisms of S; if $\gamma = (a_{ij}) \in GL(n)$ then $\gamma(x_j) = \sum_i a_{ij}x_i$. This automorphism induces a natural compatible action on F by $\gamma(\sum f_i e_i) = \sum \gamma(f_i)e_i$. Let $GL(F)$ be the group of graded S-module automorphisms of F. Then $G = GL(n) \rtimes GL(F)$ acts on F through graded k-vector space automorphisms; this action takes submodules to submodules. Let B be the subgroup of G consisting of all automorphisms taking e_i to an S-linear combination of $e_1, \ldots, e_i$ and x_i to a k-linear combination of $x_1, \ldots, x_i$. B is a Borel group of G and is naturally realized as upper triangular matrices. Let U be the unipotent subgroup of B.

PROPOSITION 4. *Let N be a submodule of F and let G, B and U be as above. Then there is a Zariski-open set $\mathcal{G} \subseteq G$ such that*

(1) *for all* $\gamma_1, \gamma_2 \in \mathcal{G}$, $\mathbf{in}(\gamma_1(N)) = \mathbf{in}(\gamma_2(N))$,
(2) *for all* $\gamma \in \mathcal{G}$, $\mathbf{in}(\gamma(N))$ *is a* B*-fixed submodule of* F *and*
(3) $\mathcal{G} \cap U \neq \emptyset$.

Proof. See Example I.7 and Proposition VII.1 of [Pa], or modify the proofs of Theorems 15.18 and 15.20 in [Ei] which are for the case $F = S$. □

A form of Proposition 4 was first proved by Galligo for the action of $GL(n)$ on $\mathbb{C}\{x_1, \dots x_n\}$. Parts (1) and (2) of Proposition 4 were first proved by Bayer and Stillman in the case $F = S$. Now I will summarize the basic properties of Borel-fixed submodules of F. Proofs of Propositions 6 and 8–10 may be found in Chapter II of [Pa] or in Section 15.9.3 of [Ei], where the case of Borel-fixed ideals is studied.

Definition 5. If p is a prime number, $k = \sum_i k_i p^i$ and $\ell = \sum_i \ell_i p^i$ where $0 \leq k_i, \ell_i \leq p-1$, then say that $k \leq_p \ell$ if $k_i \leq \ell_i$ for every i. The number p will always be the characteristic of k. If $p = 0$, then $\leq_0$ is the usual order on the natural numbers.

PROPOSITION 6. *A submodule $N \subseteq F$ is fixed by the action of B on F if and only if*

(1) $N = I_{(1)}e_1 \oplus \cdots \oplus I_{(r)}e_r$ *is a monomial submodule and*
(2) *for every monomial* $m \in N$, *if* $x_j^\ell \| m$ *and* $i < j$ *then* $(\frac{x_i}{x_j})^k m \in N$ *for every* $k \leq_p \ell$ *and*
(3) $\mathfrak{m}^{d_j - d_i} I_{(j)} \subseteq I_{(i)}$ *for every* $i < j$.

Bayer proved Proposition 6 in the case of $F = S$ and $p = 0$ [Ba].

Definition 7. A submodule $N \subseteq F$ is a *Borel-fixed submodule* if N is fixed by B. A submodule $N \subseteq F$ is a *standard Borel-fixed submodule* if N satisfies conditions (1) and (3) of Proposition 6 and furthermore for every monomial $me_i \in N$, if $x_j | m$ then $\frac{x_i}{x_j} me_i \in N$ for every $i < j$.

A standard Borel-fixed submodule is Borel-fixed. If the characteristic of k is zero, then every Borel-fixed submodule is standard. A Borel-fixed submodule which is not standard is called *nonstandard.* If $p > 0$ then an example of a nonstandard Borel-fixed ideal is (x_1^p, x_2^p). In most places in the literature, the definition of a "standard Borel-fixed" ideal is used as a definition for a "Borel-fixed" ideal. This creates difficulties if we work in positive characteristic since then there are many ideals fixed by the action of the Borel group which do not fit this definition.

PROPOSITION 8. *If N is a Borel-fixed submodule then the codimension of N is equal to the codimension of $I_{(r)}$ where $I_{(r)}$ is as in Proposition* 6. *If $I_{(r)} \neq 0$ and k is the highest index such that $I_{(r)}$ has a generator of the form x_k^s, then the codimension of $I_{(r)}$ is k. If $I_{(r)} = 0$ then the codimension of $I_{(r)}$ is* 0.

PROPOSITION 9. *If N is a Borel-fixed submodule, then $N:_F \mathfrak{m}_n^\infty = N:_F (x_n)^\infty$. So, if N is generated by $\{x^\mu e_i\}$ then $N:_F \mathfrak{m}_n^\infty$ is generated by $\{\frac{x^\mu}{x_n^{\mu_n}} e_i\}$.*

PROPOSITION 10. *If N is a nonzero Borel-fixed submodule and k is the highest index such that x_k divides some generator of N, then a maximal regular sequence on F/N is $x_{k+1}, \dots, x_n$. In particular, the depth of F/N is $n - k$.*

4. Fans

Definitions 11. Let $P = k[z_{ijk}]$ where $1 \le i \le n$, $1 \le k \le r$ and $1 \le j \le J$ where J is sufficiently large. Let $\pi\colon P \to S$ be the map given by $\pi(z_{ijk}) = x_i$. Let F' be a free P-module with basis $e'_1, \dots, e'_r$ where $\deg e'_i = \deg e_i = d_i$. Let $\pi'\colon F' \to F$ be the map $\pi'(\sum_i f_i e'_i) = \sum_i \pi(f_i) e_i$. If $I \subseteq S$ is a monomial ideal, then define $I^{(p_k)}$, the *k-polarization* of I, to be the monomial ideal in P generated by

$$\{z^{p_k(\mu)} = \prod_{i=1}^{n} \prod_{j=1}^{\mu_i} z_{ijk} \colon x^\mu \text{ is a minimal generator of } I\}.$$

If $N = I_{(1)} e_1 \oplus \cdots \oplus I_{(r)} e_r$ is a monomial submodule of F, then the *polarization* of N is

$$N^{(p)} = I_{(1)}^{(p_1)} e'_1 \oplus \cdots \oplus I_{(r)}^{(p_r)} e'_r.$$

Note that $\pi(I^{(p_k)}) = I$ and $\pi'(N^{(p)}) = N$. If $N \subseteq M$ are both monomial submodules of F, then $N^{(p)} \subseteq M^{(p)}$. Also, $I^{(p_k)}$ is a radical monomial ideal. Geometrically, the map $P/I^{(p_k)} \to S/I$ gives a representation of the scheme **Proj** S/I as a special plane section of **Proj** $P/I^{(p_k)}$, which is a union of reduced coordinate planes in a large projective space.

PROPOSITION 12. *The kernel of $\pi\colon P \to S$ is generated by $\{z_{ijk} - z_{i1k}\}$. If N is a monomial submodule of F, then these generators form a regular sequence on $F'/N^{(p)}$.*

Proof. The first statement is clear from the definition of π. For the second statement,

$$F'/N^{(p)} \simeq P/I_{(1)}^{(p_1)}(-d_1) \oplus \cdots \oplus P/I_{(r)}^{(p_r)}(-d_r).$$

So, we need to see that $\{z_{ijk} - z_{i1k}\}$ is a regular sequence on $P/I_{(s)}^{(p_s)}$ for $1 \leq s \leq r$. Since $I_{(s)}^{(p_s)}$ is generated by monomials in the variables $\{z_{ijs}\}$, we only need to see that $\{z_{ijs} - z_{i1s}\}$ is a regular sequence on $P/I_{(s)}^{(p_s)}$. This is a theorem of Fröberg and Weyman. See [Fr] or Proposition III.5 of [Pa] for a proof. □

Now we will replace the special plane section by a general plane section.

