ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 1, Spring 1998

THE NEAR RADON-NIKODYM PROPERTY IN LEBESGUE-BOCHNER FUNCTION SPACES

NARCISSE RANDRIANANTOANINA AND ELIAS SAAB

1. Introduction

Let X be a Banach space, $(\Omega, \Sigma, \lambda)$ be a finite measure space and $1 \leq p < \infty$. We denote by $L^p(\lambda, X)$ the Banach space of all (classes of) λ-measurable functions from Ω to X which are p-Bochner integrable with its usual norm $\|f\|_p = (\int \|f(\omega)\|^p \, d\lambda(\omega))^{1/p}$. If X is the scalar field then $L^p(\lambda, X)$ will be denoted by $L^p(\lambda)$.

The relationship between Radon-Nikodym type properties for Banach spaces and operators with domain $L^1[0, 1]$ is classical in theory of vector-measures. Such connections have been investigated by several authors. In [17], Kaufman, Petrakis, Riddle and Uhl introduced and studied the notion of nearly representable operators (see definition below). They isolated the class of Banach spaces X for which every nearly representable operator with range X is representable. Such Banach spaces are said to have the Near Radon-Nikodym Property (NRNP). It was shown in [17] that every Banach lattice that does not contain any copy of c_0 has the NRNP; in particular L^1-spaces have the NRNP. A question that arises naturally from this fact is whether the Lebesgue-Bochner space $L^1(\lambda, X)$ has the NRNP whenever X does. Let us recall that the answers to similar questions about related properties such as the Radon-Nikodym property (RNP), the Analytic Radon-Nikodym property (ARNP) and the complete continuity property (CCP) are known for Bochner spaces (see [24], [9] and [20] respectively). We also remark that Hensgen [14] observed that (as in the scalar case) $L^1(\lambda, X)$ has the NRNP if X has the RNP.

In this paper, we show that the Near Radon-Nikodym property can indeed be lifted from a Banach space X to the space $L^1(\lambda, X)$. Our proof relies on a representation of operators from L^1 into $L^1(\lambda, X)$ due to Kalton [16] and properties of operator-valued measurable functions along with some well known characterization of integral and nuclear operators from L^∞ into a given Banach space.

Our notation is standard Banach space terminology as may be found in the books [6], [7] and [26].

Acknowledgements. The authors would like to thank Paula Saab for her constant interest in this work. The first author also would like to thank Neal Carothers for

Received September 24, 1996.
1991 Mathematics Subject Classification. Primary 46E40, 46G10; Secondary 28B05, 28B20.

creating an enjoyable work atmosphere at the Bowling Green State University where part of this work was done. We also would like the thank the referee for many valuable suggestions.

2. Definitions and preliminary results

Throughout this note, $I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n})$ is the sequence of dyadic intervals in [0, 1] and Σ_n is the σ-algebra generated by the finite sequence $(I_{n,k})_{1\leq k\leq 2^n}$. The word operator will always mean linear bounded operator and $\mathcal{L}(E, F)$ will stand for the space of all operators from E into F. For any given Banach space E, its closed unit ball will be denoted by E_1.

Definition 1. Let X be a Banach space. An operator T: $L^1[0, 1] \to X$ is said to be representable if there is a Bochner integrable function $g \in L^\infty([0, 1], X)$ such that T(f)=$\int fg\, dm$ for all f in $L^1[0, 1]$.

Definition 2. An operator D: $L^1[0, 1] \to X$ is called a Dunford-Pettis operator if D sends weakly compact sets into norm compact sets.

It is well known [7, Example 5-III-2.11] that all representable operators from $L^1[0, 1]$ are Dunford-Pettis; but the converse is not true in general.

Definition 3. An operator T: $L^1[0, 1] \to X$ is said to be *nearly representable* if for each Dunford-Pettis operator D: $L^1[0, 1] \to L^1[0, 1]$, the composition $T \circ D$ is representable.

