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PERTURBATION OF PLANE CURVES AND SEQUENCES OF INTEGERS

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1. Perturbation of a curve

Definition 1.1. The Lebesgue measure on $\mathbb{R}^2$ is denoted by m.

Let Γ: $[0, \infty) \to \mathbb{R}^2$ be a continuous curve. For $s > 0$ and locally integrable f: $\mathbb{R}^2 \to \mathbb{C}$ we set

$$M_s f(x) = M_s(\Gamma, f)(x) = \frac{1}{s} \int_0^s f(x + \Gamma(t))dt.$$

(The measurability of $M_s f(x)$ is discussed in the appendix.)

Let $p \geq 1$. We say that Γ *differentiates* L^p_{loc} if and only if for $f \in L^p_{loc}(\mathbb{R}^2)$ we have

$$\lim_{s\to 0} M_s(\Gamma, f)(x) = f(x)$$

for m-a.e. $x \in \mathbb{R}^2$.

Let $1 \leq p < \infty$. We say that Γ is ∞-*sweeping out for* L^p if and only if there is $f \in L^p(\mathbb{R}^2)$ so that

$$\limsup_{s\to 0^+} M_s(\Gamma, f)(x) = \infty$$

for a.e. $x \in \mathbb{R}^2$.

We say that the continuous curve Δ: $[0, \infty) \to \mathbb{R}^2$ is a *perturbation* of Γ if and only if

$$\lim_{s\to 0} \frac{1}{s} \quad |\{t \mid 0 \leq t \leq s, \quad \Gamma(t) \neq \Delta(t)\}| = 0,$$

where $|A|$ means the one dimensional Lebesgue-measure of the set A.[1]

In the sequel C will denote a "generic" positive constant, which is independent of those quantities it should be independent of, but it can have different values even in the same set of inequalities.

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[1] We will use the same notation for the absolute value, but it will not cause any confusion.

It is an immediate corollary to Lebesgue's differentiation theorem that the curve (t, t) differentiates L^1_{loc}. It was shown in [13] that for fixed $a > 0$ the curve (t, t^a) differentiates L^p_{loc} for $p > 1$. The question whether this curve differentiates L^1_{loc} is open. In [7] a large class of convex plane curves, such as $(t, t/\log(1/t))$, is proved to differentiate L^p_{loc}, $p > 1$, and the same was shown in [16] for some oscillating curves like $(t, t^2 \cdot \sin(1/t))$.

In [7] the question was raised whether there exists a plane curve that differentiates L^2_{loc}, but does not differentiate L^p_{loc} for each $p < 2$. Our Corollary 1.3 below answers this question. The main result of this section is Theorem 1 which describes a general method to obtain curves that differentiate only certain L^p-classes. This method, which we may call perturbation, has its origin in [9].

THEOREM A. *Suppose that the curve Γ: $[0, \infty) \to \mathbb{R}^2$ is of the form $\Gamma(t) = (t, \gamma(t))$, where the real continuous function γ is strictly increasing on* $[0, \infty)$.

(I) *Let $1 < q \le \infty$. Suppose that Γ differentiates L^q_{loc}. Then there is a perturbation Δ of Γ which is of the same form as Γ—$\Delta(t) = (t, \delta(t))$ with a strictly increasing δ; it also differentiates L^q_{loc}, but it is ∞-sweeping out for L^p if $1 \le p < q$.*

(II) *Let $1 \le q < \infty$. Suppose that Γ differentiates L^p_{loc} for each $p > q$. Then there is a perturbation Δ of Γ which is of the same form as Γ, it also differentiates L^p_{loc} for each $p > q$, but it is ∞-sweeping out for L^q.*

Remark 1.2.

1. The reader will notice, examining the proof, that if Γ was smooth, then its perturbation Δ can also be smooth. But our method of perturbation does not preserve convexity. For convex curves that differentiate L^p_{loc} for certain values of p, but does not differentiate L^p for some other values of p M. Christ [6] has partial results. In particular, he constructs a convex curve which differentiates L^p_{loc} for each $p > 1$, but does not differentiate L^1. Also, he has an example of a convex curve which differentiates L^2_{loc} but does not differentiate L^p for *some* p, $1 < p < 2$.
2. In the above theorem we described the perturbation of curves of a special form, since then the new curve can have the same form. But it is important to note that *any* curve (continuous or not) can be perturbed to get the results in Theorem A. In particular, the "oscillating" curves described in [16] can also be perturbed.
3. In this paper we just examine curves with respect to L^p-classes but it is possible to extend the results to other Orlitz-classes. For example, if a curve differentiates $L \log L_{loc}$, then it has a perturbation which also differentiates $L \log L_{loc}$, but it is ∞-sweeping out for L^1.