Definitions 13. Let $\mathbb{L} = \{\ell_{ijk}\}$ be a collection of linear forms in S where i, j and k vary as for the indices of z_{ijk}. Define $\sigma_{\mathbb{L}}\colon P \to S$ to be the map given by $\sigma_{\mathbb{L}}(z_{ijk}) = \ell_{ijk}$. Define $\sigma'_{\mathbb{L}}\colon F' \to F$ to be the map given by $\sigma'_{\mathbb{L}}(\sum_i f_i e'_i) = \sum_i \sigma_{\mathbb{L}}(f_i)e_i$. In particular if $\ell_{ijk} = x_i$, then $\sigma_{\mathbb{L}} = \pi$ and $\sigma'_{\mathbb{L}} = \pi'$. Let $\mathcal{L} = (S_1)^{nJr}$ be the affine space of all such collections of linear forms.

PROPOSITION 14. *If $\mathbb{L} = \{\ell_{ijk}\}$ is a generic collection of linear forms in S_1 as above, and N is a monomial submodule of F, then the kernel of $\sigma_{\mathbb{L}}$ is generated by linear forms in P which are a regular sequence on $F'/N^{(p)}$.*

Proof. It is clear that the kernel of $\sigma_{\mathbb{L}}$ is generated by linear forms. The kernel of $\sigma_{\mathbb{L}}$ is generated by linear forms which are a regular sequence on $F'/N^{(p)}$ if and only if $\mathrm{Tor}_i^P(P/\ker\sigma_{\mathbb{L}}, F'/N^{(p)}) = 0$ for all $i > 0$. This is an open property on $\mathcal{L}$. By Proposition 12 above, $\{z_{ijk} - z_{i1k}\}$ has this property. So for generic $\mathbb{L}$, $\ker\sigma_{\mathbb{L}}$ is generated by a sequence of linear forms regular on $F'/N^{(p)}$. □

COROLLARY 15. *If $\mathbb{L} = \{\ell_{ijk}\}$ is a generic collection of linear forms in S_1 as above, and N is a monomial submodule of F, then the graded Betti numbers of F/N and of $F/\sigma'_{\mathbb{L}}(N^{(p)})$ are the same. Also, the Hilbert functions of F/N and of $F/\sigma'_{\mathbb{L}}(N^{(p)})$ are the same.*

Proof. By Propositions 12 and 14, the kernels of π and of $\sigma_{\mathbb{L}}$ are generated by sequences of linear forms which are regular on $F'/N^{(p)}$. These sequences of linear forms are clearly also regular on P, since the image of P is an integral domain in both cases. But $F/\pi'(N^{(p)}) \simeq F'/N^{(p)} \otimes_P P/\ker\pi$ and $F/\sigma'_{\mathbb{L}}(N^{(p)}) \simeq F'/N^{(p)} \otimes_P P/\ker\sigma_{\mathbb{L}}$. Thus, the graded Betti numbers of $F/\pi'(N^{(p)}) = F/N$ and of $F/\sigma'_{\mathbb{L}}(N^{(p)})$ are the same as those of $F'/N^{(p)}$, and are thus the same as those of each other. The Hilbert function of a module is the alternating sum of the Hilbert functions in the terms of a resolution of the module. Since F/N and $F/\sigma_{\mathbb{L}}(N^{(p)})$ have the same graded Betti numbers, the modules in the kth step of the free resolution of each are isomorphic. Thus, F/N and $F/\sigma'_{\mathbb{L}}(N^{(p)})$ have the same Hilbert function. □

5. The operations ϕ and Φ

Definitions 16. Let $\mathbb{L}$ be a generic set of linear forms in S as above. Let $U(F)$ be the subgroup of $GL(F)$ consisting of graded S-linear automorphisms of F sending each e_i to $e_i + \sum_{j<i} f_{ij} e_j$. Let γ be a generic element of $U(F)$. If N is a monomial submodule of F, then define $\phi(N) = \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))$ and define $\Phi(N)$ to be the saturation of $\phi(N)$. Define $\phi^s(N)$ and $\Phi^s(N)$ to be the s-fold applications of ϕ and Φ to N.

We will see in Proposition 18 that ϕ does not depend on the particular choice of $\mathbb{L}$ and γ. So ϕ and Φ are well defined.

Definitions 17. A *monomial* of $\bigwedge^{\ell} F_d$ is an element of the form $m_1 \wedge \cdots \wedge m_\ell$ where each m_i is a monomial of F_d and $m_1 > \cdots > m_\ell$ in lexicographic order. The monomials of $\bigwedge^{\ell} F_d$ form a basis of $\bigwedge^{\ell} F_d$. The *lexicographic order* on monomials of $\bigwedge^{\ell} F_d$ is the order in which $m_1 \wedge \cdots \wedge m_\ell > n_1 \wedge \cdots \wedge n_\ell$ if for some s we have that $m_s > n_s$ in lexicographic order and $m_i = n_i$ for $i < s$. If $f \in \bigwedge^{\ell} F_d$ is not zero, then define the *initial term* of f, denoted by $\mathbf{in}(f)$, to be the lexicographically greatest monomial to occur with nonzero coefficient when f is written as a linear combination of monomials. Note that the monomials of $\bigwedge^{\ell} F_d$ are in one to one correspondence with monomial subspaces of F_d of dimension ℓ; a monomial subspace $V \subseteq F_d$ corresponds to the wedge product of the monomials in its basis. So, we may interpret the lexicographic order on monomials of $\bigwedge^{\ell} F_d$ as an order on monomial subspaces of F_d of dimension ℓ. Extend this order to an order on all monomial subspaces of F_d by saying that $V > W$ if $\dim V > \dim W$.

PROPOSITION 18. *Let N be a monomial submodule of F. Let $\mathcal{L}$ be as in Definition* 13 *and put the Zariski topology on $\mathcal{L} \times U(F)$. Then there is a nonempty open set $\mathcal{G} \subseteq \mathcal{L} \times U(F)$ such that if $(\mathbb{L}, \gamma) \in \mathcal{G}$ and $(\mathbb{L}_0, \gamma_0) \in \mathcal{L} \times U(F)$, then for every d,*

$$\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d) \geq \mathbf{in}(\gamma_0(\sigma'_{\mathbb{L}_0}(N^{(p)}))_d).$$

Proof. First, we will see that there is an open set $\mathcal{G}_d$ which works for a fixed degree d. Then we will see that $\mathcal{G} = \bigcap_d \mathcal{G}_d$ is open and nonempty.

Say that the dimension of N_d is ℓ. Let V be the greatest monomial subspace, in lexicographic order, of F_d to occur as $\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d)$ for any $(\mathbb{L}, \gamma) \in \mathcal{L} \times U(F)$ and choose a $(\mathbb{L}_1, \gamma_1)$ which gives V in this way. Note that V has dimension ℓ. There are ℓ monomials $n_1, \ldots, n_\ell \in N_d^{(p)}$ such that $\{\gamma_1(\sigma'_{\mathbb{L}_1}(n_i))\}$ span $\gamma_1(\sigma'_{\mathbb{L}_1}(N^{(p)}))_d$. Let $m_1, \ldots, m_\ell$ be a monomial basis of V. Let M be the subspace of $\bigwedge^{\ell} F_d$ spanned by the monomial $m_1 \wedge \cdots \wedge m_\ell$ and let M' be the complimentary monomial subspace so that $\bigwedge^{\ell} F_d = M \bigoplus M'$. Consider the composition of algebraic maps

