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INFINITE DIMENSIONAL HOMOGENEOUS REDUCTIVE SPACES AND FINITE INDEX CONDITIONAL EXPECTATIONS

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1. Introduction

In recent years several papers appeared dealing with the concept of infinite dimensional homogeneous reductive space modelled on C*-algebras ([CPR1], [CPR2], [LR], [MR], [M], [Ma], [ARS], [AS1], [AS2], [ACS]). A homogeneous reductive space (abbreviated: HRS) is a differentiable manifold $\mathcal{Q}$ and a smooth transitive action of a Banach-Lie group G (generally the group of invertibles or unitaries of a C*-algebra) on $\mathcal{Q}$, L: $G \times \mathcal{Q} \to \mathcal{Q}$ with the following properties.

Homogeneous structure. For each $\rho \in \mathcal{Q}$ the map

$$\pi_\rho\colon G \to \mathcal{Q} \quad \pi_\rho(g) = L_g\rho$$

is a principal bundle with structure group $I_\rho = \{g \in G\colon L_g\rho = \rho\}$ (called the isotropy group of ρ).

Reductive structure. For each $\rho \in \mathcal{Q}$ there exists a linear subspace H_ρ of the Lie algebra $\mathcal{G}$ of G such that

$$\mathcal{G} = H_\rho \oplus \mathcal{I}_\rho \quad (\mathcal{I}_\rho = \text{ Lie algebra of } I_\rho)$$

is invariant under the natural action of I_ρ and such that the distribution $\rho \mapsto H_\rho$ is smooth.

In the cases we are interested in, G is the group of invertible (resp. unitary) elements of a C*-algebra $\mathcal{A}$, and therefore $\mathcal{G}$ identifies with $\mathcal{A}$ (resp. the real Banach space of antihermitic elements of $\mathcal{A}$). Moreover the groups I_ρ turn out to be the groups of invertible (resp. unitary) elements of certain Banach (resp. C*-algebra) algebras $\mathcal{B}_\rho \subset \mathcal{A}$, and the supplement H_ρ is realized as the kernel of some conditional expectations E_ρ: $\mathcal{A} \to \mathcal{B}_\rho$. The latter seems to be a slightly stronger condition than the one required in the general case, although it is equivalent in some cases (see [ACS]).

Perhaps the most important example of an HRS, and certainly the best understood, is that of the Grassmanians of a C*-algebra, namely, the space of selfadjoint projections of the algebra. It is a fact (see [PR1] and [CPR1]) that each connected

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component of this space is an HRS under the action (by inner automorphisms) of the unitary group of the algebra. That is, if $\mathcal{M}$ is a C*-algebra, $U_{\mathcal{M}}$ its unitary group and p is a selfadjoint projection of $\mathcal{M}$,

$$L_u p = upu^*, \qquad u \in U_{\mathcal{M}}.$$

In this example the isotropy group I_p is the unitary group of $\{p\}' \cap \mathcal{M}$. The reductive structure is given by the conditional expectation

$$\mathcal{M} \ni m \mapsto pmp + (1-p)m(1-p) \in \{p\}' \cap \mathcal{M}.$$

In this paper we present a natural representation for general HRS $\mathcal{Q}$, whose reductive structure is determined by a faithful conditional expectation, into the Grassmanian of an extension of the given C*-algebra. Any such HRS, under the action of the group of invertibles $G_{\mathcal{A}}$ of a C*-algebra $\mathcal{A}$, is mapped diffeomorphically onto the orbit

$$\mathcal{Q} \longleftrightarrow \mathcal{S}_{\mathcal{A}}(p) = \{gpg^{-1} \colon g \in G_{\mathcal{A}}\}$$

for a suitable selfadjoint projection p lying on an extension of $\mathcal{A}$ (Prop. 3.5).

These orbits were considered in [AS] in the von Neumann algebra case. This paper generalizes those results to the C*-algebra context. The basic tool is Stinespring's theorem ([Sti, Th. 1] or [Ar]) in order to obtain a Jones-like basic construction for a given conditional expectation between C*-algebras. Let $E \colon \mathcal{A} \to \mathcal{B}$ be the expectation that determines the reductive structure of $\mathcal{Q}$. By Stinespring's theorem we obtain a Hilbert space K, a representation π of $\mathcal{A}$ in $L(K)$ and a selfadjoint projection $p \in L(K)$ with analogous properties as Jones' projection (see 2.4). The extension $\mathcal{M}$ mentioned above is the C*-algebra generated by $\mathcal{A}$ and p in $L(K)$.

We prove that the orbit $\mathcal{S}_{\mathcal{A}}(p)$ ($\simeq \mathcal{Q}$) is a submanifold of the full orbit $\mathcal{S}_{\mathcal{M}}(p) = \{mpm^{-1} \colon m \in G_{\mathcal{M}}\}$ of p in $\mathcal{M}$ if and only if the index of E is finite (Th. 6.6).

Here we use the notion of index (called weak index in the von Neumann algebra framework, see [BDH]) defined as follows: E has *finite index* if

$$0 < \lambda = \sup\{\, \epsilon \in \mathbb{R}_{\geq 0} \colon \|E(a)\| \geq \epsilon \|a\|, \qquad \text{for all } a \in \mathcal{A}^+\}.$$

In that case define $\operatorname{Ind}(E) = \lambda^{-1}$. Otherwise, $\operatorname{Ind}(E) = \infty$. The finiteness of this index is a condition weaker than Watatani's finite type condition (see [Wa]).

Suppose that $\mathcal{Q}$ admits an involution (see [MR]). Let $\mathcal{P}$ be the submanifold of "selfadjoint" elements of $\mathcal{Q}$, which is itself an HRS under the action of $U_{\mathcal{A}}$. The same results hold for the space $\mathcal{P}$, replacing similarity by unitary orbit of the projection p.

We introduce a natural Finsler structure on $\mathcal{P}$ (Section 4). It turns out that the basic representation becomes *isometric* when one considers the natural Finsler metric on the Grassmanians of $\mathcal{M}$ and what we call the P-Finsler metric on $\mathcal{P}$. This metric is a kind of a 2-norm in terms of the conditional expectation.

In the case of HRS with finite index conditional expectations (denoted FHRS), we construct an equivariant distribution of projections

$$\Xi_q \colon T\mathcal{U}_{\mathcal{M}}(p)_q \to T\mathcal{U}_{\mathcal{A}}(p)_q$$

from the tangent spaces of $\mathcal{U}_{\mathcal{M}}(p)$ onto the correspondent tangent spaces of the submanifold $\mathcal{U}_{\mathcal{A}}(p)$. This enables one to define a "spatial" linear connection in $\mathcal{P}$, projecting the reductive linear connection of the Grassmanians of $\mathcal{M}$ (Section 7).

We prove that the spatial and reductive connections of $\mathcal{P}$ do not coincide but they have the same geodesics. We obtain explicit formulas for the spatial and reductive linear connections (on $\mathcal{P}$ and $\mathcal{U}_{\mathcal{A}}(p)$) and we show that the spatial connection is the average of the classifying connection (see [MR]) and the reductive connection (Section 8).

2. The basic construction

Let H be a separable Hilbert space and denote by $L(H)$ the algebra of bounded operators on H. Let $\mathcal{B} \subset \mathcal{A} \subset L(H)$ be C*-algebras, $G_{\mathcal{A}}$ the group of invertible elements of $\mathcal{A}$ and E: $\mathcal{A} \to \mathcal{B}$ a conditional expectation. In particular (see for example [Str, 9.3]) E: $\mathcal{A} \to L(H)$ is completely positive. So we can apply the Stinespring Theorem:

2.1 PROPOSITION. *Let $\mathcal{B} \subset \mathcal{A} \subset L(H)$ be C*-algebras and E: $\mathcal{A} \to \mathcal{B}$ a conditional expectation. Then there exist a Hilbert space K, a $*$-representation π: $\mathcal{A} \to L(K)$ and a partial isometry $V \in L(H, K)$ such that, for all $a \in \mathcal{A}$,*

$$E(a) = V^*\pi(a)V.$$

Proof. Apply Stinespring's theorem to the completely positive map E.

2.2 *Remarks.* (2.2.1) The operator $V \in L(H, K)$ of 2.1 is an isometry with range $K_1 = V(H)$. So the projector P_{K_1} equals VV^*.

(2.2.2) The map $a \mapsto VaV^*$ is a faithful representation of $\mathcal{A}$ into $L(K_1)$, since V: $H \to K_1$ is unitary. We shall identify H with K_1, and then we can suppose that $H \subset K$.

(2.2.3) Given $a \in \mathcal{A}$, the projection $p = VV^*$ gives a matrix representation of $\pi(a)$ in the following form:

$$\pi(a) = \begin{pmatrix} E(a) & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(2.2.4) If $b \in \mathcal{B}$, then

$$\pi(b) = \begin{pmatrix} b & 0 \\ 0 & r(b) \end{pmatrix}$$

where r: $\mathcal{B} \to L(K_2)$ is a $*$-representation (not necessarily faithful).