COROLLARY 1.3. (I) *Let $1 < q \le \infty$. There exists a continuous curve Δ of the form $\Delta(t) = (t, \delta(t))$ with a strictly increasing δ, which differentiates L^q_{loc}, but it is ∞-sweeping out for L^p if $1 \le p < q$.*

(II) *Let* $1 \leq q < \infty$. *There exists a continuous curve* Δ *of the form* $\Delta(t) = (t, \delta(t))$ *with a strictly increasing* δ, *which differentiates* L^p_{loc} *for each* $p > q$, *but it is* ∞*-sweeping out for* L^q.

For the easiest way to obtain this corollary from Theorem A we should take perturbations of the curve $\Gamma(t) = (t, t)$. For a more interesting class of examples perturb $\Gamma(t) = (t, t^a)$ with some fixed $a > 0$ (cf. [7]). As we noted earlier these curves differentiate L^p_{loc} for each $p > 1$.

Proof of Theorem A. We just prove (I), and we do that only for finite q. We shall construct a Δ of the form $\Delta(t) = (t, \delta(t))$, where the real continuous function δ will differ from γ only on a sequence of disjoint intervals $I_k \subseteq [0, \infty)$, $k = 1, 2, \ldots$. For $u = 0, 1, \ldots$, let

$$A_u = \{k \mid k = 2^u, 2^u + 1, \ldots, 2^{u+1} - 1\}.$$

The δ we construct is not going to be strictly increasing, but the reader will have no difficulty in modifying our construction to get a strictly increasing δ. Let $I_k = (a_k, b_k)$, and let

$$J_k = \bigcup_{j=k}^{\infty} I_j$$

Select the *positive* numbers a_k and b_k so that:

(i) $b_k < a_{k-1}$;
(ii) $\lim_{k\to\infty} b_k = 0$;
(iii) $\dfrac{|I_k|}{b_k} = \left(\dfrac{1}{u^2 2^u}\right)^{1/q}$, $k \in A_u$;
(iv) $|J_k| < 2|I_k|$.

Note that by (iii) we also have

(v) $\lim_{k\to\infty} \dfrac{a_k}{b_k} = 1$,

and hence by (iv) and (v),

(vi) $\dfrac{|J_k|}{a_k} < C\left(\dfrac{1}{u^2 2^u}\right)^{1/q}$, $k \in A_u$. □

Proof that Δ *differentiates* L^q_{loc}. Of course, the only fact we can use about Δ is that it coincides with Γ off the intervals I_k. Let $f \in L^q_{loc}$. We can assume f to be nonnegative, and also that it has compact support, so in fact, $f \in L^q$. We just have to prove that for a.e. $x \in \mathbb{R}^2$,

$$\lim_{s\to 0} |M_s(\Delta, f)(x) - M_s(\Gamma, f)(x)| = 0.$$

Let $0 < s < a_1$, and let k be such that $a_k \le s < a_{k-1}$. By the choice of k we have

$$\begin{aligned} |M_s(\Delta, f)(x) - M_s(\Gamma, f)(x)| &\le \tfrac{1}{s} \textstyle\int_{J_k} (f(x + \Delta(t)) + f(x + \Gamma(t)))\, dt \\ &\le \tfrac{1}{a_k} \textstyle\int_{J_k} (f(x + \Delta(t)) + f(x + \Gamma(t)))\, dt. \end{aligned}$$

Therefore it is enough to prove that for a.e. x,

$$\lim_{k\to\infty} \frac{1}{a_k} \int_{J_k} (f(x + \Delta(t)) + f(x + \Gamma(t)))\, dt = 0.$$

This would follow from

$$\sum_{u=1}^{\infty} \sum_{k\in A_u} \left\| \frac{1}{a_k} \int_{J_k} (f(x + \Delta(t)) + f(x + \Gamma(t)))\, dt \right\|_{L^q(dm)}^q \le C\|f\|_{L^q(dm)}^q. \tag{1.1}$$

By (vi), for $k \in A_u$ we have

$$\begin{aligned} \left\| \frac{1}{a_k} \int_{J_k} (f(x + \Delta(t)) + f(x + \Gamma(t)))\, dt \right\|_{L^q(dm)} &\le \frac{1}{a_k} |J_k| 2 \|f\|_{L^q(dm)} \\ &\le C \left(\frac{1}{u^2 2^u} \right)^{1/q} \|f\|_{L^q(dm)}, \end{aligned}$$

which implies (1) since the cardinality of A_u is 2^u.