$$\mathcal{L} \times U(F) \to \bigwedge^{\ell} F_d \to M$$

where the first map takes $(\mathbb{L}, \gamma)$ to $\gamma(\sigma'_{\mathbb{L}}(n_1)) \wedge \cdots \wedge \gamma(\sigma'_{\mathbb{L}}(n_\ell))$, and the second is the projection of $\bigwedge^{\ell} F_d$ onto its summand M. The image of $(\mathbb{L}, \gamma)$ is not zero if and only if $\{\gamma(\sigma'_{\mathbb{L}}(n_i))\}$ are linearly independent and $m_1 \wedge \cdots \wedge m_\ell$ is a monomial of $\gamma(\sigma'_{\mathbb{L}}(n_1)) \wedge \cdots \wedge \gamma(\sigma'_{\mathbb{L}}(n_\ell))$. This happens if and only if the set $\{\gamma(\sigma'_{\mathbb{L}}(n_i))\}$ is a basis of $\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d$ and, since no earlier term can ever occur, the initial term of $\gamma(\sigma'_{\mathbb{L}}(n_1)) \wedge \cdots \wedge \gamma(\sigma'_{\mathbb{L}}(n_\ell))$ is $m_1 \wedge \cdots \wedge m_\ell$ so that $V = \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d)$. So, the image of $(\mathbb{L}_1, \gamma_1)$ in M is nonzero. Since the map is algebraic, there is an open set $\mathcal{G}_d \subseteq \mathcal{L} \times U(F)$ such that the image is nonzero for $(\mathbb{L}, \gamma) \in \mathcal{G}_d$.

Let $\mathcal{G} = \bigcap_d \mathcal{G}_d$. We only need to see that $\mathcal{G}$ is open and nonempty. Let $V_d \subseteq F_d$ be the unique monomial subspace arising as $\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d)$ for $(\mathbb{L}, \gamma) \in \mathcal{G}_d$. Let $H \subseteq F$ be the submodule generated by all of the V_d. Since F is Noetherian, there is a number D such that H is generated in degrees $d \leq D$. Let $\mathcal{H} = \bigcap_{d \leq D} \mathcal{G}_d$. If $(\mathbb{L}, \gamma) \in \mathcal{H}$ then for every $d \leq D$, $\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))_d) = V_d$. In particular, for $d \leq D$, we have that $SV_{d-1} \subseteq V_d$ so that $H_d = V_d$. Thus in degrees $d \leq D$, H and $\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)})))$ agree. Since H has no generators in higher degree, $H \subseteq \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)})))$. But, for $d > D$, the dimension of H_d is at least the dimension of V_d which, for generic $(\mathbb{L}, \gamma)$, is the same as the dimension of $\mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)})))_d$. Thus $H_d = V_d$ and $\mathcal{H} = \mathcal{G}$ is open and nonempty. $\square$

Since $(\mathbb{L}, \gamma) \in \mathcal{G}$ is generic, it follows immediately that ϕ is well defined. Thus Φ is well defined as well. In the next proposition I summarize the main elementary properties of ϕ and Φ.

PROPOSITION 19. *Let $N \subset F$ be a monomial submodule and let ϕ and Φ be as above. Then*:

(1) *$\phi(N)$ is a Borel-fixed submodule.*
(2) *$F/\phi(N)$ has the same Hilbert function as F/N.*
(3) *$\phi(N)_d \geq N_d$ for every d.*
(4) *If L is a lexicographic submodule, then $\phi(L) = L$.*
(5) *If M is a monomial submodule of F containing N then $\phi(N) \subseteq \phi(M)$.*
(6) *$\Phi(N)$ is a saturated Borel-fixed submodule.*
(7) *$F/\Phi(N)$ has the same Hilbert polynomial as F/N.*
(8) *$\Phi^s(N) = \phi^s(N) :_F \mathfrak{m}^\infty$.*
(9) *$\Phi(N) = \Phi(N :_F \mathfrak{m}^\infty)$.*

Proof. (1) If $(\mathbb{L}, \gamma)$ is generic, then $\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))$ is in generic coordinates. $\phi(N)$ is the submodule of initial terms of $\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))$ and is Borel-fixed by Proposition 4.

(2) This follows from Corollary 15.

(3) This follows from Proposition 18.

(4) A lexicographic subspace of F_d is maximal among monomial subspaces of the same dimension with respect to lexicographic order. So, this statement follows from (2) and (3).

(5) If $N \subseteq M$ then $N^{(p)} \subseteq M^{(p)}$. For generic $(\mathbb{L}, \gamma)$,

$$\phi(N) = \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))) \subseteq \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(M^{(p)}))) = \phi(M).$$

(6) The saturation of a Borel-fixed submodule is also Borel-fixed, so this follows from (1).

(7) This follows from (2).

(8) From (5) we know that $\Phi^s(N) \supseteq \phi^s(N)$. But the quotient modules have the same Hilbert polynomial and $\Phi^s(N)$ is saturated. So, the saturation of $\phi^s(N)$ must be $\Phi^s(N)$.

(9) From (5) we know that $\phi(N) \subseteq \phi(N:_F \mathfrak{m}^\infty)$. Since their quotient modules have the same Hilbert polynomial, they must have the same saturation. □

6. Moving to L

The main result of this section, Proposition 30, is that for e sufficiently large, $\phi^e(N)$ is a lexicographic submodule. The main results of this paper, Theorems 31 and 35, will follow easily. Before proving that $\phi^e(N)$ is lexicographic, it is necessary to prove that $\Phi^e(N)$ is a saturated lexicographic submodule. My proof of this is based on Reeves' argument in [Re1], [Re2].

There are two major differences between my argument and Reeves'. The first is that there are additional arguments needed to work with submodules of free modules, rather than with ideals. The second is that Reeves' argument was restricted to characteristic zero by her reliance on an interesting algorithm which she wrote to find the components of a fan associated to a standard Borel-fixed ideal. I replace this part of her argument with a more primitive analysis using Hartshorne's description of the minimal primes of the polarization of a monomial ideal, on which her algorithm is based.

I will start by stating Hartshorne's criterion, rewritten for monomial submodules of F.

PROPOSITION 20. *If $N = I_{(1)}e_1 \oplus \cdots \oplus I_{(r)}e_r$ is a monomial submodule of F, then $N^{(p)}$ has an irredundant primary decomposition as the intersection of the submodules*

$$Pe_1 \oplus \cdots \oplus Pe_{k-1} \oplus (z_{i_1 j_1 k}, \ldots, z_{i_s j_s k})e_k \oplus Pe_{k+1} \oplus \cdots \oplus Pe_r \subseteq F'$$

where

(1) $I_{(k)} \neq S$,

(2) $0 \leq s \leq n$,

(3) $i_1 < \cdots < i_s$,

(4) $I_{(k)} \subseteq (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$, *and*

(5) *no proper subset of* $\{x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s}\}$ *generates an ideal containing* $I_{(k)}$.

Also, if N is Borel-fixed, then we always have $i_t = t$.

Proof. $N^{(p)}$ is the intersection of the submodules

$$Pe_1 \oplus \cdots \oplus Pe_{k-1} \oplus I_{(k)}^{(p_k)}e_k \oplus Pe_{k+1} \oplus \cdots \oplus Pe_r.$$

So, we only need to prove the proposition in the case $r = 1$, which is Propositions 4.4 and 4.8 in Hartshorne [Ha]. $\square$

We need good descriptions of the generators of saturated lexicographic submodules and of the components of their polarizations. The notation e_q was described at the end of Section 2. From the definition of a lexicographic submodule L, it is easy to see that if $L = L_{(1)}e_1 \oplus \cdots \oplus L_{(q)}e_q$ (so that $L_{(q)} \neq 0$), then $L_{(i)}$ has finite colength for every $i < q$. So, a saturated lexicographic submodule must be of the form $L = Se_1 \oplus \cdots \oplus Se_{q-1} \oplus L_{(q)}e_q$, where $L_{(q)}$ is a saturated lexicographic ideal.