Proof. (2.2.1) Clear since $E(1) = 1$ and therefore $1 = V^*V$.

(2.2.3) If $a \in \ker E$, by 2.1, $0 = V^*\pi(a)V$. Then $a_{11} = p\pi(a)p = 0$. On the other hand, $E(a) = E(E(a)) = V^*\pi(E(a))V$. Then using the identification 2.2.2,

$$\pi(E(a))_{11} = p\pi(E(a))p = VE(a)V^* = E(a).$$

(2.2.4) Suppose that $b \in \mathcal{B}_{sa}$ and let

$$\pi(b) = \begin{pmatrix} b & a^* \\ a & c \end{pmatrix}.$$

Then

$$\pi(b^2) = \pi(b)^2 = \begin{pmatrix} b^2 + a^*a & * \\ & * \end{pmatrix}.$$

So $b^2 = E(b^2) = b^2 + a^*a$. Therefore $a^*a = a = 0$ and $\pi(b)$ is diagonal. The general case follows using the real and imaginary parts of a general $b \in \mathcal{B}$. Finally if

$$\pi(b) = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix},$$

then it is clear that $r(b) = c$ defines a $*$-representation of $\mathcal{B}$.

Summarizing, we have proved the following theorem.

2.3 THEOREM. *Let $\mathcal{B} \subset \mathcal{A} \subset L(H)$ and $E\colon \mathcal{B} \to \mathcal{A}$ be a conditional expectation. Then there exist a Hilbert space $H \subset K$ and a $*$-representation $\pi\colon \mathcal{A} \to L(K)$ such that, if $K = H \perp K_2$, then:*

(1)

$$\pi(a) = \begin{pmatrix} E(a) & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

(2) *If $b \in \mathcal{B}$, then*

$$\pi(b) = \begin{pmatrix} b & 0 \\ 0 & r(b) \end{pmatrix}$$

where $r\colon \mathcal{B} \to L(K_2)$ is a $*$-representation.

If the conditional expectation E is faithful, it is easy to verify that the representation π of the Theorem 2.3 is also faithful.

2.4 We shall now prove a list of nice properties of the representation π of 2.3 when E is *faithful*. Using the same notations as in 2.2 (recall that $p = P_H \in L(K)$):

(2.4.1) If $a \in \mathcal{A}$ then $p\pi(a)p = \pi(E(a))p$.

(2.4.2) Denote by $\{p\}'$ the commutant of p in $L(K)$. Then $\{p\}' \cap \pi(\mathcal{A}) = \pi(\mathcal{B})$.

(2.4.3) For $a \in \mathcal{A}$, $\pi(a)p = 0 \Rightarrow a = 0$.

(2.4.4) Let $\mathcal{M}$ be the C*-algebra generated in $L(K)$ by $\pi(\mathcal{A}) \cup \{p\}$ (if $\mathcal{B}$ and $\mathcal{A}$ are W*-algebras, we shall consider $\mathcal{M} = (\pi(\mathcal{A}) \cup \{p\})''$). Then

$$\mathcal{M}_0 = \left\{ a_0 + \sum_{i=1}^{n} a_i p b_i \qquad : a_i, b_i \in \pi(\mathcal{A}) \right\}$$

is a norm dense $*$-subalgebra of $\mathcal{M}$ (σ-weak dense for W*-algebras).

(2.4.5) The map $\mathcal{B} \ni b \to \pi(b)p \in p\mathcal{M}p$ defines a $*$-isomorphism from $\mathcal{B}$ onto $p\mathcal{M}p \subset L(H) = L(p(K))$.

Proof. (2.4.1) Clear using the matricial picture of Th. 2.3.

(2.4.2) Suppose that $g \in \mathcal{A}$ and $\pi(g)$ diagonal (i.e., $\pi(g) \in \{p\}'$). Then $\pi(g) = \begin{pmatrix} E(g) & 0 \\ 0 & c \end{pmatrix}$. Let $h = g - E(g)$,

$$\pi(g - E(g)) = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \pi(h) \Rightarrow \pi(h^*h) = \begin{pmatrix} 0 & 0 \\ 0 & d^*d \end{pmatrix} \Rightarrow E(h^*h) = 0.$$

The assumption that E is faithful implies that $h = 0$ and $g \in \mathcal{B}$.

(2.4.3) Let $a \in \mathcal{A}$ such that $\pi(a)p = 0$. Then

$$\pi(a) = \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}, \quad \pi(a^*a) = \begin{pmatrix} 0 & 0 \\ 0 & b^*b + c^*c \end{pmatrix} \quad \text{and } E(a^*a) = 0 \Rightarrow a = 0$$

(2.4.4) Property 2.4.1 implies that $\mathcal{M}_0$ is a $*$-subalgebra of $\mathcal{M}$ and the density follows easily.

(2.4.5): $b \to \pi(b)p$ is a $*$-isomorphism by 2.4.1 and 2.4.3. It remains to prove the surjectivity. Let $m_0 = \pi(a_0) + \sum \pi(a_i)p\pi(b_i) \in \mathcal{M}_0$. Then by 2.4.1,

$$pm_0p = \pi(E(a_0))p + \sum \pi(E(a_i)E(b_i))p = \pi\left(E(a_0) + \sum E(a_i)E(b_i)\right)p$$

and therefore $pm_0p \in \pi(\mathcal{B})p$. Since $b \to \pi(b)p$ is $*$-morphism of C*-algebras, then

$$\pi(\mathcal{B})p \supset \overline{(p\mathcal{M}_0p)}^{\|\ \|} = p\mathcal{M}p.$$

For the W*-algebra case see [AS2].

3. Representation of homogeneous reductive spaces

3.1. Let $\mathcal{A} \subset L(H)$ be a C*-algebra with invertible group $G_{\mathcal{A}}$ and unitary group $U_{\mathcal{A}}$. Let $\mathcal{Q}$ be a homogeneous reductive space (HRS) with involution under the action of $G_{\mathcal{A}}$, named L_g for $g \in G_{\mathcal{A}}$. Fix $\rho \in \mathcal{Q}$ selfadjoint such that the map

$$\pi_\rho : G_{\mathcal{A}} \to \mathcal{Q} \quad \text{given by} \quad \pi_\rho(g) = L_g\rho, \quad g \in G_{\mathcal{A}}$$

is surjective (i.e., the action is transitive). Let $\mathcal{P} \subset \mathcal{Q}$ by the unitary orbit of ρ; that is $\mathcal{P} = \{\pi_\rho(u)\colon\ u \in U_{\mathcal{A}}\}$. So $\mathcal{P}$ is also a homogeneous reductive space. Actually $\mathcal{P}$ consists of the selfadjoint points of $\mathcal{Q}$ (see [MR]). Suppose that the isotropy group of ρ is the group $G_{\mathcal{B}}$ of invertibles of a C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and that the reductive structure is given by a conditional expectation $E_\rho\colon\ \mathcal{A} \to \mathcal{B}$. This means that the horizontal space at the identity of $G_{\mathcal{A}}$ is obtained by taking the kernel of E_ρ.

Using Theorems 2.3 and 2.4 we obtain a Hilbert space $K \supset H$, a $*$-representaion π of $\mathcal{A}$ on K, the projector $p = P_H$ and the C*-algebra $\mathcal{M} = C^*(\pi(\mathcal{A}), p)$. We want to represent the space $\mathcal{Q}$ into the well studied HRS $\mathcal{Q}_{\mathcal{M}}$ of idempotents in $\mathcal{M}$. Let $K_2 = K \ominus H = K \cap H^\perp$. Consider the 2×2 matrix representation of $L(K)$ given by $p = P_H$ and $I - p = P_{K_2}$. The assignment

$$\rho \mapsto p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

can be extended to $\mathcal{Q}$ in the following way: given $g \in G_{\mathcal{A}}$,

$$\begin{array}{lccccc} & L_g\rho & \mapsto & \wp(L_g\rho) & = \pi(g)p\pi(g)^{-1} & \\ (3.2) & \cap & \wp & \cap & & \\ & \mathcal{Q} & \hookrightarrow & \mathcal{S}_{\mathcal{M}}(p) & \subset & \mathcal{Q}_{\mathcal{M}} \end{array}$$

where $\mathcal{S}_{\mathcal{M}}(p) = \{hph^{-1}\colon\ h \in G_{\mathcal{M}}\}$ denotes the $\mathcal{M}$-similarity orbit of p (which coincides with the connected component of p in $\mathcal{Q}_{\mathcal{M}}$; see [CPR2]). That it is well-defined follows from the fact that $g \in G_{\mathcal{B}} \Leftrightarrow L_g\rho = \rho$, and then, by 2.3, $\pi(g)$ is a diagonal matrix which commutes with p. Note that since $p \in \mathcal{P}_{\mathcal{M}}$ (the space of orthogonal projectors of $\mathcal{M}$), then $\wp(\mathcal{P}) \subset \mathcal{U}_{\mathcal{M}}(p) = \{upu^*\colon\ u \in U_{\mathcal{M}}\} \subset \mathcal{P}_{\mathcal{M}}$.