The actual construction of Δ. Let us define the function δ on the intervals I_k. Denoting by c_k the midpoint of I_k we let $\delta(t) = \gamma(a_k)$ for $a_k \le t \le c_k$, and linear for $c_k \le t \le b_k$, which means that the graph of δ for $c_k \le t \le b_k$ is a straight line segment connecting the points $(c_k, \gamma(c_k))$ and $(b_k, \gamma(b_k))$ of $\mathbb{R}^2$.

Proof that Δ *is* ∞*-sweeping out for* L^p, $0 < p < q$. So let us fix $0 < p < q$. We would like to use the theorem of Sawyer [12, Corollary 1.1] but to be able to use it we need to have a finite measure-space. Here is what we are going to do.

Let S denote the square $[-1/2, 1/2) \times [-1/2, 1/2)$. We will prove that there is $f_0 \in L^p$, $f_0 \ge 0$, supported on S for which

$$\limsup_{s\to 0^+} M_s(\Delta, f_0)(x) = \infty \quad \text{for a.e. } x \in S. \tag{1.2}$$

Now to get an L^p-function f for which

$$\limsup_{s\to 0^+} M_s(\Delta, f)(x) = \infty$$

holds for a.e. $x \in \mathbb{R}^2$, just let

$$f(x) = \sum_{(i,j)\in\mathbb{Z}^2} \left(\frac{1}{2^{|i|+|j|}}\right)^{1/p} \cdot f_0(x-(i,j)).$$

Let us prove the existence of $f_0 \in L^p$ with support in S and satisfying (2). Consider a "mod S version" of the operators $M_s(\Delta, f)$. In other words, for f with support in S, consider its periodic extension to $\mathbb{R}^2$ defined by

$$\overline{f}(x) = \sum_{(i,j)\in\mathbb{Z}^2} f(x-(i,j)),$$

and define

$$\overline{M}_s(\Delta, f)(x) = M_s(\Delta, \overline{f})(x).$$

We just need to prove that there is $f_0 \in L^p$, $f_0 \geq 0$, with support in S such that for a.e. $x \in S$,

$$\limsup_{s\to 0^+} \overline{M}_s(\Delta, f_0)(x) = \infty, \tag{1.3}$$

because for $x \in \operatorname{int} S$ and small enough s we have $\overline{M}_s f_0(x) = M_s f_0(x)$.

For the rest of the proof we will work on $\mathbb{T}^2$ which we identify with S in the usual way, and so everything—functions, translation etc.—is understood mod S. We also drop the "bar" notation.

We are going to prove that for each large enough u there is $f = f_u$ supported on a narrow rectangle of the form $[-1/2, 1/2) \times [0, d_u)$ so that

$$m\left(\max_{k\in A_u} M_{b_k} f \geq \frac{1}{2}\right) > \frac{2^u}{(u^2 2^u)^{p/q}} \|f\|^p_{L^p(\mathbb{T}^2)}, \tag{1.4}$$

which denies the existence of a weak (p, p) maximal inequality since $p < q$ and u is arbitrary. To be able to use Sawyer's theorem we need a mixing family B of measure preserving transformations on $\mathbb{T}^2$ which commutes with M_s. Let us show that B can be taken to be the set of translations.

Translations (mod S!) certainly commute with each M_s. We are left to show that given $\varrho > 1$, and measurable sets P, Q of positive measure, there is a t so that $m((P+t)\cap Q) \leq \varrho m(P)m(Q)$. But this follows readily from the observation that, by Fubini's theorem,

$$\int_{\mathbb{T}^2} m((P+t)\cap Q)dt = m(P)m(Q).$$

Now, let

$$d = d_u = \frac{1}{2}\min_{k\in A_u} \{\delta(a_{k-1}) - \delta(a_k)\} \cup \left\{\delta(a_{2^{u+1}-1})\right\}.$$