The notion of nearly representable operators was introduced by Kaufman, Petrakis, Riddle and Uhl in [17]. It should be noted that since the class of Dunford-Pettis operators from $L^1[0, 1]$ into $L^1[0, 1]$ is a Banach lattice [3], if an operator $T \in \mathcal{L}(L^1[0, 1], X)$ fails to be nearly representable then one can find a positive Dunford-Pettis operator $D \in \mathcal{L}(L^1[0, 1], L^1[0, 1])$ such that $T \circ D$ is not representable.

The following definition isolates the main topic of this paper.

Definition 4. A Banach space X has the *Near Radon-Nikodym Property* (*NRNP*) if every nearly representable operator from $L^1[0, 1]$ into X is representable.

Examples of Banach spaces with the NRNP are spaces with the RNP, L^1-spaces, L^1/H^1. For more detailed discussion on the NRNP and nearly representable operators, we refer to [1], [11] and [17].

We now collect a few well known facts about operators from $L^1[0, 1]$ that we will need in the sequel. Our references for these facts are [2], [3] and [7].

FACT 1. *For a Banach space X, there is a one to one correspondence between the space of operators from $L^1[0,1]$ into X and all uniformly bounded X-valued martingales. This correspondence is given by*:

(*) $T(f) = \lim_{n\to\infty} \int \psi_n(t) f(t)\, dt$ *if* $(\psi_n)_n$ *is a uniformly bounded martingale.*
(**) $\psi_n(t) = 2^n \sum_{k=1}^{2^n} \chi_{I_{n,k}}(t)\ T(\chi_{I_{n,k}})$ *if* $T \in \mathcal{L}(L^1[0,1], X)$.

FACT 2. *A uniformly bounded X-valued martingale is Pettis-Cauchy if and only if the corresponding operator $T \in \mathcal{L}(L^1[0,1], X)$ is Dunford-Pettis.*

As an immediate consequence of Fact 2, we get:

FACT 3. *An operator $T \in \mathcal{L}(L^1[0,1], X)$ is nearly representable if and only if it maps uniformly bounded Pettis-Cauchy martingales to Bochner-Cauchy martingales.*

Definition 5. Let E and F be Banach spaces and suppose $T\colon E \to F$ is a bounded linear operator. The operator T is said to be an *absolutely summing operator* if there is a constant C such that for any finite sequence $(x_m)_{1\leq m\leq n}$ in E, the following holds:

$$\sum_{m=1}^{n} \|Tx_m\| \leq C \ \sup\left\{\sum_{m=1}^{n} |x^*(x_m)|;\ x^* \in E^*\ ;\ \|x^*\| \leq 1\right\}.$$

The least constant C for the inequality above to hold will be denoted by $\pi_1(T)$. It is well known that the class of all absolutely summing operators from E to F is a Banach space under the norm $\pi_1(T)$. This Banach space will be denoted by $\Pi_1(E, F)$.

Definition 6. We say that an operator $T\colon E \to F$ is an *integral operator* if it admits a factorization

$$\begin{array}{ccc} E & \xrightarrow{i\circ T} & F^{**} \\ \big\downarrow{\scriptstyle\alpha} & & \big\uparrow{\scriptstyle\beta} \\ L^\infty(\mu) & \xrightarrow{J} & L^1(\mu) \end{array}$$

where i is the inclusion from F into F^{**}, μ is a probability measure on a compact space K, J is the natural inclusion and α and β are bounded linear operators.

We define the integral norm $i(T) = \inf\{\|\alpha\| \cdot \|\beta\|\}$ where the infimum is taken over all such factorization. We denote by $I(E, F)$ the space of integral operators from E into F.

If $E = C(K)$ where K is a compact Hausdorff space or $E = L^\infty(\mu)$, then it is well known that T is absolutely summing (equivalently T is integral) if and only if its representing measure G (see [7], p. 152) is of bounded variation and in this case $\pi_1(T) = i(T) = |G|(K)$ where $|G|(K)$ denotes the total variation of G.

Definition 7. We say that an operator $T\colon E \to F$ is a *nuclear operator* if there exist sequences $(e_n^*)_n$ in E^* and $(f_n)_n$ in F such that $\sum_{n=1}^{\infty} \|e_n^*\|\ \|f_n\| < \infty$ and

$$T(e) = \sum_{n=1}^{\infty} e_n^*(e) f_n$$

for all $e \in E$.