3.3 *Definition.* Given an HRS $\mathcal{Q}$ as in 3.1 , we denote by the *basic representation* of $\mathcal{Q}$ the map $\wp$ of 3.2. The following diagram is commutative:

$$(3.4) \qquad \begin{array}{ccccc} & G_{\mathcal{A}} & \xrightarrow{\pi} & G_{\mathcal{M}} & \\ \pi_\rho & \downarrow & & \downarrow & \pi_p\ . \\ & \mathcal{Q} & \xrightarrow[\wp]{} & \mathcal{Q}_{\mathcal{M}} & \end{array}$$

3.5 PROPOSITION. *The basic representation $\wp$ of $\mathcal{Q}$ is a C^∞ map. It is also one to one if the conditional expectation E_ρ is faithful.*

Proof. Recall from 2.4.2. that if $g \in \mathcal{A}$ and $\pi(g)$ is diagonal, then $g \in \mathcal{B}$. If

$$\pi(g_1)p\pi(g_1)^{-1} = \pi(g_2)p\pi(g_2)^{-1}$$

then $\pi(g_1^{-1}g_2)$ commutes with p and therefore $g_1^{-1}g_2 \in \mathcal{B}$. So $\wp$ is one to one. Since π_ρ has C^∞ local cross sections ($\mathcal{Q}$ is an HRS), it is clear from the diagram (3.4) that $\wp$ is a C^∞ map.

3.6 *Remark.* The image of $\mathcal{Q}$ by the basic representation is the orbit of p by the action of $G_{\mathcal{A}}$. Let

$$\mathcal{S}_{\mathcal{A}}(p) = \{\, \pi(g)p\pi(g)^{-1}\colon\ g \in G_{\mathcal{A}}\} = \wp(\mathcal{Q}).$$

Analogously, $\wp(\mathcal{P}) = \mathcal{U}_{\mathcal{A}}(p) = \{\, \pi(u)p\pi(u)^{-1}\colon\ u \in U_{\mathcal{A}}\}$.

4. The P-Finsler structure on $\mathcal{P}$

From now on we suppose that E_ρ is faithful. Denote by $T\mathcal{Q}_\rho$ the tangent space of $\mathcal{Q}$ at ρ and let $K_\rho\colon\ T\mathcal{Q}_\rho \to H_\rho = \ker E_\rho$ be the natural isomorphism $K_\rho = (\,(T\pi_\rho)_1|_{H_\rho})^{-1}$. If $\rho \in \mathcal{P}$, denote also by K_ρ the analogous linear isomorphism for the unitary case,

$$K_\rho\colon\ T\mathcal{P}_\rho \to \tilde{H}_\rho = \{h \in \ker E_\rho\colon\ h^* = -h\,\}, \qquad K_\rho = (T(\pi_\rho)_1|_{\tilde{H}_\rho})^{-1}.$$

4.1 Given $\rho \in \mathcal{P}$ and $X \in T\mathcal{P}_\rho$ we define the P-norm of X as

$$\|X\|_P = \|E_\rho(K_\rho(X)^*K_\rho(X))\|^{1/2}.$$

In matrix form,

$$\pi(K_\rho(X)) = \begin{pmatrix} 0 & -a_{21}^* \\ a_{21} & a_{22} \end{pmatrix}.$$

So $E_\rho(K_\rho(X)^*K_\rho(X)) = a_{21}^*a_{21}$, and

$$\|X\|_P = \left\|\begin{pmatrix} 0 & a_{21}^* \\ a_{21} & 0 \end{pmatrix}\right\| = \|a_{21}^*a_{21}\|^{1/2} = \|a_{21}\|. \tag{4.2}$$

4.3 *Remark.* With the P-Finsler structure defined above, the group $U_{\mathcal{A}}$ acts isometrically on $\mathcal{P}$. That is, if $X \in T\mathcal{P}_\rho$ and $\varsigma = L_u\rho$ for a fixed $u \in U_{\mathcal{A}}$, then

$$\|T(L_u)_\rho(X)\|_P = \|X\|_P.$$

Indeed, note that $K_\varsigma(T(L_u)_\rho(X)) = Ad(u)\circ K_\rho(X)$ and $E_\varsigma = Ad(u)\circ E_\rho \circ Ad(u^*)$ (see [MR]). Therefore

$$\begin{aligned} \|T(L_u)_\rho(X)\|_P &= \|E_\varsigma(K_\varsigma(T(L_u)_\rho(X))^*K_\varsigma(T(L_u)_\rho(X)))\| \\ &= \left\|uE_\rho(K_\rho(X)^*K_\rho(X))u^*\right\| \\ &= \|X\|_P. \end{aligned}$$

Let $(T\wp)_\rho\colon\ T\mathcal{Q}_\rho \to T\mathcal{Q}(K)_p$ be the derivative of the C^∞ map $\wp\colon\ \mathcal{Q} \to \mathcal{Q}(K)$ at ρ. Recall that $T\mathcal{Q}(K)_p$ consists of the "antidiagonal" matrices $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. It is easy to verify that for $X \in T\mathcal{Q}_\rho$ such that $\pi(K_\rho(X)) = \left(\begin{smallmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$,

$$(T\wp)_\rho(X) = \pi(K_\rho(X))p - p\pi(K_\rho(X)) \quad \text{and then} \tag{4.4}$$

$$\|(T\wp)_\rho(X)\| = \left\|\begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix}\right\|.$$

If moreover $X \in T\mathcal{P}_\rho$, then, as in 4.2, $\pi(K_\rho(X)) = \begin{pmatrix} 0 & -a_{21}^* \\ a_{21} & a_{22} \end{pmatrix}$. Therefore using 4.4,

$$\|(T\wp)_\rho(X)\| = \left\| \begin{pmatrix} 0 & a_{21}^* \\ a_{21} & 0 \end{pmatrix} \right\| = \|X\|_P. \tag{4.5}$$

In other words $(T\wp)_\rho$ is an isometry in the selfadjoint case. So we have the following:

4.6 Proposition. *Let $\mathcal{P}$ be a selfadjoint HRS under the action of the unitary group $U_\mathcal{A}$ of a C^*-algebra $\mathcal{A}$ with reductive structure defined at $\rho \in \mathcal{P}$ by a faithful conditional expectation E_ρ. Then the basic representation $\wp$: $\mathcal{P} \to \mathcal{P}(K)$ of 3.3 is an isometry from the P-Finsler metric on $\mathcal{P}$ to the natural Finsler metric of $\mathcal{P}(K)$ (see [CPR1]).*

There is another natural Finsler metric on $\mathcal{P}$. It is given by

$$\|X\| = \|K_\rho(X)\|_\mathcal{A}, \qquad X \in T\mathcal{P}_\rho. \tag{4.7}$$

4.8 Proposition. *The norms $\| \ \|_P$ and $\| \ \|$ of 4.7 are equivalent on $T\mathcal{P}_\rho$ if and only if there exists $0 < \lambda \in \mathbb{R}$ such that $\|E_\rho(a)\| \geq \lambda\|a\|$ for all positive $a \in \mathcal{A}^+$.*

Proof. Since E_ρ is a contraction it is clear that for all $X \in T\mathcal{P}_\rho$, $\|X\|_P \leq \|K_\rho(X)\|_\mathcal{A}$. Put $x = K_\rho(X)$. Recall that in this case $x^* = -x$.

If there exists $0 < \lambda \in \mathbb{R}$ such that $\|E_\rho(a)\| \geq \lambda\|a\|$ for all $0 \leq a \in \mathcal{A}$, then

$$\|X\|_P = \|E_\rho(x^*x)\|^{1/2} \geq \lambda^{1/2}\|x^*x\|^{1/2} = \lambda^{1/2}\|x\|$$

and the two norms are equivalent.

Conversely, suppose that for all $n \in \mathbb{N}$, there exists $0 \leq a_n \in \mathcal{A}$ such that $\|a_n\| = 1$ and $\|E_\rho(a_n^2)\| \leq 1/n$. Let $b_n = E_\rho(a_n)$. Therefore, since $E_\rho(a_n^2) \geq E_\rho(a_n)^2$, also $\|E_\rho(a_n)\| = \|b_n\| \leq (1/n)^{1/2}$. Let $h_n = (I - E_\rho)(a_n) = a_n - b_n \in H_\rho = \ker E_\rho$. Then $h_n^* = h_n$. For some n_0 we have $n \geq n_0 \Rightarrow \|h_n\| \geq 1/2$. Since $a_n^2 = b_n^2 + h_n^2 + h_n b_n + b_n h_n$,

$$\|E_\rho(h_n^2)\| = \|E_\rho(a_n^2) - b_n^2\| \underset{n\to\infty}{\longrightarrow} 0.$$

Since $K_\rho(T\mathcal{P}_\rho) = \{h \in \ker E_\rho \colon h^* = -h\}$, there exist $X_n \in T\mathcal{P}_\rho$ such that $K_\rho(X_n) = ih_n$. Then

$$\|X_n\|_P = \|E(h_n^2)\|^{1/2} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{but} \quad \|ih_n\| \geq 1/2$$

for $n \geq n_0$. So the two norms are not equivalent. This completes the proof.