Note that $d > 0$ since γ is strictly increasing and for each k we have $\delta(a_k) = \gamma(a_k)$. Let $f(x) = (u^2 2^u)^{1/q}$ on the rectangle $R = [-1/2, 1/2) \times [0, d)$, and let $f(x) = 0$ everywhere else. Then we have

$$\|f\|^p_{L^p(\mathbb{T}^2)} = d \cdot (u^2 2^u)^{p/q}. \tag{1.5}$$

For each $k \in A_u$ let $E_k = R - (0, \delta(a_k))$ (so we just shift R down by the amount $\delta(a_k)$), and set $E = \bigcup_{k \in A_u} E_k$. Note that by the definition of d the E_k's, $k \in A_u$, are disjoint (provided u is large enough), and $m(E_k) = m(R)$. Hence we have

$$m(E) = d \cdot 2^u. \tag{1.6}$$

In order to have (4) we just have to prove, by (5) and (6), that if $x \in E_k$ for some $k \in A_u$, then $M_{b_k} f \geq \frac{1}{2}$. Let $x \in E_k$. By the definitions of E_k and δ for every t satisfying $a_k \leq t \leq c_k$ we have $x + \Delta(t) \in R \bmod S$. Hence we can estimate

$$\begin{aligned} M_{b_k} f(x) &= \frac{1}{b_k} \int_0^{b_k} f(x + \Delta(t))dt \geq \frac{1}{b_k} \int_{a_k}^{c_k} f(x + \Delta(t))dt \\ &= \frac{1}{2} \frac{|I_k|}{b_k} (u^2 2^u)^{1/q} = \frac{1}{2}, \end{aligned}$$

where in the last equality we used property (iii). □

2. Perturbation of a sequence

Definitions 2.1. Let Γ be a strictly increasing sequence of positive integers. The *counting function* of Γ is $\Gamma(n) = \#\{\gamma \mid \gamma \in \Gamma, \gamma \leq n\}$. The *lower density* $d(\Gamma)$ of Γ is

$$d(\Gamma) = \liminf_{n \to \infty} \frac{\Gamma(n)}{n}.$$

Let (X, B, m, U) be a σ-finite measure-preserving system. For an a.e. finite $f \colon X \to \mathbb{C}$ we let

$$M_n f(x) = M_n(\Gamma, f)(x) = \frac{1}{\Gamma(n)} \sum_{\substack{\gamma \leq n \\ \gamma \in \Gamma}} f(U^\gamma x).$$

Let $p \geq 1$. We say that Γ is *good for* L^p in (X, B, m, U) if and only if for $f \in L^p(X)$ we have that

$$\lim_{n \to \infty} M_n(\Gamma, f)(x)$$

exists and is finite for a.e. $x \in X$. We say that Γ is *universally good for* L^p if it is good for L^p in every σ-finite measure preserving system.

Let $1 \leq p < \infty$. We say that Γ is *∞-sweeping out for L^p* in (X, B, m, U) if and only if there is $f \in L^p(X)$ such that

$$\sup_n M_n(\Gamma, f)(x) = \infty$$

for a.e. $x \in X$. We say that Γ is *universally ∞-sweeping out for L^p* if and only if it is ∞-sweeping out for L^p in every *aperiodic* (*free*), probability measure preserving system. (The "probability" part here means that $m(X) = 1$).

The strictly increasing sequence of positive integers Δ is called a *perturbation* of Γ if and only if

$$\lim_{n\to\infty} \frac{\#\{\alpha \mid \alpha \in (\Delta \setminus \Gamma) \cup (\Gamma \setminus \Delta),\ \alpha \leq n\}}{\Gamma(n)} = 0.$$

Note that if Δ is a perturbation of Γ then

$$\lim_{n\to\infty} \frac{\Delta(n)}{\Gamma(n)} = 1. \tag{2.1}$$

In the sequel, C will denote a "generic" positive constant, which is independent of those quantities it should be independent of, but it can have different values even in the same set of inequalities.

The existence of a sequence of integers with 0 density that is universally good for L^1 was proved in [2]. In [4], it was proved that the sequence of squares is universally good for L^p, $p > 1$. That the sequence of primes is universally good for L^p, $p > 1$, was proved in [14]. Other sequences that are universally good for L^p, $p > 1$, are given in [5] and [3]. For example, the sequences $([n^{3/2}])$ and $([n \log n])$, or the sequence of those integers the decimal expansion of which contain only 0's and 1's, are universally good for L^p, $p > 1$.