We define the nuclear norm $n(T) = \inf\{\sum_{n=1}^{\infty} \|e_n^*\|\ \|f_n\|\}$ where the infimum is taken over all sequences $(e_n^*)_n$ and $(f_n)_n$ such that $T(e) = \sum_{n=1}^{\infty} e_n^*(e) f_n$ for all $e \in E$. We denote by $N(E, F)$ the space of all nuclear operators from E into F under the norm $n(\cdot)$.

FACT 4. *An operator $T \in \mathcal{L}(L^1[0,1], X)$ is representable if and only if its restriction to $L^\infty[0,1]$, $T|_{L^\infty[0,1]} \in \mathcal{L}(L^\infty[0,1], X)$ is nuclear.*

Throughout this paper, we will identify the two function spaces $L^p(\lambda, L^p(\mu, X))$ and $L^p(\lambda \otimes \mu, X)$ for $1 \leq p < \infty$ (see [10], p. 198).

The following representation theorem of Kalton [16] is essential for the proof of the main result. We denote by $\beta(K)$ the σ-algebra of Borel subsets of K in the statement of the theorem.

THEOREM 1 (KALTON [16]). *Suppose that*:

(i) *K is a compact metric space and μ is a Radon probability measure on K*;
(ii) *Ω is a Polish space and λ is a Radon measure on Ω*;
(iii) *X is a separable Banach space*;
(iv) *$T\colon L^1(\mu) \longrightarrow L^1(\lambda, X)$ is a bounded linear operator.*

Then there is a map $\omega \to T_\omega$ ($\Omega \to \Pi_1(C(K), X)$) such that for every $f \in C(K)$, the map $\omega \to T_\omega(f)$ is Borel measurable from Ω into X and:

(α) *If μ_ω is the representing measure of T_ω then*

$$\int_\Omega |\mu_\omega|(B)\, d\lambda(\omega) \leq \|T\| \mu(B) \quad \textit{for every } B \in \beta(K);$$

(β) *If $f \in L^1(\mu)$, then for λ a.e. ω, one has $f \in L^1(|\mu_\omega|)$*;
(γ) *$Tf(\omega) = T_\omega(f)$ for λ a.e. ω and for every $f \in L^1(\mu)$.*

The following proposition gives a characterization of representable operators in connection with Theorem 1.

PROPOSITION 1 [21]. *Under the assumptions of Theorem* 1, *the following two statements are equivalent*:

(i) *The operator T is representable*;
(ii) *For λ a.e. ω, μ_ω has a Bochner integrable density with respect to μ.*

For the next result, we need the following definition.

Definition 8. Let E and F be Banach spaces. A map $T\colon (\Omega, \Sigma, \lambda) \to \mathcal{L}(E, F)$ is said to be strongly measurable if $\omega \to T(\omega)e$ is measurable for every $e \in E$.

We observe that if E and F are separable Banach spaces and $T\colon (\Omega, \lambda) \to \mathcal{L}(E, F)$ with $\sup_\omega \|T(\omega)\| \leq 1$, then T is strongly measurable if and only if $T^{-1}(B)$ is λ-measurable for each Borel subset B of $\mathcal{L}(E, F)_1$ endowed with the strong operator topology.

The following selection result will be needed for the proof of the main theorem.

PROPOSITION 2. *Let X be a separable Banach space and $T\colon (\Omega, \lambda) \to \mathcal{L}(L^1[0, 1], X)$ be a strongly measurable map with*:

(1) $\|T(\omega)\| \leq 1$ *for every* $\omega \in \Omega$;
(2) $T(\omega)$ *is not nearly representable for* $\omega \in A$, $\lambda(A) > 0$.

Then one can choose a strongly measurable map $D\colon (\Omega, \lambda) \to \mathcal{L}(L^1[0, 1], L^1[0, 1])$ with the following properties:

(i) $\|D(\omega)\| \leq 1$ *for every* $\omega \in \Omega$;
(ii) $T(\omega) \circ D(\omega)$ *is not representable for every* $\omega \in A$;
(iii) $D(\omega)$ *is Dunford-Pettis for every* $\omega \in \Omega$;
(iv) $D(\omega)$ *is a positive operator for every* $\omega \in \Omega$.