4.9 *Remark.* Let $b \in K_\rho(T\mathcal{P}_\rho) = \{h \in \ker E_\rho \colon h^* = -h\}$. Using 2.3 we have

$$\pi(b) = \begin{pmatrix} 0 & -a^* \\ a & c \end{pmatrix} \tag{4.10}$$

where $c = -c^* \in L(K_2)$ and $a \in L(H, K_2)$. It is easy to see that another condition equivalent to those of 4.8 is the following: with the notation of 4.10, the linear operator $a \mapsto c$ is bounded; i.e., there exists $0 \le \lambda \in \mathbb{R}$ such that $\|c\| \le \lambda \|a\|$ for all possible $a \in L(H, K_2)$. Recall that if E_ρ is faithful, this map is well defined as was shown in the proof of 3.5.

5. Conditional expectations with finite Jones index

There are several definitions of the index of a conditional expectation $E \colon \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ for $\mathcal{B} \subset \mathcal{A}$, an inclusion of general C*-algebras. See, for example, [J], [PP], [K], [L], [BDH], [Wa] and, in particular, [AS2] where the concept is applied in this context.

5.1 *Definition.* Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of C*-algebras and $E \colon \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ a conditional expectation. Then E has *finite index* if

$$0 < \lambda = \sup\{\,\epsilon \in \mathbb{R}_{\ge 0} \colon \ \|E(a)\| \ge \epsilon \|a\|, \text{for all} \quad a \in \mathcal{A}^+\}.$$

In that case define $\operatorname{Ind}(E) = \lambda^{-1}$. Otherwise, $\operatorname{Ind}(E) = \infty$.

This definition agrees with the general definition when $\mathcal{A}$ and $\mathcal{B}$ are factors (see [L]). For general von Neumann algebras $\operatorname{Ind}(E)$ is known as the "weak index" of E (see [BDH]). For general C*-algebras our definition of finite index is weaker than Watatani's condition (see [Wa]) of conditional expectations of "finite type". Actually Watatani's index is not necessarily a scalar but an element of the center of $\mathcal{A}$. Let

$$\mathcal{M}_1 = \{\pi(a)p\pi(b) \colon \ a,\ b \in \mathcal{A}\} \subset \mathcal{M}_0$$

where π, p, $\mathcal{M}$and $\mathcal{M}_0$ are related to E as in 2.4. Then E is of *finite type* (see [Wa]) iff

(i) $\operatorname{Ind} E < \infty$ (5.1) and
(ii) $\mathcal{M}_1$ is dense in $\mathcal{M}$, that is, 1 lies in the norm closure of $\mathcal{M}_1$.

However, for applications concerning HRS, the weaker Definition 5.1 is appropriate. In what follows we shall use just this condition.

5.2. We are interested in HRS's whose reductive structure is given by finite index conditional expectations. Let us call these spaces *finite homogeneous reductive spaces* (denoted FHRS). In view of 4.8, selfadjoint HRS's are finite iff the P and the usual Finsler metrics are equivalent. In [AS2], an FHRS associated to every finite index conditional expectation between von Neumann algebras is constructed. A similar construction will be done for general finite index expectations between C*-algebras. Also, the following statement proves that a large class of HRS's are finite.

5.3 PROPOSITION. *Let G be a compact Hausdorff group, $\mathcal{A}$ a C*-algebra and $\beta: G \to U_{\mathcal{A}}$ a* **norm continuous** *unitary representation of G on $\mathcal{A}$. Consider $\mathcal{A}^G \subset \mathcal{A}$ the subalgebra of elements of $\mathcal{A}$ invariant for the action of G, and $E_G: \mathcal{A} \to \mathcal{A}^G$ the conditional expectation defined by*

$$E_G(a) = \int_G \beta(u)a\beta(u)^*\, dm(u), \qquad a \in \mathcal{A},$$

where $dm(u)$ means integraton with the invariant Haar measure m of G.
Then $\operatorname{Ind}(E_G) < \infty$.

Proof. Let $a \in \mathcal{A}^+$. Take $\epsilon, \delta > 0$ and an open set $V \subset G$ such that $1 \in V$ and

(1) $m(V) = \delta$, and
(2) if $u \in V$ then $\|1 - \beta(u^*)\| < \epsilon$.

A convenient $*$-representation of $\mathcal{A}$ on a Hilbert space H (via GNS) allows one to choose $x \in H$ such that $\|x\| = 1$ and $\|a\| = \langle ax, x\rangle$. Then

$$\begin{aligned}
\|E_G(a)\| &\geq \left\langle \left(\int_G \beta(u)a\beta(u)^*\, dm(u) \right) x, x \right\rangle \\
&\geq \int_V \langle a\beta(u)^*x, \beta(u)^*x\rangle\, dm(u) \\
&\geq \int_V \|a\|\, dm(u) - 3\epsilon \int_V \|a\|\, dm(u) \\
&= (1 - 3\epsilon)\delta\|a\|
\end{aligned}$$

Since $(1 - 3\epsilon)\delta$ is independent of a, this proves that $\operatorname{Ind}(E_G) \leq ((1 - 3\epsilon)\delta)^{-1} < \infty$ and the prooof is complete.

5.4 PROPOSITION. *With the notations of* 2.4, $\operatorname{Ind}(E) < \infty \Leftrightarrow \pi(\mathcal{A})p$ *is norm closed in $\mathcal{M}$. In that case the map $\kappa: \pi(\mathcal{A})p \to \pi(\mathcal{A})$ given by $\kappa(\pi(a)p) = \pi(a)$ is bounded with norm* $(\operatorname{Ind}(E))^{1/2}$.

Proof. Let $a \in \mathcal{A}$, $a' = \pi(a)$ and $0 < \lambda = (\operatorname{Ind}(E))^{-1}$. Then

$$\|a'p\|^2 = \|pa'^*a'p\| = \|E(a^*a)\| \geq \lambda\|a^*a\| = \lambda\|a'\|^2. \tag{5.5}$$

Therefore the map $a' \mapsto a'p$ is bounded from below and $\pi(\mathcal{A})p$ is norm closed. If $\operatorname{Ind}(E) = \infty$, let $a_n \in \mathcal{A}^+$ such that $\|a_n\| = 1$ and $\|E(a_n)\| \leq 1/n$. As in 5.5, κ is unbounded. On the other hand if $\pi(\mathcal{A})p$ were norm closed κ should be bounded by the open mapping theorem. Using 5.5 it is easy to see that $\|\kappa\| = \lambda^{-1/2}$.

In what follows, as in 2.4 we consider an inclusion of C*-algebras $\mathcal{B} \subset \mathcal{A}$, a conditional expectation $E\colon \mathcal{A} \to \mathcal{B}$ and the corresponding K, $\pi\colon \mathcal{A} \to L(K)$, $p = P_H \in \mathcal{P}(K)$, $\mathcal{M}$ and $\mathcal{M}_0$ (see 2.4). The following results are generalizations to C*-algebras of identical results appearing in [AS2, Section 2.].

From now on let us write $x' = \pi(x)$ for any $x \in \mathcal{A}$.

5.6 PROPOSITION. *$\pi(\mathcal{A})p$ is closed in norm iff $\mathcal{M}p = \pi(\mathcal{A})p$.*

Proof. Clearly if $\mathcal{M}p = \pi(\mathcal{A})p$, then $\pi(\mathcal{A})p$ is norm closed. Conversely suppose that $\pi(\mathcal{A})p$ is closed. Then, as in the preceeding result, $\|a'\| \leq \lambda^{-1/2}\|a'p\|$ for all $a \in \mathcal{A}$. By 2.4.4,

$$\mathcal{M}_0 = \left\{ a_0' + \sum_{1\leq j\leq n} a_{1,j}'\, p\, a_{2,j}' \colon n \in \mathbb{N}, a_{i,j} \in \mathcal{A} \right\}$$

is norm dense in $\mathcal{M}$. Let $m \in \mathcal{M}$ with $\|m\| \leq 1$ and $x_k = a_{0,k}' + \sum_{1\leq j\leq n} a_{1,j,k}'\, p\, a_{2,j,k}'$ be a sequence in $\mathcal{M}_0$ converging to m. Clearly, $x_k p$ converges to mp and

$$x_k p = a_0' p + \sum_{1\leq j\leq n} a_{1,j,k}'\, E(a_{2,j,k})' p = c_k' p \in \pi(\mathcal{A})p.$$

Therefore the sequence $a_0' + \sum_{1\leq j\leq n} a_{1,j,k}'\, E(a_{2,j,k})' = c_k'$ is a Cauchy sequence, since $\|c_k' - c_{k+i}'\| \leq \lambda^{-1/2}\|x_k - x_{k+i}\|$. Let $a' \in \pi(\mathcal{A})$ be the limit of the sequence c_k'. Then $mp = a'p$ and the proof is complete.