It was shown in [1] that there exists a sequence of integers which is universally good for, say, L^2 but not good for L^p, $p < 2$. Finally, it was in [10] (and in [11]) that the existence of a sequence that is universally good for L^p, $p > 1$, but not good for L^1 was proved. The method used in [1] and [10] is perturbation. Our purpose is to describe a more flexible and technically simpler version of the perturbation method, and to use it to prove a result, Theorem B below, that cannot be improved in the sense of the first remark after the enunciation of the theorem.

THEOREM B. *Let Γ be a strictly increasing sequence of positive integers with $d(\Gamma) = 0$.*

(I) *Let $1 < q \leq \infty$. Suppose that Γ is universally good for L^q. Then there is a perturbation Δ of Γ which is also universally good for L^q, but it is universally ∞-sweeping out for L^p if $1 \leq p < q$.*

(II) *Let $1 \le q < \infty$. Suppose that Γ is universally good for L^p for each $p > q$. Then there is a perturbation Δ of Γ which is also universally good for L^p for each $p > q$, but it is universally ∞-sweeping out for L^q.*

Remark 2.2. 1. Note that if the sequence Γ has positive lower density then the ergodic averages along Γ satisfy a weak (1,1) inequality. But then any perturbation Δ of Γ will have the same property, hence Δ will be good for some irrational rotation of the interval $[0, 1)$.
2. We get interesting instances of Theorem B if we perturb the sequence of squares, the primes, or the sequence $([n^{3/2}])$. As we see our theorem applies to such irregular sequences as the primes, while the perturbation used in [1] and [11] does not seem to be effective enough to handle these sequences.
3. It is possible, using our method, to construct a sequence of integers that is universally good for $L \log L$ (of course, for finite measure-spaces), but it is ∞-sweeping out for L^1. Specific examples could be perturbations of the sequences $([n^{3/2}])$ or $([n \log n])$, since they are known to be universally good for $L \log L$ (cf. [15]).

Proof of Theorem B. Since we proved part (I) of Theorem A, here we just prove part (II); the reader will have no difficulty proving the other part.

The idea of the proof is similar to the previous proof's: the new sequence Δ is formed by adding segments of "bad" sequences to Γ, and we shall do this so that the cardinality of these "perturbations" is not big enough to effect the good behaviour in L^p, $p > q$, but the perturbation is strong enough to destroy the (q, q)-maximal inequality. We will make this more quantitative in a minute, but first let us indicate what we mean by a "bad" sequence.

There are numerous ways to construct a bad sequence, but probably the easiest is to give a sequence which is not uniformly distributed among residue classes for infinitely many modulus . This means that fixing a modulus Q and a residue v, at one point sufficiently many elements of our sequence will be congruent to $v \bmod Q$.

Below we shall define integers n_k, $k = 1, 2, \ldots$. For $u = 0, 1, \ldots$ set

$$A_u = \{k \mid k = 2^u, 2^u + 1, \ldots, 2^{u+1} - 1\}.$$

The new sequence Δ will contain Γ, and is formed by adding to Γ a certain number of integers from the interval $[n_k, 2n_k)$ so that these added integers will be congruent to $k \bmod 2^u$ if $k \in A_u$. To be specific, the cardinality of these numbers will be

$$\left(\frac{u}{2^u}\right)^{1/q} \cdot \Gamma(n_k), \qquad k \in A_u. \tag{2.2}$$

In order to be able to find this many integers in $[n_k, 2n_k)$, each congruent to $k \bmod 2^u$, we need n_k large enough: $n_k > u^{1/q} \cdot 2^{u(1-1/q)} \cdot \Gamma(n_k)$. We certainly achieve this if

$$n_k > u2^u \cdot \Gamma(n_k). \tag{2.3}$$

We also have to make sure that these new numbers are numerous enough to destroy the weak (q, q) maximal inequality, hence we have to make sure that Γ does not have many elements in $[n_k, 2n_k)$. Indeed, we will choose n_k so that the number of elements of Γ in $[n_k, 2n_k)$ will not exceed $2\Gamma(n_k)$.

To sum up, the n_k will satisfy:

(i) $n_k > 2n_{k-1}$;
(ii) $\frac{\Gamma(n_k)}{n_k} < \frac{1}{u2^u}$, $\quad k \in A_u$;
(iii) $\Gamma(2n_k) \leq 3\Gamma(n_k)$.