We will need several steps for the proof.

LEMMA 1. *The space $\mathcal{L}(L^1[0, 1], X)_1$, the closed unit ball of the space $\mathcal{L}(L^1[0, 1], X)$ endowed with the strong operator topology is a Polish space.*

Proof. Let us consider the Polish space $\Pi_n\{X^{2^n}\}$. We will show that $\mathcal{L}(L^1[0, 1], X)_1$ is homeomorphic to a closed subspace of $\Pi_n\{X^{2^n}\}$.

Let C be the following subset of $\Pi_n\{X^{2^n}\}$: $(x_{n,k})_{k \leq 2^n; n \in \mathbb{N}}$ belongs to C if and only if

(a) $x_{n,k} = \frac{1}{2}(x_{n+1,2k-1} + x_{n+1,2k})$ for all $k \leq 2^n$ and $n \in \mathbb{N}$,
(b) $\|x_{n,k}\| \leq 1$ for all $k \leq 2^n$ and $n \in \mathbb{N}$.

It is evident that $\mathcal{C}$ is closed in $\Pi_n\{X^{2^n}\}$.

Consider the map Γ: $\mathcal{L}(L^1[0,1],X)_1 \to \Pi_n\{X^{2^n}\}$ given by $T \to (2^n T(\chi_{I_{n,k}}))_{k\leq 2^n, n\in\mathbb{N}}$.

The map Γ is clearly continuous, one to one and its range is contained in $\mathcal{C}$. We claim that $\Gamma(\mathcal{L}(L^1[0,1],X)_1) = \mathcal{C}$ and $\Gamma|_{\mathcal{C}}^{-1}$ is continuous: to see this claim, let $x = (x_{n,k}) \in \mathcal{C}$ and $T \in \mathcal{L}(L^1[0,1],X)$ defined by the martingale $\psi_n(t) = \sum_{k=1}^{2^n} x_{n,k}\chi_{I_{n,k}}(t)$. The operator T is well defined (see Fact 1) and $T(\chi_{I_{n,k}}) = (1/2^n)x_{n,k}$ so $\Gamma(T) = x$. Using the fact that the span of $\{\chi_{I_{n,k}}, k \leq 2^n, n \in \mathbb{N}\}$ is dense in $L^1[0,1]$, the continuity of $\Gamma|_{\mathcal{C}}^{-1}$ follows. The lemma is proved. □

Consider $\mathcal{L}(L^1[0,1],X)_1$ with the strong operator topology and $L^1([0,1], L^1[0,1])$ with the norm-topology.

The fact that the natural injection from $L^\infty([0,1], L^1[0,1])$ into $L^1([0,1], L^1[0,1])$ is a semi-embedding and the unit ball of $L^\infty([0,1], L^1[0,1])$ (that we will denote by Z) is a closed subset of the Polish space $L^1([0,1], L^1[0,1])$ implies that Z with the relative topology is a Polish space.

The space $\mathcal{L}(L^1[0,1],X)_1 \times Z^{\mathbb{N}}$ with the product topology is a Polish space.

Let $\mathcal{A}$ be the subset of $\mathcal{L}(L^1[0,1],X)_1 \times Z^{\mathbb{N}}$ defined as follows.

$\{T, (\phi_n)_n\} \in \mathcal{A}$ if and only if:

(i) $\mathbb{E}(\phi_{n+1}/\Sigma_n) = \phi_n$ for every $n \in \mathbb{N}$;

(ii) $\lim_{n,m} \sup_{g\in L^\infty, \|g\|_\infty \leq 1} \int |\int(\phi_m(t,s) - \phi_n(t,s))g(s)\,ds|\,dt = 0$;

(iii) $\lim_{j\to\infty} \sup_{n,m\geq j} \int \|T(\phi_n(t) - \phi_m(t))\|\,dt > 0$;

(iv) $\phi_n \geq 0$ as an element of the Banach lattice $L^\infty([0,1], L^1[0,1])$.