6. The basic representation for FHRS

Let $\mathcal{Q}$ be an HRS with involution under the transitive action of $G_{\mathcal{A}}$ and let $\mathcal{P}$ be its selfadjoint part. It is a fact (see [MR]) that $\mathcal{P}$ is an HRS under the action of $U_{\mathcal{A}}$. Fix $\rho \in \mathcal{P}$ and suppose that the reductive structure of $\mathcal{Q}$ is determined by the conditional expectation $E_\rho\colon \mathcal{A} \to \mathcal{B}$. Consider the basic representation $\wp\colon \mathcal{Q} \to \mathcal{Q}_{\mathcal{M}}$ as in (3.3). Recall from (3.6) that

$$\mathcal{S}_{\mathcal{A}}(p) = \{\, \pi(g)p\pi(g)^{-1}\colon\ g \in G_{\mathcal{A}}\} = \wp(\mathcal{Q}),$$

and

$$\wp(\mathcal{P}) = \mathcal{U}_{\mathcal{A}}(p) = \{\, \pi(u)p\pi(u)^{-1}\colon\ u \in U_{\mathcal{A}}\}.$$

In this section we will study the topological and differentiable structures of these orbits. Using a result by Herrero ([AFHV], Th. 16.3), since $\mathcal{B}$ is complemented in $\mathcal{A}$

(using $\ker E_\rho$ as a supplement), the quotient map $G_{\mathcal{A}} \to G_{\mathcal{A}}/G_{\mathcal{B}} \simeq \mathcal{S}_{\mathcal{A}}(p)$ admits continuous local cross sections. The following statements show that when the index is finite the quotient topology agrees with the norm topology induced on $\mathcal{S}_{\mathcal{A}}(p)$ by $\mathcal{M}$.

6.1 PROPOSITION. *If the index* $\mathrm{Ind}(E_\rho) < \infty$ *then the map* $\pi_p\colon G_{\mathcal{A}} \to S_{\mathcal{A}}(p)$ *has continuous local cross sections.*

Proof. Let $g \in G_{\mathcal{A}}$ and $g'pg'^{-1} \in S_{\mathcal{A}}(p)$ such that $\|g'pg'^{-1} - p\| < \mathrm{Ind}(E)^{-1/2}$. Then $\|g'pg'^{-1}p - p\| = \|g'E(g^{-1})'p - p\| < \mathrm{Ind}(E)^{-1/2}$ and since the norm of the map $a'p \mapsto a', a \in \mathcal{A}$ is $\mathrm{Ind}(E)^{1/2}$, we have $\|gE(g^{-1}) - 1\| < 1$ and $gE(g^{-1}) \in G_{\mathcal{A}}$.

Define $\phi(g'pg'^{-1}) = gE(g^{-1})$. Clearly ϕ is well defined and continuous at p (and therefore continuous at every point of its domain by a standard argument of homogeneous spaces). It is a cross section for π_p:

$$\phi(g'pg'^{-1})p\phi(g'pg'^{-1})^{-1} = g'pg'^{-1}.$$

6.2 *Remark.* With the same techniques it can be shown that $\mathrm{Ind}(E) < \infty$ implies $\mathcal{U}_{\mathcal{A}}(p)$ has unitary local cross sections. Indeed

$$\phi_U(u'pu'^*) = \phi(u'pu'^*)\ |\phi(u'pu'^*)|^{-1} \in U_{\mathcal{A}}$$

is a unitary local cross section for $\pi_p\colon U_{\mathcal{A}} \to \mathcal{U}_{\mathcal{A}}(p)$.

The existence of continuous local cross sections guarantees that $\mathcal{S}_{\mathcal{A}}(p)$ and $\mathcal{U}_{\mathcal{A}}(p)$ are analytic (resp. C^∞) homogeneous spaces in the general case ($\mathrm{Ind}(E)$ not necessarily finite). However these spaces may not be submanifolds of $\mathcal{M}$. We have shown that the finite index condition implies that in these orbits the quotient and the norm topology coincide. The next results show that this condition also implies that $\mathcal{S}_{\mathcal{A}}(p)$ and $\mathcal{U}_{\mathcal{A}}(p)$ are submanifolds of $\mathcal{M}$. Moreover, $\mathrm{Ind}(E) < \infty$ is necessary for these orbits to be submanifolds of $\mathcal{M}$:

6.3 PROPOSITION. *If the index of* $E_\rho\colon \mathcal{A} \to \mathcal{B}$ *is finite, then* $\mathcal{S}_{\mathcal{A}}(p)$ *is an analytic submanifold of* $\mathcal{M}$.

Proof. If the index is finite, then $\pi(\mathcal{A})p$ is closed in norm. By 5.6, this implies that $\pi(\mathcal{A})p = \mathcal{M}p$. Recall that the cross section defined in 6.1 is given by

$$\mathcal{S}_{\mathcal{A}}(p) \ni g'pg'^{-1} \mapsto g'pg'^{-1}p = g'E_\rho(g^{-1})'p \mapsto gE(g^{-1}).$$

Since $\pi(\mathcal{A})p = \mathcal{M}p$, this map can be extended to a neighborhood of p in $\mathcal{M}$:

$$m \mapsto mp \in \mathcal{M}p = \pi(\mathcal{A})p \quad \text{followed by the linear isomorphism} \quad \pi(\mathcal{A})p \to \mathcal{A}.$$

Clearly, this extension is an analytic map. Therefore the proposition follows (see [Ra] and [AS1]).

6.4 *Remark.* With the same hypothesis, $\mathcal{U}_{\mathcal{A}}(p)$ is a C^∞ submanifold of $\mathcal{M}$. In fact, the unitary cross section defined in 6.2, can be extended in an analogous way.

6.5 PROPOSITION. *If the unitary orbit $\mathcal{U}_{\mathcal{A}}(p)$ is a C^∞ submanifold of $\mathcal{M}$ and a C^∞ homogeneous space under the action of $U_{\mathcal{A}}$, then the index of E_ρ is finite.*

Proof. Note that $\mathcal{U}_{\mathcal{A}}(p)$ is also a C^∞ homogeneous space under the action of $U_{\pi(\mathcal{A})} = \pi(U_{\mathcal{A}})$. Therefore there exists a C^∞ map θ,

$$\theta\colon W \subset \mathcal{M} \to \mathcal{M},$$

from an open neighborhood W of p in $\mathcal{M}$, such that θ restricted to $W \cap \mathcal{U}_{\mathcal{A}}(p)$ is a cross section of π_p. Let us consider the action of $U_{\pi(\mathcal{A})}$ on p by left multiplication. We claim that when such a map θ exists, then the orbit $U_{\pi(\mathcal{A})}p = \{u'p \in \mathcal{M}\colon u' \in U_{\pi(\mathcal{A})}\}$ is also a homogeneous space under this new action of $U_{\pi(\mathcal{A})}$. In particular, this implies that its tangent space at p, $\{a'p\colon a \in \mathcal{A}, a^* = -a\}$, is closed and complemented in $\mathcal{M}$. This implies that $\pi(\mathcal{A}_s)p$ is closed in $\mathcal{M}$ ($\mathcal{A}_s$= selfadjoint elements of $\mathcal{A}$), and therefore, as in 5.4, the index is finite.

So it remains to prove that the orbit $U_{\pi(\mathcal{A})}p$ is a homogeneous space. Let Δ be the isomorphism from $p\mathcal{M}p$ to $\pi(\mathcal{B})$, and let $\sigma\colon V \subset \mathcal{M} \to \mathcal{M}$,

$$\sigma(x) = \theta(xx^*)\Delta(p\theta(xx^*)^*xp).$$

Clearly, σ is a C^∞ map defined on the open subset V consisting of all x in $\mathcal{M}$ such that xx^* lies in the domain of θ. Observe that if $x = u'p$ then $xx^* = u'pu'^*$, so that $\sigma(u'p) = \theta(u'pu'^*)\Delta(p\theta(u'pu'^*)^*u'p)$. On the other hand, since θ restricted to the unitary orbit $\mathcal{U}_{\mathcal{A}}(p)$ of p is a cross section for π_p, $u'pu'^* = \theta(u'pu'^*)p\theta(u'pu'^*)^*$, which implies that $\theta(u'pu'^*)^*u' \in \pi(\mathcal{B})$. Therefore $\Delta(p\theta(u'pu'^*)^*u'p) = \theta(u'pu'^*)^*u'$, and $\sigma(u'p) = u'$. In other words, we have found a C^∞ map defined around p, whose restriction to $U_{\pi(\mathcal{A})}p$ is a cross section for the action of $U_{\pi(\mathcal{A})}$. Then $U_{\pi(\mathcal{A})}p$ is a homogeneous space and the proof is complete.

We can summarize the information in the following theorem.