PROPOSITION 21. *Let $L = Se_1 \oplus \cdots \oplus Se_{q-1} \oplus L_{(q)}e_q$ be a saturated lexicographic submodule of F. Then there are nonnegative integers $b_1, \ldots, b_s$, with $0 \leq s < n$, such that L is generated by $e_1, \ldots, e_{q-1}$ and $x_1^{b_1} \cdots x_{k-1}^{b_{k-1}} x_k^{b_k+1} e_q$ for $1 \leq k \leq s$. If $L_{(q)}$ has codimension c, then $b_1 = \cdots = b_{c-1} = 0$.*

Proof. In [Ba], Bayer proved for the case in which L is an ideal a stronger statement, which we will not need, in which the exponents are interpreted in terms of the decomposition of the Hilbert polynomial as a sum of binomial coefficients. The first statement, which follows trivially from the case in which $q = 1$, is easily proved by induction on the number of generators of $L_{(q)}$. That $s < n$ follows from Proposition 9. The last statement follows from Proposition 8. $\square$

COROLLARY 22. *If L is a saturated lexicographic submodule with generators as above, then $L^{(p)}$ is the irredundant intersection of the primary submodules*

$$Pe_1 \oplus \cdots \oplus Pe_{k-1} \oplus 0e_k \oplus Pe_{k+1} \oplus \cdots \oplus Pe_r$$

where $k > q$, and

$$Pe_1 \oplus \cdots \oplus Pe_{q-1} \oplus (z_{1(b_1+1)q}, \ldots, z_{(t-1)(b_{t-1}+1)q}, z_{tuq})e_q$$

where

(1) $c \leq t \leq s$,
(2) $1 \leq u \leq b_t$ *if* $t < s$, *and*
(3) $1 \leq u \leq b_s + 1$ *if* $t = s$.

Proof. We will use the criterion of Proposition 20. We only need to worry about the primary decomposition of $L_{(q)}^{(p_q)}$, since the rest is trivial.

Note that $x_i \in L_{(q)}$ for $i < c$. So, if $J = (x_1^{j_1}, \ldots, x_t^{j_t}) \supseteq L_{(q)}$, then $j_1, \ldots, j_{c-1} = 1$. Also, some power of x_c is in J, so $t \geq c$. If J also satisfies condition (5) of Proposition 20, then $t \leq s$ since otherwise $x_t^{j_t}$ is not required for the inclusion $J \supseteq L_{(q)}$. This establishes (1).

Now we will establish the indices of the z_{ij_iq} for $1 \leq i \leq t-1$. It is clear that $j_i \leq b_i + 1$, for otherwise J would not contain $L_{(q)}$. If $j_i < b_i + 1$ then by induction, $(x_1^{j_1}, \ldots, x_i^{j_i}) = (x_1^{b_1+1}, \ldots, x_{i-1}^{b_{i-1}+1}, x_i^{j_i})$ which contains $L_{(q)}$ by Proposition 21. Then, since $i < t$, J does not satisfy condition (5) of Proposition 20. So, $j_i = b_i + 1$ for $1 \leq i \leq t-1$.

Now it's easy to see that J satisfies the conditions of Proposition 20 if and only if $j_t \leq b_t$ if $t < s$, and $j_s \leq b_s + 1$ if $t = s$. $\square$

LEMMA 23. *Let $I \subseteq S$ be a radical homogeneous ideal, other than* 0 *or* $\mathfrak{m}$. *If the degree of the union of the irreducible components of $Z(I)$ of codimension j is m_j, then the lexicographically last minimal generator of the generic ideal of initial terms,* $\mathbf{Gin}(I)$, *is* $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$.

Proof. If $I = S$, then each $m_j = 0$ and $\mathbf{Gin}(S) = S$ is generated by 1. Now we will proceed by induction on the dimension of $Z(I)$. We may assume that I is in generic coordinates so that $\mathbf{Gin}(I) = \mathbf{in}(I)$.

Let $d = \dim Z(I)$. Let $I_{(1)}$ be the ideal of the union of the d dimensional irreducible components of $Z(I)$, and let $I_{(2)}$ be the ideal of the union of the lower dimensional irreducible components. $I = I_{(1)} \cap I_{(2)}$ and we have already proved the lemma for $I_{(2)}$ by induction.

I has codimension $n-d-1$ and $\mathbf{in}(I)$ is Borel-fixed. So by Proposition 8, the last minimal generator of $\mathbf{in}(I)$ is in the subring $S' = k[x_{n-d-1}, \ldots, x_n]$. Also, since we are using lexicographic order, $(\mathbf{in}(I)) \cap S' = \mathbf{in}(I \cap S')$. (See [Ei], Proposition 15.4.) So the last minimal generator of $\mathbf{in}(I)$ is also the last minimal generator of $\mathbf{in}(I \cap S')$.

Now, since I is in generic coordinates, $I \cap S'$ is the ideal of the projection of $Z(I)$ onto a generic plane of dimension $d+1$. So, the image of $Z(I_{(1)})$ is a (reducible) hypersurface of degree m_{n-d-1}. Thus $I_{(1)} \cap S'$ is generated by a homogeneous polynomial $f \in S'$ of degree m_{n-d-1}. Since we are working in generic coordinates, $\mathbf{in}(f) = x_{n-d-1}^{m_{n-d-1}}$.

I claim that $I \cap S' = f(I_{(2)} \cap S')$. This is certainly true if $I_{(2)} = S$. Otherwise, since all of the components of $Z(I)$ remain distinct under our projection, f is not in any associated prime of $I_{(2)} \cap S'$ and $I \cap S' = fS' \cap (I_{(2)} \cap S') = f(I_{(2)} \cap S')$.

It follows that $\mathbf{in}(I \cap S') = x_{n-d-1}^{m_{n-d-1}} \mathbf{in}(I_{(2)} \cap S')$. The last minimal generator of $\mathbf{in}(I_{(2)} \cap S')$ is $x_{n-d}^{m_{n-d}} \cdots x_{n-1}^{m_{n-1}}$. So, the last minimal generator of $\mathbf{in}(I \cap S')$, and thus of $\mathbf{in}(I)$, is $x_{n-d-1}^{m_{n-d-1}} \cdots x_{n-1}^{m_{n-1}}$ and we are done. $\square$

PROPOSITION 24. *If $N = N_{(1)}e_1 \oplus \cdots \oplus N_{(q)}e_q \subseteq F$ is a saturated monomial submodule with $N_{(i)} \neq 0$ for $1 \leq i \leq q$, and for $1 \leq j \leq n-1$ $N^{(p)}$ has m_j components of codimension j among those listed in Proposition* 20, *then $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$ is the lexicographically last minimal generator of both $\phi(N)$ and $\Phi(N)$.*

Proof. Note that none of the $N_{(i)}$ are $\mathfrak{m}$ since N is saturated. Say that $N_{(i)}^{(p_i)}$ has m_{ij} components of codimension j. Then $m_j = \sum_i m_{ij}$. As in Definitions 13, let $\mathbb{L}$

be a generic set of linear forms, let $\sigma = \sigma_{\mathbb{L}}$ and let $\sigma' = \sigma'_{\mathbb{L}}$. Let $J_{(i)} = \sigma(N_{(i)}^{(p_i)})$. Then $J_{(i)}$ is a radical homogeneous ideal, not equal to 0 or $\mathfrak{m}$, and $Z(J_{(i)})$ has m_{ij} irreducible components of codimension j, all of which are linear.

First I will show that the lexicographically last minimal generator of $\phi(N)$ is $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$. Note that by definition,

$$\phi(N) = \mathbf{in}(\gamma(\sigma'(N^{(p)})) = \mathbf{in}\left(\sum_{i=1}^{q} J_{(i)} \sum_{\ell=1}^{i} f_{i\ell} e_\ell\right)$$

where $f_{ii} = 1$ and $f_{i\ell}$ is a generic polynomial of degree $d_i - d_\ell$ for $\ell < i$.