LEMMA 2. *The set $\mathcal{A}$ is a Borel subset of $\mathcal{L}(L^1[0,1],X)_1 \times Z^{\mathbb{N}}$.*

Proof. (i) Let $\mathcal{A}_1$ be the subset of $Z^{\mathbb{N}}$ given by $\phi = (\phi_n)_n \in \mathcal{A}_1$ if and only if

$$\mathbb{E}(\phi_{n+1}/\Sigma_n) = \phi_n \quad \forall n \in \mathbb{N}.$$

We claim that $\mathcal{A}_1$ is a Borel subset of $Z^{\mathbb{N}}$: if we denote by P_n the n^{th} projection of $Z^{\mathbb{N}}$ and $\mathbb{E}_n$ the conditional expectation with respect to Σ_n, then the map θ_n: $L^1([0,1], L^1[0,1])^{\mathbb{N}} \to L^1([0,1], L^1[0,1])$ given by $\theta_n(\phi) = (\mathbb{E}_n \circ P_{n+1} - P_n)(\phi)$ is continuous and therefore $\mathcal{A}_1 = \bigcap_{n\in\mathbb{N}} \theta_n^{-1}(\{0\}) \cap Z^{\mathbb{N}}$ is Borel measurable.

(ii) Let $g \in L^\infty$ be fixed. For every $m, n \in \mathbb{N}$, the map

$$L^1([0,1], L^1[0,1])^{\mathbb{N}} \longrightarrow \mathbb{R}$$

$$\phi \longrightarrow \int \left| \int (\phi_m(t,s) - \phi_n(t,s))g(s)\,ds \right| dt$$

is continuous so $\phi \to \Gamma_{n,m}(\phi) = \sup_{g\in L^\infty, \|g\|\leq 1} \int |\int \phi_m(t,s) - \phi_n(t,s))g(s)\,ds|\,dt$ is lower semi-continuous and therefore $\phi \to \Gamma(\phi) = \lim_{j\to\infty} \sup_{n,m\geq j} \Gamma_{n,m}(\phi)$ is

Borel measurable and

$$\mathcal{A}_2 = \left\{ \phi\colon \lim_{n,m} \sup_{g \in L^\infty, \|g\| \le 1} \int \left| \int (\phi_m(t,s) - \phi_n(t,s)) g(s)\, ds \right| dt = 0 \right\} \cap Z^{\mathbb{N}}$$

is a Borel measurable subset of $Z^{\mathbb{N}}$.

(iii) For each n and m in $\mathbb{N}$, the map

$$\begin{aligned} \theta_{n,m}\colon \mathcal{L}(L^1[0,1], X)_1 \times L^1([0,1], L^1[0,1])^{\mathbb{N}} &\longrightarrow \mathbb{R} \\ (T, \phi) &\longrightarrow \int \|T(\phi_n(t)) - T(\phi_m(t))\|\, dt \end{aligned}$$

is continuous and then the set $\mathcal{B} = \{(T,\phi);\ \limsup_{n,m} \theta_{n,m}(T,\phi) > 0\}$ is a Borel measurable subset of $\mathcal{L}(L^1[0,1], X)_1 \times L^1([0,1], L^1[0,1])^{\mathbb{N}}$.

(iv) The set $\mathcal{P}$ of sequences of positive functions is a closed subspace of $Z^{\mathbb{N}}$.

Now $\mathcal{A} = \mathcal{B} \cap \{\mathcal{L}(L^1[0,1], X)_1 \times (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{P})\}$ so $\mathcal{A}$ is Borel measurable. The lemma is proved. □

Proof of Proposition 2. Let U be the restriction on $\mathcal{A}$ of the first projection. The set $U(\mathcal{A})$ is an analytic subset of $\mathcal{L}(L^1[0,1], X)_1$ and by Theorem 8.5.3 of [5], there is a universally measurable map $\theta\colon U(\mathcal{A}) \to Z^{\mathbb{N}}$ such that the graph of θ is contained in $\mathcal{A}$.