6.6 THEOREM. (6.6.1) *Let $\mathcal{Q}$ be an HRS with involution under the transitive action of $G_{\mathcal{A}}$ and let $\mathcal{P}$ be its selfadjoint part. Fix $\rho \in \mathcal{P}$ and suppose that the reductive structure of $\mathcal{Q}$ is determined by the conditional expectation $E_\rho\colon \mathcal{A} \to \mathcal{B}$. Consider the basic representation $\wp\colon \mathcal{Q} \to \mathcal{Q}_{\mathcal{M}}$ as in* (3.3). *Then the following conditions are equivalent*:

(*a*) *$\mathcal{Q}$ is a finite homogeneous space (that is* $\mathrm{Ind}(E_\rho) < \infty$).

(*b*) *$\mathcal{S}_{\mathcal{A}}(p) = \wp(\mathcal{Q})$ is an analytic Banach homogeneous space under the action of $G_{\mathcal{A}}$ and an analytic submanifold of $\mathcal{M}$.*

(*c*) $\mathcal{U}_{\mathcal{A}}(p) = \wp(\mathcal{P})$ *is a* C^∞ *Banach homogeneous space under the action of* $U_{\mathcal{A}}$ *and a* C^∞ *submanifold of* $\mathcal{M}$.
(*d*) *The norms* $\| \; \|_P$ *of* 4.1 *and* $\| \; \|$ *of* 4.7 *are equivalent on the tangent space* $T\mathcal{P}_\rho$.

(6.6.2) *Let* $\mathcal{B} \subset \mathcal{A}$ *be* C^**-algebras and let* $E\colon \mathcal{A} \to \mathcal{B}$ *be a faithful conditional expectation. Consider* K, π, p *and* $\mathcal{M}$ *as in* 2.4. *Then, the following statements are equivalent*:

(*a*) *The index of* $E\colon \mathcal{A} \to \mathcal{B}$ *is finite.*
(*b*) $\pi(\mathcal{A})p$ *is norm closed in* $\mathcal{M}$.
(*c*) $\pi(\mathcal{A})p = \mathcal{M}p$
(*d*) $\mathcal{S}_{\mathcal{A}}(p)$ *is an analytic Banach homogeneous space under the action of* $G_{\mathcal{A}}$ *and an analytic submanifold of* $\mathcal{M}$.
(*e*) $\mathcal{U}_{\mathcal{A}}(p)$ *is a* C^∞ *Banach homogeneous space under the action of* $\mathcal{U}_{\mathcal{A}}$ *and a* C^∞ *submanifold of* $\mathcal{M}$.

7. Projecting $T\mathcal{S}_{\mathcal{M}}(p)$ onto $T\mathcal{S}_{\mathcal{A}}(p)$

Let $\mathcal{Q}$ be an HRS with involution under the transitive action of $G_{\mathcal{A}}$ and let $\mathcal{P}$ be its selfadjoint part. Fix $\rho \in \mathcal{P}$ and suppose that the reductive structure of $\mathcal{Q}$ is determined by the conditional expectation $E = E_\rho\colon \mathcal{A} \to \mathcal{B}$. Consider the basic representation $\wp\colon \mathcal{Q} \to \mathcal{Q}_{\mathcal{M}}$ as in (3.3).

In what follows, we shall *identify* $\mathcal{A}$ with $\pi(\mathcal{A}) \subset \mathcal{M}$ and also suppose that $\operatorname{Ind}(E) < \infty$. Using 6.6 and the basic facts about HRS (see [MR] or [AS2]), it is easy to obtain the following result.

7.1 *Remark.* With the hypothesis considered above there is a natural HRS structure with involution on $\mathcal{S}_{\mathcal{A}}(p)$, determined by the expectation E. Then $\wp$ turns out to be a HRS diffeomorphism

$$\wp\colon \mathcal{Q} \to \mathcal{S}_{\mathcal{A}}(p)$$

which preserves the involution. Therefore it is also a diffeomorphism between the selfadjoint parts $\mathcal{P}$ and $\mathcal{U}_{\mathcal{A}}(p)$.

In this case it is possible to define the bounded linear map $R\colon \mathcal{M} \to \mathcal{A}$ given by

$$\mathcal{M} \ni m \mapsto R(m) = a \quad \text{where} \quad a \in \mathcal{A} \tag{7.2}$$
$$\text{is the unique element such that} \quad mp = ap.$$

In other words, using $\mathcal{M}p = \mathcal{A}p$ one can define $R(m) = \kappa(mp) \in \mathcal{A}$. It is straightforward to verify that R has the following properties:

7.3 *Remark.* (7.3.1) $\|R\| = \mathrm{Ind}(E)^{1/2}$.

(7.3.2) R restricted to $\mathcal{S}_{\mathcal{A}}(p)$ near p is the local cross section ϕ of $\pi_p\colon G_{\mathcal{A}} \to \mathcal{S}_{\mathcal{A}}(p)$ constructed in 6.1.

(7.3.3) If $X \in T\mathcal{S}_{\mathcal{A}}(p)_p$ then $R(X) \in H_p$. Therefore $R|_{T\mathcal{S}_{\mathcal{A}}(p)_p} = K_p$ in the sense of 4.

(7.3.4) Let $R_U(m) = R(m)|R(m)|^{-1}$ be the C^∞ map defined in the open neighbourhood $\mathcal{V} = \{m \in \mathcal{M}\colon R(m) \in G_{\mathcal{M}}\}$ of p in $\mathcal{M}$. Then $R_U|_{\mathcal{V}\cap\mathcal{U}_{\mathcal{A}}(p)} = \phi_U$, the local cross section of the action of $U_{\mathcal{A}}$ on $\mathcal{U}_{\mathcal{A}}(p)$ constructed in 6.2.
The natural consequence of these facts is the following:

7.4 PROPOSITION. *The map $P \in L(\mathcal{M})$ defined by*

$$P(m) = R(m)p - pR(m) = [R(m), p], \qquad m \in \mathcal{M},$$

defines a projection from $T\mathcal{S}_{\mathcal{M}}(p)_p$ onto $T\mathcal{S}_{\mathcal{A}}(p)_p$ with norm $\|P\|_{L(\mathcal{M})} \leq 2\,\mathrm{Ind}(E)^{1/2}$. Also, the map $P_u\colon T\mathcal{U}_{\mathcal{M}}(p)_p \to T\mathcal{U}_{\mathcal{A}}(p)_p$ given by

$$P_u(m) = \frac{P(m) + P(m)^*}{2} = \left[\frac{R(m) - R(m)^*}{2}, p\right]$$

is a projection with $\|P_u\|_{T\mathcal{U}_{\mathcal{M}}(p)_p} \leq \mathrm{Ind}(E)^{1/2}$.

Proof. The first statement can be easily derived using the fact that P restricted to the tangent space $T\mathcal{S}_{\mathcal{M}}(p)_p$ coincides with $T(\pi_p)_1 \circ R$ and using $\|R\| = \mathrm{Ind}(E)^{1/2}$. On the other hand

$$T\mathcal{U}_{\mathcal{M}}(p)_p = \left\{m = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in \mathcal{M}\colon a \in (1-p)\mathcal{M}p\right\}.$$

Let $m = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in T\mathcal{U}_{\mathcal{M}}(p)_p$. If

$$R(m) = \begin{pmatrix} 0 & b \\ a & d \end{pmatrix} \in \mathcal{A},$$

then

$$P_u(m) = \begin{pmatrix} 0 & \frac{a^*-b}{2} \\ \frac{a-b^*}{2} & 0 \end{pmatrix}.$$

Note that the tangent vectors of $T\mathcal{U}_{\mathcal{A}}(p)_p$ are of the form $Z = [z, p]$ with $-z^* = z \in H_p = \ker E$, or in matrix form

$$z = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix} \quad \text{with} \quad d^* = -d. \tag{7.5}$$

It follows that $R(Z) = z$ because they have the same first column. So $P_u(Z) = Z$ for $Z \in T\mathcal{U}_{\mathcal{A}}(p)_p$ and $P_u(T\mathcal{U}_{\mathcal{M}}(p)_p) = T\mathcal{U}_{\mathcal{A}}(p)_p$ since $\frac{R(m)-R(m)^*}{2} \in H_p$ and is skew-symmetric for all $m \in T\mathcal{U}_{\mathcal{M}}(p)_p$. To prove the norm inequality, note that if $m \in T\mathcal{U}_{\mathcal{M}}(p)_p$ then $\frac{R(m)-R(m)^*}{2} = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix}$ and therefore

$$\left\| \left[\frac{R(m) - R(m)^*}{2}, p \right] \right\| \leq \left\| \frac{R(m) - R(m)^*}{2} \right\|.$$

Finally

$$\begin{aligned} \|P_u(m)\| &= \left\| \frac{R(m) - R(m)^*}{2} p - p \frac{R(m) - R(m)^*}{2} \right\| \\ &\leq \left\| \frac{R(m) - R(m)^*}{2} \right\| \leq \|R(m)\|. \end{aligned}$$

The statement follows using $\|R\| = \mathrm{Ind}(E)^{1/2}$.