As a last requirement on the n_k, so Δ becomes a perturbation of Γ, we will have

(iv) $\left(\frac{u}{2^u}\right)^{1/q} \cdot \Gamma(n_k) > \sum_{i=1}^{k-1} \Gamma(n_i), \quad k \in A_u.$

The recursive construction of the n_k satisfying the above four properties is quite simple. Since $d(\Gamma) = 0$, there is a sequence $\{m_j\}$ of positive integers such that:

(v) $\lim_{j\to\infty} \frac{\Gamma(m_j)}{m_j} = 0$;
(vi) $\frac{\Gamma(m_j)}{m_j} \leq \frac{\Gamma(m)}{m}$ for $m \leq m_j$.

Having constructed $n_1, \ldots, n_{k-1}$, we just take $n_k = [m_j/2]$ for large enough j. It is clear that we can choose j so that (i) and (iv) hold. To have (ii), we use (v) and the estimate

$$\frac{\Gamma(n_k)}{n_k} = \frac{\Gamma([m_j/2])}{[m_j/2]} \leq 3\frac{\Gamma(m_j)}{m_j}.$$

Finally, to see that we can choose j to have (iii), use (vi) and estimate

$$\begin{aligned}\Gamma(2n_k) &\leq \Gamma(m_j) = \frac{\Gamma(m_j)}{m_j} \cdot m_j \leq \frac{\Gamma([m_j/2])}{[m_j/2]} \cdot m_j \\ &\leq 3\Gamma([m_j/2]) = 3\Gamma(n_k).\end{aligned}$$

□

Proof that Δ is a perturbation of Γ. Since Δ is formed by *adding* new terms to Γ we need to prove that

$$\lim_{n\to\infty} \frac{\Delta(n) - \Gamma(n)}{\Gamma(n)} = 0. \tag{2.4}$$

Let n be arbitrary. Then for some k and u, $k \in A_u$, we have $n_k \leq n < n_{k+1}$. By property (iv), we have the following estimate (recall that the cardinality of the new numbers in the interval $[n_k, n_{k+1})$ is given in (2)):

$$\Delta(n) - \Gamma(n) \leq 2\left(\frac{u}{2^u}\right)^{1/q} \Gamma(n_k).$$

Since $\Gamma(n) \geq \Gamma(n_k)$, we have

$$\Delta(n) - \Gamma(n) \leq 2\left(\frac{u}{2^u}\right)^{1/q} \Gamma(n).$$

This implies (4) for as $n \to \infty$ so do k and u. □

Proof that Δ is universally good for L^p, $p > q$. Fix $p > q$, and the measure-preserving system (X, B, m, U). Since we have property (1), we just need to prove that for $f \in L^p$,

$$\frac{\Delta(n)}{\Gamma(n)} \cdot M_n(\Delta, f)(x) = \frac{1}{\Gamma(n)} \sum_{\substack{\delta \leq n \\ \delta \in \Delta}} (U^\delta x)$$

converge a.e. Without loss of generality we can assume $f \geq 0$. Since Γ is a good sequence for L^p it is enough to prove that

$$\limsup_{n \to \infty} \frac{1}{\Gamma(n)} \sum_{\substack{\delta \leq n \\ \delta \in \Delta \setminus \Gamma}} f(U^\delta x) = 0.$$

Let n be arbitrary. Then for some k and u, $k \in A_u$, we have $n_k \leq n < n_{k+1}$. Noting that the extra elements of $\Delta \cap [n_k, n_{k+1})$ are taken from the interval $[n_k, 2n_k)$, we can estimate

$$\frac{1}{\Gamma(n)} \sum_{\substack{\delta \leq n \\ \delta \in \Delta \setminus \Gamma}} f(U^\delta x) \leq \frac{1}{\Gamma(n_k)} \sum_{\substack{\delta \leq 2n_k \\ \delta \in \Delta \setminus \Gamma}} f(U^\delta x) \overset{\text{def}}{\longrightarrow}= B_k f(x).$$

So we just need to prove that

$$B_k f(x) \to 0 \quad \text{a.e.}$$

This will follow if we prove

$$\int_X \left(\sum_{k=1}^{\infty} (B_k f(x))^p \right) \mathrm{d}m(x) = \sum_{k=1}^{\infty} \|B_k f(x)\|_{L^p}^p < \infty. \tag{2.5}$$