We will compute the initial submodule in two stages, the first only taking into account the basis elements of F. To this end, if $h = \sum_{i=k}^{r} h_i e_i \in F$ and $h_k \neq 0$, then define $\mathbf{in}'(h) = h_k e_k$. (Also define $\mathbf{in}'(0) = 0$). If H is a submodule of F, then define $\mathbf{in}'(H)$ to be the submodule of F generated by all $\mathbf{in}'(h)$ where $h \in H$. Then $\mathbf{in}'(H) = K_{(1)}e_1 \oplus \cdots \oplus K_{(r)}e_r$ where each $K_{(i)}$ is an ideal in S. Also, $\mathbf{in}(H) = \mathbf{in}(\mathbf{in}'(H)) = \mathbf{in}(K_{(1)})e_1 \oplus \cdots \oplus \mathbf{in}(K_{(r)})e_r$.

Set $\mathbf{in}'(\gamma(\sigma'(N^{(p)}))) = K_{(1)}e_1 \oplus \cdots \oplus K_{(q)}e_q$. We need to see that the lexicographically last minimal generator of $\mathbf{in}(K_{(q)})$ is $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$. First, I claim that $K_{(q)} = J_{(1)} \cap \cdots \cap J_{(q)}$. The general element of $\gamma(\sigma'(N^{(p)}))$ is of the form

$$h = \sum_{i=1}^{q} g_i \sum_{\ell=1}^{i} f_{i\ell} e_\ell = \sum_{\ell=1}^{q} \left(\sum_{i=\ell}^{q} g_i f_{i\ell}\right) e_\ell$$

where $g_i \in J_{(i)}$. Say that $\mathbf{in}'(h) \in K_{(q)}e_q$. Then $\mathbf{in}'(h) = g_q e_q$ and $\sum_{i=\ell}^{q} g_i f_{i\ell} = 0$ for $\ell < q$. This implies that $g_\ell = \alpha_\ell g_q$ where α_ℓ is inductively defined by $\alpha_q = 1$ and $\alpha_\ell = -\sum_{i=\ell+1}^{q} \alpha_i f_{i\ell}$ for $1 \le \ell \le q-1$. So $g_q \in J_{(\ell)} \colon \alpha_\ell$. But α_ℓ is a generic polynomial of degree $d_q - d_\ell$. (In fact, α_ℓ is the sum of $-f_{q\ell}$ and other terms of the same degree, and $f_{q\ell}$ is generic.) Since $\mathfrak{m}$ is not an associated prime of $J_{(\ell)}$, we have that $J_{(\ell)} \colon \alpha_\ell = J_{(\ell)}$. Thus $\mathbf{in}'(h) = g_q e_q \in (J_{(1)} \cap \cdots \cap J_{(q)})e_q$ and $J_{(1)} \cap \cdots \cap J_{(q)} \supseteq K_{(q)}$.

For the opposite inclusion, let $g \in J_{(1)} \cap \cdots \cap J_{(q)}$. For $1 \le i \le q$ let α_i be as in the last paragraph and let $g_i = \alpha_i g$. Then $g_i \in J_{(i)}$. Let $h = \sum_{i=1}^{q} g_i \sum_{\ell=1}^{i} f_{i\ell} e_\ell \in \gamma(\sigma'(N^{(p)}))$. By the choice of α_i, we see that $\mathbf{in}'(h) = g_q e_q = g e_q$. So $g \in K_{(q)}$.

Now $Z(J_{(1)} \cap \cdots \cap J_{(q)}) = \bigcup_{i=1}^{q} Z(J_{(i)})$ and since no component of one $Z(J_{(i)})$ contains a component of another, $Z(J_{(1)} \cap \cdots \cap J_{(q)})$ has m_j components of codimension j, all of which are linear. Lemma 23 tells us that the last minimal generator of $\mathbf{in}(J_{(1)} \cap \cdots \cap J_{(q)})$ is $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$. So the lexicographically last minimal generator of $\phi(N)$ is $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$.

Let $x^\mu e_s$ be a minimal generator of $\Phi(N)$. Then by Proposition 9, $\phi(N)$ has a minimal generator of the form $x^\mu x_n^k e_s$. If $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$ precedes $x^\mu e_s$ lexicographically, then it also precedes $x^\mu x_n^k e_s$, which is a contradiction. There must be a minimal generator $x^\mu e_q$ of $\Phi(N)$ such that x^μ divides $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$. If x^μ properly divides $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$, then $x^\mu e_q$ comes after $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$ lexicographically, contradicting what was just proved. Thus $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} e_q$ is a minimal generator of $\Phi(N)$ and is the lexicographically last one. □

COROLLARY 25. *If $N = N_{(1)}e_1 \oplus \cdots \oplus N_{(q)}e_q \subseteq F$ is a saturated monomial submodule with $N_{(j)} \neq 0$ for $1 \leq j \leq q$, and $N^{(p)}$ has m_j components of codimension j for $1 \leq j \leq n-1$, then the following submodules of F' are among the primary components of $\Phi(N)^{(p)}$ described in Proposition* 20:

$$Pe_1 \oplus \cdots \oplus Pe_{q-1} \oplus (z_{1(m_1+1)q}, \ldots, z_{(t-1)(m_{t-1}+1)q}, z_{tuq})e_q$$

where $\operatorname{codim}(N_{(q)}) \leq t \leq n-1$ *and* $1 \leq u \leq m_t$.

Proof. Let $\Phi(N) = I_{(1)}e_1 \oplus \cdots \oplus I_{(q)}e_q$. By the last proposition, the lexicographically last minimal generator of $I_{(q)}$ is $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$. By Proposition 20, we need to see that $J = (x_1^{m_1+1}, \ldots, x_{t-1}^{m_{t-1}+1}, x_t^u) \supseteq I_{(q)}$ and that for each generator of J there is some monomial of $I_{(q)}$ which is divisible by that generator and no other. If $J \not\supseteq I_{(q)}$ then let x^μ be a monomial in $I_{(q)}$, but not in J. Then $\mu_i \leq m_i$ for $i < t$ and $\mu_t < u \leq m_t$. So, if x^ν is a minimal generator of $I_{(q)}$ dividing x^μ then $x^\nu < x_c^{m_c} \cdots x_{n-1}^{m_{n-1}}$ in lexicographic order. This contradicts that $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}}$ is the lexicographically last minimal generator of $I_{(q)}$.

Now I will show that every generator of J is essential. $I_{(q)}$ is Borel-fixed and $x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \in I_{(q)}$. For every $k \leq t$ let $E_k = \sum_{i=k}^{n-1} m_i$. Then

$$x^\mu = x_1^{m_1} \cdots x_{k-1}^{m_{k-1}} x_k^{E_k} \in I_{(q)}$$

by Proposition 6. The monomial $x_k x^\mu$ is divisible only by the kth generator of J. □

LEMMA 26. *If $N_{(1)} = G \cap H_{(1)}$ and $N_{(2)} = G \cap H_{(2)}$ are submodules of F, the Hilbert polynomials of $F/N_{(1)}$ and $F/N_{(2)}$ are the same, the codimension of $G + H_{(1)}$ is greater than the codimension of $H_{(1)}$, and the codimension of $G + H_{(2)}$ is greater than the codimension of $H_{(2)}$, then $H_{(1)}$ and $H_{(2)}$ have the same codimension and multiplicity.*

Proof. We need to see that the Hilbert polynomials of $F/H_{(1)}$ and $F/H_{(2)}$ have the same degree and leading coefficient. From the short exact sequence

$$0 \to F/N_{(i)} \to F/G \oplus F/H_{(i)} \to F/(G + H_{(i)}) \to 0$$

we see that $p_{F/H_{(i)}}(z) = p_{F/N_{(i)}}(z) + p_{F/(G+H_{(i)})}(z) - p_{F/G}(z)$. Because of the hypothesis on codimensions, the highest degree term of $p_{F/H_{(i)}}(z)$ is the same as that of $p_{F/N_{(i)}}(z) - p_{F/G}(z)$, which is the same for $i = 1$ and $i = 2$. □

PROPOSITION 27. *If $N \subseteq F$ is a saturated monomial submodule, F/N has Hilbert polynomial $p(z)$ of degree d, and $L \subseteq F$ is the saturated lexicographic submodule such that F/L has Hilbert polynomial $p(z)$, then $\Phi^{d+1}(N) = L$.*

Proof. In this proof, we only consider primary components such as those given in Proposition 20. Each of these components has multiplicity one. We need to see that if the primary components of $N^{(p)}$ and $L^{(p)}$ agree in codimensions less than c, then the primary components of $\Phi(N)^{(p)}$ and $L^{(p)}$ agree in codimensions less than $c+1$. Since all of the components of $N^{(p)}$ and $L^{(p)}$ have codimensions between $n-1-d$ and $n-1$, the theorem will follow.