7.6. We can transfer the projections P and P_u through the similarity (resp. unitary) orbits $\mathcal{S}_{\mathcal{A}}(p)$ (resp. $\mathcal{U}_{\mathcal{A}}(p)$) in the usual way. Put

$$\text{(7.6.1)} \quad \Pi_p = P \quad \text{and} \quad \Pi_{gpg^{-1}} = Ad(g) \circ \Pi_p \circ Ad(g^{-1}), \qquad g \in G_{\mathcal{A}},$$

and

$$\text{(7.6.2)} \quad \Xi_p = P_u \quad \text{and} \quad \Xi_{wpw^*} = Ad(w) \circ \Xi_p \circ Ad(w^*), \qquad w \in U_{\mathcal{A}}.$$

In order that these distributions are well defined we need the following lemma.

7.7 Lemma. *If $h \in G_{\mathcal{B}}$ (resp. $v \in U_{\mathcal{B}}$) then $Ad(h) \circ \Pi_p \circ Ad(h^{-1}) = \Pi_p$ (resp. $Ad(v) \circ \Xi_p \circ Ad(v^*) = \Xi_p$).*

Proof. Recall that if $b \in \mathcal{B}$, then its matrix is of the form $b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$. Therefore $hmh^{-1}p = hmph^{-1} = hR(m)ph^{-1} = hR(m)h^{-1}p$ and the first equality holds since h commutes with p. For the unitary case, the same argument shows that $vmv^*p = vR(m)v^*p$. Therefore

$$\begin{aligned} 2\Xi_p(vmv^*) &= [R(vmv^*) - R(vmv^*)^*, p] \\ &= v[R(m) - R(m)^*, p]v^* \end{aligned}$$

and $Ad(v) \circ \Xi_p \circ Ad(v^*) = \Xi_p$.

7.8 PROPOSITION. *The distributions $\mathcal{S}_{\mathcal{A}}(p) \ni r \mapsto \Pi_r$ and $\mathcal{U}_{\mathcal{A}}(p) \ni q \mapsto \Xi_q$ with range in $L(\mathcal{M})$ have the following properties.*

(*a*) *They are well defined C^∞ maps.*
(b_1) *For each $r \in \mathcal{S}_{\mathcal{A}}(p)$, Π_r projects $T(\mathcal{S}_{\mathcal{M}}(p))_r$ onto $T(\mathcal{S}_{\mathcal{A}}(p))_r$.*
(b_2) *For each $q \in \mathcal{U}_{\mathcal{A}}(p)$, Ξ_q projects $T(\mathcal{U}_{\mathcal{M}}(p))_q$ onto $T(\mathcal{U}_{\mathcal{A}}(p))_q$.*
(*c*) *For each $q \in \mathcal{U}_{\mathcal{A}}(p)$, $\|\Xi_q\|_{L(\mathcal{M})} \leq \mathrm{Ind}(E)^{1/2}$.*

Proof. Part (a) follows from the preceeding lemma. Smoothness of the distributions can be proved using the existence of C^∞ local cross sections in both the similarity and unitary orbits.

If $q = gpg^{-1}$ then $T(\mathcal{S}_{\mathcal{M}}(p))_q = gT(\mathcal{S}_{\mathcal{M}}(p))_p g^{-1}$ (see [MR]) and (b_1) follows. (b_2) follows from analogous considerations. Property (c) is completely apparent using 7.4.

8. Linear connections in $T\mathcal{U}_{\mathcal{A}}(p)$

In this section we shall introduce three covariant derivatives on the tangent bundle $T\mathcal{U}_{\mathcal{A}}(p)$: the reductive connection D^r (8.3), the clasifying connection D^c (8.12) and the spatial connection D^s (8.2). It is a remarkable fact that the three share the same geodesic curves (8.8). The first two come from the general theory of HRS's and we refer the reader again to [MR].

In order to define the third one we recall the definition of the covariant derivative of the Grassmannians. In [CPR1], Corach, Porta and Recht introduced the natural connection in the tangent bundle of the space of projections of a C*-algebra. Let γ be a smooth curve in $\mathcal{U}_{\mathcal{M}}(p)$ with $\gamma(0) = p$ and $X = X_t$ a smooth vector field along γ. Put $V = V_t = \dot{\gamma}(t)$. The covariant derivative of X along γ (in $\mathcal{U}_{\mathcal{M}}(p)$) is

$$\frac{DX}{dt} = \dot{X} + [X, [V, \gamma]]. \tag{8.1}$$

Using this connection and the family of projections Ξ we can define a "spatial" connection in the subamnifold $\mathcal{U}_{\mathcal{A}}(p) \subset \mathcal{U}_{\mathcal{M}}(p)$.

8.2 *Definition.* Let γ be a smooth curve in $\mathcal{U}_{\mathcal{A}}(p)$ and X a smooth vector field along γ. Put $V = \dot{\gamma}$. Define the spatial covariant derivative of X along γ:

$$\begin{aligned} \frac{D^s X}{dt} &= \Xi_\gamma\left(\frac{DX}{dt}\right) \\ &= \Xi_\gamma(\dot{X} + [X, [V, \gamma]]). \end{aligned}$$

8.3 *Remark.* Using the fact that $\mathcal{U}_{\mathcal{A}}(p)$ is an HRS, it has a linear connection induced by the reductive structure. Recall (Section 4) the map $K_\gamma\colon T\mathcal{U}_{\mathcal{A}}(p)_\gamma \to$

$\tilde{H}_\gamma = \{h \in H_\gamma : h^* = -h\}$ which is the inverse of $T\pi_\gamma|_{\tilde{H}_\gamma}$. We have proved that $K_p(m) = \frac{R(m)-R(m)^*}{2}$. From the general theory of HRS we know that the covariant derivative $\frac{D^r}{dt}$ is given by

$$K_\gamma\left(\frac{D^r X}{dt}\right) = (K_\gamma^{\cdot} X) + [K_\gamma X, K_\gamma V]$$

where γ, X and V are as in 8.2. In order to explicitly compute $\frac{D^r}{dt}$ we need the following lemma:

8.4 LEMMA. *Let $q = upu^* \in \mathcal{U}_{\mathcal{A}}(p)$. Then $K_q(X) = uR(u^*Xu)u^*$.*

Proof. That it is well defined was proven in 7.6. Since $K_{upu^*} = Ad(u) \circ K_p \circ Ad(u^*)$ it suffices to verify that $K_p = R$. Note that this was proved in (7.3.3) for the space $\mathcal{S}_{\mathcal{A}}(p)$. To see that the same holds for $\mathcal{U}_{\mathcal{A}}(p)$, it is enough to verify that for $X \in T\mathcal{U}_{\mathcal{A}}(p)_p$, $R(X) \in \tilde{H}_p$. This is easily deduced from 7.5.

In what follows we shall use capital letters to denote elements of the tangent spaces and lower case letters to denote the corresponding elements in $\mathcal{A}$. Typically, $x = K_q(X)$ for $X \in T\mathcal{U}_{\mathcal{A}}(p)_q$.

8.5 PROPOSITION. $\frac{D^r X}{dt} = \dot{X} + [X, K_\gamma V] = \dot{X} + [X, v]$.

Proof. Let X be a smooth vector field in $\mathcal{U}_{\mathcal{A}}(p)$ along the smooth curve γ with velocity V. Using the previous lemma, it is easy to see that

$$[x,\ \gamma] = X$$

and in particular

$$[v,\ \gamma] = V.$$

Taking derivatives, we have

$$[\dot{x},\ \gamma] + [x,\ V] = \dot{X} \Rightarrow [\dot{x},\ \gamma] = \dot{X} - [x,\ V].$$

Then

$$\begin{aligned}
\frac{D^r X}{dt} &= \left[K_\gamma\left(\frac{D^r X}{dt}\right), \gamma\right] \\
&= [\dot{x},\ \gamma] + [\,[x, v],\ \gamma] \\
&= \dot{X} - [x, V] - [\,[\gamma, x], v] - [\,[v, \gamma], x] \text{ by the Jacobi identity} \\
&= \dot{X} - [x, V] - [V, x] + [X, v] \\
&= \dot{X} + [X, v].
\end{aligned}$$

8.6. Let $v = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix}$ be a typical element of $\tilde{H}_p = \{v \in \ker E\colon\ v^* = -v\}$. Then the unique geodesic γ (relative to D^r) such that $\gamma(0) = p$ and $\dot\gamma(0) = [v, p] = V_0$ is the curve (see [MR]):

$$\gamma\colon \mathbb{R} \to \mathcal{U}_{\mathcal{A}}(p) \qquad \text{given by} \qquad \gamma(t) = e^{tv}\, p\, e^{-tv}\,, \quad t \in \mathbb{R}. \tag{8.7}$$

We shall prove that γ is also a geodesic for D^s, the spatial covariant derivative of 8.2. Note that:

(1) $V_t = \dot\gamma(t) = e^{tv}[v, p]e^{-tv} \in T\mathcal{U}_{\mathcal{A}}(p)_p$.
(2) $\dot V_t = e^{tv}[v, [v, p]\,]e^{-tv}$.
(3) $[V_t, [V_t, \gamma(t)]\,] = e^{tv}[\,[v, p], [\,[v, p], p]\,]e^{-tv} = e^{tv}[V_0, [V_0, p]\,]e^{-tv}$.