By the triangle inequality and by (iii) we can estimate

$$\begin{aligned}
\sum_{k=1}^{\infty} \|B_k f(x)\|_{L^p}^p &\leq \sum_{k=1}^{\infty} \left(\frac{1}{\Gamma(n_k)} \sum_{\substack{\delta \leq 2n_k \\ \delta \in \Delta \setminus \Gamma}} \|f(U^\delta x)\|_{L^p} \right)^p \\
&= \|f\|_{L^p}^p \sum_{k=1}^{\infty} \left(\frac{1}{\Gamma(n_k)} \sum_{\substack{\delta \leq 2n_k \\ \delta \in \Delta \setminus \Gamma}} 1 \right)^p
\end{aligned}$$

$$= \|f\|_{L^p}^p \sum_{u=1}^{\infty} \sum_{k \in A_u} \left(\frac{1}{\Gamma(n_k)} \sum_{\substack{\delta \le 2n_k \\ \delta \in \Delta \setminus \Gamma}} 1 \right)^p \le \quad \text{by (iii)}$$

$$\le \|f\|_{L^p}^p \sum_{u=1}^{\infty} \sum_{k \in A_u} \left(\frac{\left(\frac{u}{2^u}\right)^{1/q} \cdot 3\Gamma(n_k)}{\Gamma(n_k)} \right)^p$$

$$= \|f\|_{L^p}^p \sum_{u=1}^{\infty} 2^u \cdot 3^p \left(\frac{u}{2^u}\right)^{p/q} = C_p \|f\|_{L^p}^p < \infty,$$

since $p > q$. Therefore we proved (5). □

Proof that Δ is universally ∞-sweeping out for L^q. By the lemma below we just need to disprove the existence of a maximal inequality on $\mathbb{Z}$.

In the rest of the proof for $f\colon \mathbb{Z} \to \mathbb{R}$ and integer sequence Δ we use the notation

$$M_n f(x) = M_n(\Delta, f)(x) = \frac{1}{\Delta(n)} \sum_{\substack{\delta \le n \\ \delta \in \Delta}} f(x + \delta).$$

We also introduce the following definition for $f\colon \mathbb{Z} \to \mathbb{R}$:

$$D(f) = \limsup_{L \to \infty} \frac{1}{2L+1} \sum_{x=-L}^{L} f(x).$$

For a set A of integers $D(A)$ will mean $D(\chi_A)$, where χ_A is the characteristic function of A.

LEMMA 2.3. *Let $0 < q < \infty$, and let Δ be a strictly increasing sequence of positive integers. Suppose that for every positive K and ϵ, there is $f\colon \mathbb{Z} \to \mathbb{R}$, $D(|f|^q) \le 1$, and a finite set of integers Λ with*

$$D\left\{x \mid \max_{n \in \Lambda} M_n(\Delta, f)(x) \ge K\right\} \ge 1 - \epsilon.$$

Then Δ is universally ∞-sweeping out for L^q.

We remark that this lemma is inspired by similar results in [8].

Proof of Lemma 2.3. Let (X, B, m, U) be an aperiodic, probability measure preserving system. By the assumption of the lemma, and using Rokhlin's tower construction, we conclude that for each positive K and ϵ, there is $\overline{f}\colon X \to \mathbb{R}$, $\|\overline{f}\|^q_{L^q(X)} \le 1$ with

$$m\left\{x \mid \sup_n \overline{M}_n(\Delta, \overline{f})(x) \ge K\right\} \ge 1 - \epsilon,$$

where

$$\overline{M}_n(\Delta, \overline{f})(x) = \frac{1}{\Delta(n)} \sum_{\substack{\delta \le n \\ \delta \in \Delta}} \overline{f}(U^\delta x).$$

It then follows that for each positive integer N there is $\overline{g} = \overline{g}_N\colon X \to \mathbb{R}$ with

$$\|\overline{g}\|^q_{L^q(X)} \le 2^{-N},$$

and

$$m\left\{x \mid \sup_n \overline{M}_n(\Delta, \overline{g})(x) \ge N\right\} \ge 1 - \frac{1}{N}.$$

Let us set

$$\overline{g}_0(x) = \sup_N \overline{g}_N(x).$$

Then

$$\|\overline{g}_0\|^q_{L^q(X)} \le 1$$

so $\overline{g}_0 \in L^q(X)$. Let

$$E_N = \left\{x \mid \sup_n \overline{M}_n(\Delta, \overline{g})(x) \ge N\right\},$$

and set

$$E = \bigcap_{J=1}^{\infty} \bigcup_{N=J}^{\infty} E_N.$$

It is clear that $m(E) = 1$, and also that if $x \in E$ then

$$\sup_n \overline{M}_n(\Delta, \overline{g}_0)(x) = \infty. \qquad \square$$

Let us now fix u, and assume it is large—large enough to satisfy (2.9) below. Define $f\colon \mathbb{Z} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 2^{u/q}, & \text{if } 2^u \mid x; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,

$$D(|f|^q) = 1.$$

We are going to show that

$$\left\{x \mid \max_{k\in A_u} M_{2n_k}(\Delta, f)(x) \geq \frac{1}{4}u^{1/q}\right\} = \mathbb{Z}, \tag{2.6}$$

which, by the lemma, would finish the proof.