If $N^{(p)}$ and $L^{(p)}$ have the same primary components in codimensions less than c, then set G to be the intersection of these components. Lemma 26 shows that $N^{(p)}$ and $L^{(p)}$ have the same number of primary components of codimension c. A comparison of the components of $\Phi(N)^{(p)}$ listed in Corollary 25 and the components of $L^{(p)}$ listed in Corollary 22 shows that the components of $L^{(p)}$ of codimension less than or equal to c are among the components of $\Phi(N)^{(p)}$. (Note that $b_j = m_j$ for $j \leq c$.) By applying Lemma 26 inductively on the codimensions of these components, we see that $\Phi(N)^{(p)}$ has no more components of codimension less than or equal to c than the components of $L^{(p)}$, and we are done. □

I need a few more simple facts about standard Borel-fixed submodules of F before proving the main proposition of this section.

LEMMA 28. *If N is a standard Borel-fixed submodule of F generated in degree d, then $(N :_F \mathfrak{m}^\infty)_d = N_d$.*

Proof. If the generators of N are $\{x^\mu e_i\}$ then, by Proposition 9, a set of generators for $I : \mathfrak{m}^\infty$ is $\{\frac{x^\mu}{x_n^{\mu_n}} e_i\}$. So the monomials of $N :\ _F\mathfrak{m}^\infty$ in degree d are of the form $\frac{x^\nu}{x_n^{\mu_n}} x^\mu e_i$. But these are in N_d by the standard Borel-fixed property. □

COROLLARY 29. *If N is a standard Borel-fixed submodule of F generated in degree d and is not lexicographic, then N_e is not lexicographic for any $e \geq d$.*

Proof. If N_e is lexicographic then SN_e is a lexicographic submodule of F and $SN_e :_F \mathfrak{m}^\infty$, which is equal to $N :_F \mathfrak{m}^\infty$, is lexicographic as well. Then, by Lemma 28, N_d is lexicographic and N is lexicographic, which is a contradiction. □

PROPOSITION 30. *If N is a monomial submodule of F and L is the lexicographic submodule with the same Hilbert function, then for $e \gg 0$, $\phi^e(N) = L$.*

Proof. By Proposition 27, $\Phi^f(N)$ is a saturated lexicographic submodule for large enough f. But since $\Phi^f(N) = \phi^f(N) :_F \mathfrak{m}^\infty$, $\phi^f(N)$ is lexicographic in high degrees. So we may assume that N is Borel-fixed and lexicographic in high degrees. If N_d is lexicographic then $\phi(N)_d = N_d$. What we must prove is that if N_d is not lexicographic then $\phi(N)_d > N_d$. Then for large enough e, $\phi^e(N)$ is lexicographic in all degrees.

Say that N_d is not lexicographic and let $G = SN_d$. Then, since $G \subseteq N$, $\phi(G) \subseteq \phi(N)$. Since they have the same dimension in degree d, $\phi(G)_d = \phi(N)_d$. So, we only need to see that $\phi(G)_d > G_d$.

Note that G is a Borel-fixed submodule of F. First, assume that G is not a standard Borel-fixed submodule. By Proposition 18, we only need to see that there is some $\mathbb{L}$ such that $\mathbf{in}(\sigma'_{\mathbb{L}}(G^{(p)}))_d > G_d$. Say that the monomial generators of G are $m_1e_{k(1)}, \ldots, m_te_{k(t)}$ where $m_1e_{k(1)} > \cdots > m_te_{k(t)}$. Let $s \geq 2$ be the smallest index such that $m_1e_{k(1)}, \ldots, m_se_{k(s)}$ do not generate a standard Borel-fixed submodule. So, $m_1e_{k(1)}, \ldots, m_{s-1}e_{k(s-1)}$ satisfy the standard Borel-fixed property and there is a variable x_b dividing m_s and $a < b$ such that $\frac{x_a}{x_b}m_se_{k(s)} \notin G$. For each $u \in k$ let ρ_u: $P \to S$ and ρ'_u: $F' \to F$ be the homomorphisms sending $z_{b1k(s)}$ to $ux_a + x_b$ and sending every other z_{ijk} to x_i and every e_i to e'_i. Since $\rho_0 = \pi$ has a kernel generated by a regular sequence on $F'/G^{(p)}$, for generic u the kernel of ρ_u is generated by a regular sequence on $F'/G^{(p)}$. Then $\rho'_u(G^{(p)})_d$ is spanned by $m'_1e_{k(1)}, \ldots, m'_te_{k(t)}$ where $m'_q = u\frac{x_a}{x_b}m_q + m_q$ if $x_b|m_q$ and $k(q) = k(s)$, and $m'_q = m_q$ otherwise. It is easy to see that $m'_1e_{k(1)}, \ldots, m'_te_{k(t)}$ are linearly independent and that if $u \neq 0$ then $m_1e_{k(1)}, \ldots, m_{s-1}e_{k(s-1)}, \frac{x_a}{x_b}m_se_{k(s)} \in \mathbf{in}(\rho'_u(G^{(p)}))_d$ so that $\mathbf{in}(\rho'_u(G^{(p)}))_d > G_d$.

Now assume that G is a standard Borel-fixed submodule, but is not lexicographic. By part (3) of Proposition 19, we only need to see that $\phi^e(G)_d \neq G_d$ for some $e > 0$. But, for $e \gg 0$, $\Phi^e(G)$ is lexicographic. So $\phi^e(G)$ is lexicographic in high degrees. By Corollary 29, G is not lexicographic in high degrees. Therefore $\phi^e(G) \neq G$. If nonetheless $\phi^e(G)_d = G_d$ then, since G is generated in degree d, $\phi^e(G)$ properly contains G. But this is impossible because they have the same Hilbert function. So $\phi^e(G)_d \neq G_d$. □

7. Betti numbers

In this section I prove that the quotient of F by the lexicographic submodule has the largest graded Betti numbers of any quotient module of F with a given Hilbert function. Then, after some definitions and a lemma, I derive the analogous results for modules over regular local rings.