By assumption, $T\colon (\Omega, \lambda) \to \mathcal{L}(L^1([0,1], X)_1$ is measurable for the strong operator topology and $T(\omega) \in U(\mathcal{A})$ for every $\omega \in A$. So the map

$$\begin{aligned} \Omega &\longrightarrow L^1([0,1], L^1[0,1])^{\mathbb{N}} \\ \omega &\longrightarrow \begin{cases} \theta(T(\omega)) & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is well defined. The above map is the composition of the measurable map $T(\cdot)$ with the universally measurable map $\theta(\cdot)$ so it is λ-measurable. Moreover for every $\omega \in A$, $\{T(\omega), \theta(T(\omega))\}$ belongs to $\mathcal{A}$.

For every $n \in \mathbb{N}$, let Q_n be the n^{th} projection from $Z^{\mathbb{N}}$ onto Z and set $\phi_n(\omega) = Q_n(\theta(T(\omega))$. By construction, the sequence $(\phi_n(\omega))_n$ is a uniformly bounded $L^1[0,1]$-valued martingale so it defines an operator from $L^1[0,1]$ into $L^1[0,1]$ by

$$D(\omega)(f) = \lim_{n\to\infty} \int \phi_n(\omega)(t) f(t)\, dt.$$

Notice that for every $f \in L^1[0,1]$, the map $M_f\colon Z \to L^1([0,1], L^1[0,1])$ defined by $M_f(h) = f.h$ is continuous and $D(\omega)(f) = \lim_{n\to\infty} \int M_f(Q_n(\theta(T(\omega)))\, dt$. The measurability of the map $\theta(T(\cdot))$ and the continuity of M_f and Q_n show that the map $\omega \to D(\omega)(f)$ $(\Omega \to L^1[0,1])$ is measurable. Now condition (iii) implies that $T(\omega) \circ D(\omega)$ is not representable for $\omega \in A$ and condition (iv) insures that $D(\omega) \geq 0$ for every $\omega \in \Omega$. □

The following proposition is crucial for the proof of our main result and could be of independent interest.

PROPOSITION 3. *Let* $\omega \to D(\omega)$ $(\Omega \to \mathcal{L}(L^1[0,1], L^1[0,1])_1)$ *be a strongly measurable map such that* $D(\omega)$ *is positive and Dunford-Pettis for every* $\omega \in \Omega$. *If we denote by* $\theta(\omega)$ *the restriction of* $D(\omega)$ *on* $L^\infty[0,1]$, *then* $\omega \to \theta(\omega)$ *is norm-measurable as a map from* Ω *into* $I(L^\infty[0,1], L^1[0,1])$.

We will begin by proving the following simple lemma.

LEMMA 3. *Let* $D\colon L^1[0,1] \to L^1[0,1]$ *be a positive Dunford-Pettis operator and* $\theta = D|_{L^\infty}$. *Then* θ *is compact integral and is weak* to weakly continuous. Moreover* $i(\theta) = \|\theta\|$.

Proof. The fact that θ is compact integral is trivial. For the weak* to weak continuity, we observe that $\theta^*(L^\infty[0,1]) \subset L^1[0,1]$. For the identity of the norms, we will use the fact that $i(\theta)$ is equal to the total variation of the representing measure of θ.

Let G be the representing measure of θ and π be a finite measurable partition of $[0,1]$. We have

$$\begin{aligned}
\sum_{A\in\pi} \|G(A)\|_{L^1} &= \sum_{A\in\pi} \|D(\chi_A)\| \\
&\le \sum_{A\in\pi} \|\, |D|(\chi_A)\, \| \\
&= \sum_{A\in\pi} \|\, |\theta|(\chi_A)\, \| \\
&= \sum_{A\in\pi} \int |\theta|(\chi_A)(t)\, dt \\
&= \int |\theta|(\chi_{[0,1]})(t)\, dt \le \|\, |\theta|\, \|
\end{aligned}$$

where $|D|$ and $|\theta|$ denote the modulus of D and θ respectively (see [18]). So by taking the supremum over all finite measurable partitions of [0,1], we get $i(\theta) \le \|\, |\theta|\, \|$ and since θ is a positive operator, $|\theta| = \theta$. The lemma is proved. □