Let $x \in \mathbb{Z}$. Then for some $k \in A_u$ we have $x \equiv -k \bmod 2^u$. Recall, from (2), that there are

$$\left(\frac{u}{2^u}\right)^{1/q} \cdot \Gamma(n_k)$$

numbers in $\Delta \cap [n_k, 2n_k)$ that are congruent to $k \bmod 2^u$. Let us denote the set of these numbers by Θ. So we have

$$\#\Theta = \left(\frac{u}{2^u}\right)^{1/q} \cdot \Gamma(n_k), \tag{2.7}$$

and

$$f(x+\delta) = 2^{u/q} \quad \text{for} \quad \delta \in \Theta, \tag{2.8}$$

since $2^u \mid x + \delta$. By property (iii) and since Δ is a perturbation of Γ, we have, for large enough u,

$$\Delta(2n_k) \leq 4\Gamma(n_k). \tag{2.9}$$

We can now estimate as

$$\begin{aligned} M_{2n_k}(\Delta, f)(x) &= \frac{1}{\Delta\,(2n_k)} \sum_{\substack{\delta\leq 2n_k\\ \delta\in\Delta}} f(x+\delta) \geq \quad \text{by (9)} \\ &\geq \frac{1}{4\Gamma\,(n_k)} \sum_{\delta\in\Theta} f(x+\delta) = \quad \text{by (8)} \\ &= \frac{1}{4\Gamma\,(n_k)} \sum_{\delta\in\Theta} 2^{u/q} = \quad \text{by (7)} \\ &= 2^{u/q} \cdot \frac{\left(\frac{u}{2^u}\right)^{1/q} \cdot \Gamma(n_k)}{4\Gamma(n_k)} = \frac{1}{4} \cdot u^{1/q}, \end{aligned}$$

which proves (6). □

Appendix

Here we just want to reprove the following result from [13, Lemma 8.1]:

PROPOSITION. *Let Γ: $[0, \infty) \to \mathbb{R}^2$ be a continuous curve, and let f: $\mathbb{R}^2 \to \mathbb{R}$ be a locally (Lebesgue) integrable function.*

Then the $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ function $f(x + \Gamma(t))$ is measurable. Also, for almost every $x \in \mathbb{R}^2$ the $\mathbb{R} \to \mathbb{R}$ function $f(x + \Gamma(t))$ is locally integrable (in t).

The proof of this proposition appears in [13], but the proof contains a minor gap which we wish to fill here.

Proof. The main step is to show that the $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ map F defined by $F(x, t) = x + \Gamma(t)$ is measurable. This means that we have to show that for each Lebesgue measurable $U \subseteq \mathbb{R}^2$ the set

$$F^{-1}(U) = \{(x, t) \mid (x, t) \in \mathbb{R}^2 \times \mathbb{R},\ F(x, t) \in U\}$$

is Lebesgue measurable. Let us write $U = V \cup W$ where V is a Borel set and W is a set of Lebesgue measure 0. Since F is clearly Borel measurable (being the sum of two Borel measurable maps) we have that $F^{-1}(V)$ is Borel measurable. So we just have to show that $F^{-1}(W)$ is measurable. Because of the completeness of the Lebesgue measure, it is sufficient to show that $F^{-1}(W)$ is of Lebesgue measure 0. Let $X \supseteq W$ be Borel measurable and of measure 0. Then $F^{-1}(X)$ is (Borel) measurable, and we can use Fubini's theorem to conclude that $F^{-1}(X)$ is of measure 0. As a consequence, $F^{-1}(W)$ is of measure 0.

Now that the measurability of the map F is established, the measurability of the $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ function $f(x + \Gamma(t))$ follows readily. The fact that the function $f(x + \Gamma(t))$ is locally integrable in t for almost every $x \in \mathbb{R}^2$ follows now from Fubini's theorem. $\square$

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