THEOREM 31. *If $N \subseteq F$ is a graded submodule of a graded free module over a polynomial ring over any field, and $L \subseteq F$ is a lexicographic submodule such that F/N and F/L have the same Hilbert function, then the graded Betti numbers of F/N are no greater than those of F/L.*

Proof. Since extending the ground field is a faithfully flat functor, we may assume that the field is infinite. Graded Betti numbers are homological invariants and are thus upper-semicontinuous in flat families. In the flat family taking F/N to $F/\mathbf{in}(N)$, every fiber is isomorphic to F/N except for $F/\mathbf{in}(N)$. Thus, the graded Betti numbers of $F/\mathbf{in}(N)$ are at least as large as those of F/N and we may assume from the beginning that N is a monomial submodule. By Proposition 30, we only need to see that the graded Betti numbers of F/N are no greater than the graded Betti numbers of $F/\phi(N)$. But $\phi(N) = \mathbf{in}(\gamma(\sigma'_{\mathbb{L}}(N^{(p)})))$ for generic γ and $\mathbb{L}$ and so it suffices to see that the graded Betti numbers of F/N and of $F/\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))$ are the same. But $F/\gamma(\sigma'_{\mathbb{L}}(N^{(p)}))$ is isomorphic to $F/\sigma'_{\mathbb{L}}(N^{(p)})$ which we saw to have the same graded Betti numbers as F/N in Corollary 15. $\square$

In order to state the analogous theorems for regular local rings, I must first state the relevant definitions. Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Fix a minimal system of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$. Let F be a free R-module with a fixed basis $e_1, \ldots, e_r$. Define *monomials of* F, the *lexicographic order* on these monomials, and *monomial submodules of* F exactly as for the polynomial ring case. A *lexicographic submodule* L of F is a monomial submodule such that if $x^\mu e_i \in L$, $x^\nu e_j > x^\mu e_i$ in lexicographic order and x^μ and x^ν have the same degree, then $x^\nu \in L$ as well. Let $\beta_i(F/M)$ be the *ith Betti number* of F/M, which is the rank of the ith free module in a minimal free resolution of F/M.

Let $S = \mathrm{gr}_{\mathfrak{m}} R = k[\overline{x_1}, \ldots, \overline{x_n}]$, and $\overline{F} = \mathrm{gr}_{\mathfrak{m}} F$, where $\overline{x_i}$ is the class of x_i modulo $\mathfrak{m}^2$, and $\overline{F}$ is a free S-module with basis $\overline{e_1}, \ldots, \overline{e_r}$, the residue classes of e_i modulo $\mathfrak{m}F$. If $M \subseteq F$ is a monomial submodule generated by $\{x^\mu e_i\}$, then $\mathrm{gr}_{\mathfrak{m}}(F/M) = \overline{F}/\overline{M}$ where $\overline{M}$ is generated by $\{\overline{x}^\mu \overline{e}_i\}$.

The Hilbert function of a finitely generated R-module is, by definition, the Hilbert function of its associated graded module. (See, for example, Chapter 5 of [Ei].) From the last paragraph and Proposition 2, it follows that for any submodule $M \subseteq F$, there is a lexicographic submodule $L \subseteq F$ such that F/M and F/L have the same Hilbert function.

LEMMA 32. *Let $M \subseteq F$ be a submodule. Then $\beta_i(F/M) \leq \beta_i(\overline{F}/\overline{M})$. If M is a monomial submodule, then $\beta_i(F/M) = \beta_i(\overline{F}/\overline{M})$.*

Proof. The first statement is a special case of Corollary 3.2 of [HeRoVa].

Let $Q = R[y_1, \ldots, y_n]$ and $F' = F \otimes_R Q$. If M is generated by $\{x^\mu e_i\}$, then let M' be the submodule of F' generated by the monomials $\{y^\mu(e_i \otimes 1)\}$. Since M' is homogeneous and R is a local ring, the Betti numbers

$$\beta_i(F'/M') = \dim_{R/\mathfrak{m}} \mathrm{Tor}_i^Q(F'/M', Q/\mathfrak{m} + (y_1, \ldots, y_n))$$

are well defined. Note that $Q/(x_1, \ldots, x_n) = k[y_1, \ldots, y_n] \simeq S$ while $(F'/M') \otimes_Q Q/(x_1, \ldots, x_n) \simeq \overline{F}/\overline{M}$. Furthermore, $x_1, \ldots, x_n$ is a regular sequence on Q and on

F'/M'. So, $\beta_i(F'/M') = \beta_i(\overline{F}/\overline{M})$. On the other hand, $y_1 - x_1, \ldots, y_n - x_n$ is also a regular sequence on Q and on F'/M', and $Q/(y_i - x_i) \simeq R$ while $(F'/M') \otimes_Q Q/(y_i - x_i) \simeq F/M$. So, $\beta_i(F'/M') = \beta_i(F/M)$. □

COROLLARY 33. *Let R be a regular local ring and F be a finitely generated free module as above. Let $M \subseteq F$ be a submodule and let $L \subseteq F$ be the lexicographic submodule such that F/M and F/L have the same Hilbert function. Then $\beta_i(F/M) \le \beta_i(F/L)$ for every i.*

Proof. By Theorem 31 and Lemma 32 we have

$$\beta_i(F/M) \le \beta_i(\overline{F}/\overline{M}) \le \beta_i(\overline{F}/\overline{L}) = \beta_i(F/L). \qquad \square$$

8. Deformations

In this section I study the deformation classes of quotient modules of a free graded S-module F and the deformation classes of quotient sheaves of the sheafification of F, a direct sum of line bundles on $\mathbb{P}_k^{n-1}$. Theorem 34, which follows easily from Proposition 30, says that the deformation classes of quotient modules of F are the sets of quotient modules with a fixed Hilbert function. Theorem 36 says that the deformation classes of quotient sheaves of a direct sum of line bundles are the sets of quotient sheaves with a fixed Hilbert polynomial; furthermore, I bound the number of deformations over $\mathbb{A}^1$ required to take one coherent sheaf to another with the same Hilbert polynomial.

A *deformation of quotient modules of F over $\mathbb{A}^1$* is a graded quotient module $\tilde{F}/\tilde{N}$ of the $S[t]$-free module $\tilde{F} = F \otimes_S S[t]$, where t is a new variable of degree 0, such that $\tilde{F}/\tilde{N}$ is flat as a $k[t]$-module. If $\alpha \in k$, then the *fiber over $t = \alpha$* of the deformation is the S-module $(\tilde{F}/\tilde{N})_\alpha = (\tilde{F}/\tilde{N}) \otimes_{k[t]} (k[t]/(t-\alpha))$. For any two $\alpha, \beta \in k$, $\tilde{F}/\tilde{N}$ is a *deformation from $(\tilde{F}/\tilde{N})_\alpha$ to $(\tilde{F}/\tilde{N})_\beta$*. Since $\tilde{F}/\tilde{N}$ is graded and flat over $k[t]$, in each degree $(\tilde{F}/\tilde{N})_d$ is a free $k[t]$-module of finite rank. So, any two fibers of a deformation over $\mathbb{A}^1$ must have the same Hilbert function.

Now, I will show how to construct a sequence of deformations over $\mathbb{A}_k^1$ between any two quotient modules of F with the same Hilbert function. It suffices to deform F/N to F/L, where L is the unique lexicographic submodule of F such that F/N and F/L have the same Hilbert function.

It is well known that there is a deformation over $\mathbb{A}^1$ taking F/N to $F/\mathbf{in}(N)$. That is, there is a module $\tilde{N} \subseteq F \otimes_S S[t]$ such that $(F \otimes_S S[t])/\tilde{N}$ is free over $k[t]$, the fiber over $t = 1$ is isomorphic to F/N, and the fiber over $t = 0$ is $F/\mathbf{in}(N)$. See Theorem 15.7 of [E] for a full description of this deformation in the case $F = S$; the generalization to an arbitrary free graded S-module is immediate.

So, we may assume that N is a monomial submodule of F. In light of Proposition 30, it is enough to find a sequence of deformations from F/N to $F/\phi(N)$. We may deform F/N to $F/\phi(N)$ in two steps. Let $\mathcal{L}$ be as in Definition 13. Let $\mathbb{L} \in \mathcal{L}$ be a generic collection of linear forms and let $\mathbb{L}_0$ be the collection of linear forms $\ell_{ijk} = x_i$. Then from an affine line $\mathbb{A}^1_k \subseteq \mathcal{L}$ containing $\mathbb{L}_0$ and $\mathbb{L}$, we may construct a deformation from $F/N = F/\sigma'_{\mathbb{L}_0}(N^{(p)})$ to $F/\sigma'_{\mathbb{L}}(N^{(p)})$. Then, as above, there is a deformation from $F/\sigma'_{\mathbb{L}}(N^{(p)})$ to $F/\mathbf{in}(\sigma'_{\mathbb{L}}(N^{(p)})) = F/\phi(N)$. This proves the following theorem.