Since the distribution $\gamma \mapsto \Xi_\gamma$ is equivariant, in order to prove that $\Xi_{\gamma(t)}\left(\frac{D^s V_t}{dt}\right) = 0$ for all $t \in \mathbb{R}$, it suffices to verify that

$$\Xi_p\left([v, V_0] + [V_0, [V_0, p]\,]\right) = 0.$$

But $V_0 = [v, p] = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$, so

$$\begin{aligned} [v, V_0] &= vV_0 - V_0 v \\ &= \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix}\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix} \\ &= \begin{pmatrix} -a^*a & 0 \\ da & aa^* \end{pmatrix} - \begin{pmatrix} a^*a & a^*d \\ 0 & -aa^* \end{pmatrix} \\ &= \begin{pmatrix} -2a^*a & -a^*d \\ da & 2aa^* \end{pmatrix}. \end{aligned}$$

Similarly,

$$[V_0, [V_0, p]\,] = \begin{pmatrix} 2a^*a & 0 \\ 0 & -2aa^* \end{pmatrix}.$$

In other words, we have

$$\left.\frac{DV}{dt}\right|_{t=0} = \begin{pmatrix} 0 & -a^*d \\ da & 0 \end{pmatrix}.$$

CLAIM. $R\begin{pmatrix} 0 & -a^*d \\ da & 0 \end{pmatrix} = (I - E)(v^2)$.

Indeed,

$$v^2 = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix}^2 = \begin{pmatrix} -a^*a & -a^*d \\ da & -aa^* + d^2 \end{pmatrix}$$

and

$$(I - E)(v^2) = \begin{pmatrix} 0 & -a^*d \\ da & ? \end{pmatrix}.$$

Then $(I - E)(v^2)$ and $\frac{DV}{dt}|_{t=0}$ have the same first column and the claim is proved.

Note that v^2 and $(I - E)(v^2)$ are self-adjoint. Therefore, since $\Xi_p = \frac{1}{2}[(R - R^*), p]$, $\Xi_p([v, V_0] + [V_0, [V_0, p]]) = 0$. So γ is the *unique* geodesic relative to D^s such that $\gamma(0) = p$ and $\dot\gamma(0) = [v, p]$. We have proven the following theorem:

8.8 THEOREM. *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras and let $E\colon \mathcal{A} \to \mathcal{B}$ be a conditional expectation of finite index. Consider the induced HRS structure on $\mathcal{U}_{\mathcal{A}}(p)$ as in* 6.6. *Then the spatial covariant derivative D^s and D^r, the one induced by the reductive structure, have the same geodesics. Moreover, D^s is the unique covariant derivative on the tangent space $T\mathcal{U}_{\mathcal{A}}(p)$ without torsion with this property.*

Now let us compute an explicit formula for the "spatial" covariant derivative D^s of 8.2.

8.9 LEMMA. *Let γ be a smooth curve in $\mathcal{U}_{\mathcal{A}}(p)$ with $\gamma(0) = p$ and X a smooth vector field along γ. Put $V = \dot\gamma$. If $K_p(V_0) = v = \left(\begin{smallmatrix} 0 & -a^* \\ a & d \end{smallmatrix}\right)$, then*

$$\left.\frac{D^r X}{dt}\right|_{t=0} = \left.\frac{DX}{dt}\right|_{t=0} + \left[X, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right].$$

Proof. Recall from 8.1 and 8.5 that

$$\frac{DX}{dt} = \dot X + [X, [V, p]] \quad \text{and} \quad \frac{D^r X}{dt} = X + [X, v].$$

Therefore

$$\frac{D^r X}{dt} - \frac{DX}{dt} = [X, v - [V, p]].$$

But $V = [v, p]$, and then $V = \left(\begin{smallmatrix} 0 & a^* \\ a & 0 \end{smallmatrix}\right)$. Then

$$v - [V, p] = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix} - \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

and the proof is complete.

8.10 PROPOSITION. *Let γ be a smooth curve in $\mathcal{U}_{\mathcal{A}}(p)$ and X a smooth vector field along γ. Put $V = \dot\gamma$, $x = K_\gamma(X)$, and $v = K_\gamma(V)$. Then*

$$\frac{D^s X}{dt} = \dot X + \frac{1}{2}\{[X, v] + [V, x]\}.$$

Proof. Let $v = \begin{pmatrix} 0 & -a^* \\ a & d \end{pmatrix}$ and $x = \begin{pmatrix} 0 & -b^* \\ b & c \end{pmatrix}$. Using the previous lemma and the fact that $\Xi_\gamma(\frac{D^r X}{dt}) = \frac{D^r X}{dt}$ it follows that

$$\frac{D^r X}{dt} = \frac{D^s X}{dt} + \Xi_\gamma\left(\left[X, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right]\right). \tag{8.11}$$

On the other hand,

$$X = \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \quad \text{and} \quad \left[\begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right] = \begin{pmatrix} 0 & b^*d \\ -db & 0 \end{pmatrix}$$

and

$$vx = \begin{pmatrix} -a^*b & * \\ db & * \end{pmatrix} \Rightarrow R_\gamma\left(\left[X, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right]\right) = -(I - E)(vx).$$

Recall from 7.3 that $\Xi_\gamma(m) = \frac{1}{2}[R_\gamma(m) - R_\gamma(m)^*, \gamma]$. Since $x = -x^*$ and $v = -v^*$, it follows that

$$\Xi_\gamma\left(\left[X, \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\right]\right) = \frac{1}{2}[\,(I - E)([x, v]), \gamma] = \frac{1}{2}[\,[x, v], \gamma].$$

Therefore, by 8.11,

$$\begin{aligned} \frac{D^s X}{dt} &= \frac{D^r X}{dt} - \frac{1}{2}[\,[x, v], \gamma] \\ &= \dot{X} + [X, v] + \frac{1}{2}[\,[v, x], \gamma] \\ &= \dot{X} + [X, v] + \frac{1}{2}\{\,[\,[v, \gamma], x] - [\,[x, \gamma], v]\,\} \\ &= \dot{X} + \frac{1}{2}\{\,[X, v] + [V, x]\,\}. \end{aligned}$$

8.12 *Remark.* Another natural connection on the tangent bundle of a general homogeneous reductive space $\mathcal{Q}$ is considered in [MR]. It is called the "classifying connection" and can be briefly described as follows: Let X be a smooth vector field along a smooth curve γ on $\mathcal{Q}$. Then the classifying covariant derivative $\frac{D^c X}{dt}$ of X along γ is given by the relation

$$K_\gamma\left(\frac{D^c X}{dt}\right) = (I - E_\gamma)\left(\frac{d}{dt}K_\gamma(X)\right)$$

where E_γ denotes the projection onto the horizontal space of γ. It was shown in [MR] that this connection has the same geodesics as the reductive connection. Moreover, the average of these two connections gives rise to a new connection which has the same geodesics as the previous ones, with vanishing torsion tensor.

In our context, $\mathcal{Q} = \mathcal{U}_\mathcal{A}(p)$ D^c can be explicitly computed:

8.13 PROPOSITION. *Let X be a smooth vector field along a smooth curve γ on $\mathcal{U}_{\mathcal{A}}(p)$. Let $V = \dot{\gamma}$. Then the classifying covariant derivative $\frac{D^c X}{dt}$ of X along γ is given by the formula*

$$\frac{D^c X}{dt} = \dot{X} + [V, K_\gamma(X)] = \dot{X} + [V, x].$$

Moreover, the spatial connection D^s constructed via the basic representation realizes the average of D^r and D^c mentioned above.

Proof. Recall that for every tangent vector Y at q one has $[K_q(Y), q] = Y$. Then

$$\begin{aligned} \frac{D^c X}{dt} &= \left[K_\gamma \frac{D^c X}{dt}, \gamma\right] \\ &= [(I - E_\gamma)(\dot{x}), \gamma] \\ &= [\dot{x}, \gamma] \end{aligned}$$

since $E_\gamma(\dot{x})$ commutes with γ. Differentiating the relation $X = [x, \gamma]$ one obtains $\dot{X} = [\dot{x}, \gamma] + [x, V]$. Therefore

$$\frac{D^c X}{dt} = \dot{X} - [x, V] = \dot{X} + [V, x].$$

Finally, combining the description of D^r given in 8.5 and D^s given in 8.10 with this last formula one obtains

$$\frac{D^s X}{dt} = \frac{1}{2}\left\{\frac{D^r X}{dt} + \frac{D^c X}{dt}\right\}.$$

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