THEOREM 34. *If F is a finitely generated free S-module, where S is a polynomial ring over an infinite field k, and F/M and F/N are graded quotient modules of F, then there is a sequence of deformations from F/M to F/N, each of which is defined over $\mathbb{A}^1_k$, if and only if F/M and F/N have the same Hilbert function.*

The construction of the sequence of deformations from F/N to $F/\phi(N)$ can be refined so that only one deformation is needed to get from F/N to $F/\phi(N)$. I use this more efficient deformation in the proof of Theorem 36. The existence of this deformation follows from a more general construction.

Let $A = k[y_1, \dots, y_m]$ be the coordinate ring of $\mathbb{A}^m_k$, and let $S' = S \otimes_k A$ be the coordinate ring of $\mathbb{A}^m_k \times_k \mathbb{P}^{n-1}_k$; S' is a graded ring in which each x_i has degree 1 and each y_i has degree 0. Let $F' = F \otimes_k A$, a free S'-module. A *monomial* of F' is an element of the form $m \otimes a$ where m is a monomial of F and $a \in A$. An element $f \in F$ may be uniquely written as $\sum_i m_i \otimes a_i$ where the m_i are distinct monomials. The *initial term* of f is the term $m_i \otimes a_i$ for which m_i is lexicographically maximal among those m_i such that $a_i \neq 0$. If $N' \subseteq F'$ is a graded submodule, then let $\mathbf{in}(N')$ be the submodule of F' generated by the initial terms of elements of N'. As above, there is a submodule $\tilde{N}' \subseteq F' \otimes_A A[t]$ such that $(F' \otimes_A A[t])/\tilde{N}'$ is free over $k[t]$, the fiber over $t = 1$ is isomorphic to F'/N', and the fiber over $t = 0$ is $F'/\mathbf{in}(N')$.

The computation of initial submodules commutes with tensoring with a residue field $k(\mathfrak{p})$ for most prime ideals $\mathfrak{p}$ in A. Specifically, say that $\mathbf{in}(N')$ is generated by the finite set $\{m_1 \otimes a_1, \dots, m_s \otimes a_s\}$ and let a be the product of $a_1, \dots, a_s$. If we write $M(\mathfrak{p})$ for the image of a submodule M of F' in $F' \otimes_A k(\mathfrak{p})$, then whenever $a \notin \mathfrak{p}$ we have that $\mathbf{in}(N')(\mathfrak{p}) = \mathbf{in}(N'(\mathfrak{p}))$. This monomial submodule of $F' \otimes_A k(\mathfrak{p})$ is generated by the monomials $m_1, \dots, m_s$. See Chapter I of [Pa] for a more general discussion of this principle.

Since k is an infinite field, the k-rational points in $\mathbb{A}^m_k$ are dense. Let P_1 and P_2 be two k-rational points of $\mathbb{A}^m_k$ and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the corresponding maximal ideals in A. Assume also that $a \notin \mathfrak{p}_2$ and that $N'(\mathfrak{p}_1)$ and $N'(\mathfrak{p}_2)$ have the same Hilbert function. This Hilbert function is shared by $N'(\mathfrak{p})$ for a dense set of $\mathfrak{p}$, in particular for those $\mathfrak{p}$ which do not contain a. Viewing $(F' \otimes_A A[t])/\tilde{N}'$ as a deformation of quotient modules over $\mathbf{Spec}\, A[t] = \mathbb{A}^m_k \times_k \mathbb{A}^1_k$, the fiber over $(P_1, 1)$ is $F/N'(\mathfrak{p}_1)$, while the fiber over $(P_2, 0)$ is $F/\mathbf{in}(N'(\mathfrak{p}_2))$. Choose an affine line T in $\mathbb{A}^m_k \times \mathbb{A}^1_k$ containing the points $(P_1, 1)$ and $(P_2, 0)$.

A family of graded modules over an open subset of T is flat if and only if every fiber has the same Hilbert function. This is the case for an open subset of T containing

$(P_1, 1)$ and $(P_2, 0)$. But, such a flat family over an open subset of T may always be completed to a flat family over all of T, with all of the fibers quotients of the original free module. (See Proposition II-25 of [EiHa], which gives the local construction of this family in the case $F = S$; the general case is the same.) Thus, there is a flat family over T whose fibers over k-rational points are quotient modules of F, such that the fiber over $t = 1$ is $F/N'(\mathfrak{p}_1)$, and whose fiber over $t = 0$ is $F/\mathbf{in}(N'(\mathfrak{p}_2))$.

From this general construction, we get the following proposition.

PROPOSITION 35. *Let N be a monomial submodule of F. Then there is a flat family over T of quotient modules of F such that the fiber over $t = 1$ is F/N and the fiber over $t = 0$ is $F/\phi(N)$.*

Proof. The affine space in the construction above is $\mathcal{L} \times U(F)$ where $U(F)$ is the unipotent subgroup of $GL(F)$ and $\mathcal{L}$ is the space of sets of linear forms from Definition 13. N' is the submodule of F' such that if $\mathfrak{p}$ is the maximal ideal corresponding to the point $(\mathbb{L}, \gamma)$ then $N'(\mathfrak{p}) = \gamma(\sigma_{\mathbb{L}}(N^{(p)}))$. Let $\mathfrak{p}_1$ be the maximal ideal of the pair $\{\mathbb{L}_0, 1\}$ where $\mathbb{L}_0$ is the collection of linear forms $\ell_{ijk} = x_i$ and 1 is the identity element of $U(F)$. Let $\mathfrak{p}_2$ be the maximal ideal corresponding to a general pair $\{\mathbb{L}, \gamma\}$. Then the construction above gives the desired deformation. □

THEOREM 36. *If $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-d_i)$ is a direct sum of line bundles over $\mathbb{P} = \mathbb{P}_k^{n-1}$, where k is an infinite field, and $\mathcal{E}/\mathcal{N}$ is a quotient sheaf of $\mathcal{E}$ with Hilbert polynomial $p(z)$ of degree d, then there is a sequence of $d + 2$ deformations, all defined over $\mathbb{A}_k^1$, taking $\mathcal{E}/\mathcal{N}$ to $\mathcal{E}/\mathcal{L}$, where $\mathcal{L}$ is a lexicographic subsheaf of $\mathcal{E}$. In particular $\mathbf{Quot}^{p(z)}(\mathcal{E})$ is linearly connected, and there is a sequence of no more than $2d + 4$ deformations over $\mathbb{A}_k^1$ taking any coherent sheaf over $\mathbb{P}$ with Hilbert polynomial $p(z)$ to any other.*

Proof. $\mathcal{E}$, $\mathcal{N}$, and $\mathcal{L}$ are the sheafifications of a free S-module F, a submodule N and a lexicographic submodule L. The first deformation takes F/N to $F/\mathbf{in}(N)$. Then, this theorem follows from Propositions 27 and 35 and part (8) of Proposition 19. □

See [Gr] for the definition and construction of quot schemes. Baptista de Campos showed in [BdC] that $\mathbf{Quot}^{p(z)}(\mathcal{O}_{\mathbb{P}}^r)$ is connected, but not that it is linearly connected. In a slightly different direction, Sositaisvili showed in [So] that the Hilbert scheme of a nongraded local artinian ring is linearly connected.

It would be interesting to know if it is possible to construct deformations between modules over a regular local ring R which have the same Hilbert function. The techniques of this paper may be used to prove that there is a sequence of deformations between any two finitely generated R-modules with the same Hilbert function so long as one can construct deformations from R/I to R/J where J is a monomial ideal. This is the case if R is a power series ring, or a convergent power series ring. It would be useful to have such deformations in general.

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