Proof of Proposition 3. Notice that $\theta(\omega) \in K_{w^*}(L^\infty[0,1], L^1[0,1])$ for every $\omega \in \Omega$ where $K_{w^*}(L^\infty[0,1], L^1[0,1])$ denotes the space of compact operators from $L^\infty[0,1]$ into $L^1[0,1]$ that are weak* to weakly continuous. So $\omega \to \theta(\omega)$ is strongly measurable and is separably valued ($K_{w^*}(L^\infty[0,1], L^1[0,1]) = L^1[0,1]\widehat{\otimes}_\epsilon L^1[0,1]$ where $\widehat{\otimes}_\epsilon$ is the injective tensor product). By the Pettis measurability theorem (see Theorem II-1.2 of [7]), the map $\omega \to \theta(\omega)$ is measurable for the norm operator topology.

For each $n \in \mathbb{N}$, let $\mathbb{E}_n$ be the conditional expectation operator with respect to Σ_n. The sequence $(\mathbb{E}_n)_n$ satisfies the following properties: $(\mathbb{E}_n)_n$ is a sequence of finite rank operators in $\mathcal{L}(L^1[0,1], L^1[0,1])_1$, $\mathbb{E}_n \geq 0$ for every $n \in \mathbb{N}$ and $(\mathbb{E}_n)_n$ converges to the identity operator I for the strong operator topology. Consider $S_n = \mathbb{E}_n \wedge I$. Since $S_n \leq \mathbb{E}_n$ and $\mathbb{E}_n$ is integral (it is of finite rank), one can deduce from Grothendieck's characterization of integral operators with values in $L^1[0,1]$ (for instance, see [7], p. 258) that S_n is also integral.

SUBLEMMA. *For each $n \in \mathbb{N}$, there exists $K_n \in \operatorname{conv} S_n, S_{n+1}, \ldots\}$ such that the sequence $(K_n)_n$ converges to I for the strong operator topology.*

For this, we first observe that $(S_n(f))_n$ converges weakly to f for every $f \in L^1[0,1]$; in fact, if $f \geq 0$ and $n \in \mathbb{N}$ then $S_n(f) = \inf\{\mathbb{E}_n(g) + (f-g);\ 0 \leq g \leq f\}$. Choose $0 \leq g_n \leq f$ such that $\|S_n(f) - (\mathbb{E}_n(g_n) + (f - g_n))\|_1 \leq 1/n$. Since $[0, f]$ is weakly compact, we can assume (by taking a subsequence if necessary) that $(g_n)_n$ converges weakly to a function g. To conclude that $S_n(f)$ converges weakly, notice that if $\varphi \in L^\infty[0,1]$ then $\lim_{n\to\infty} \mathbb{E}_n^*(\varphi) = \varphi$ a.e. ($\mathbb{E}_n^* = \mathbb{E}_n$). So for every $n \in \mathbb{N}$, $|\langle S_n(f) - f, \varphi\rangle| \leq 1/n + |\langle \mathbb{E}_n(g_n) - g_n, \varphi\rangle|$ and

$$|\langle \mathbb{E}_n(g_n) - g_n, \varphi\rangle| = |\langle g_n, \mathbb{E}_n(\varphi) - \varphi\rangle| \leq \langle f, |\mathbb{E}_n(\varphi) - \varphi|\rangle.$$

By the Lebesgue dominated convergence theorem, we have $\lim_{n\to\infty}\langle \mathbb{E}_n(g_n) - g_n, \varphi\rangle = 0$. Now fix $(f_k)_k$, a countable dense subset of the closed unit ball of $L^1[0,1]$. For $k = 1$, by Mazur's theorem we can choose a sequence $(S_n^{(1)})_n$ with $S_n^{(1)} \in \operatorname{conv}\{S_n, S_{n+1}, \ldots\}$ for every $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} \|S_n^{(1)}(f_1) - f_1\| = 0$. By induction, one can use the same argument to construct $S_n^{(k+1)} \in \operatorname{conv}\{S_n^{(k)}, S_{n+1}^{(k)}, \ldots\}$ such that $\lim_{n\to\infty} \|S_n^{(k+1)}(f_j) - f_j\| = 0$ for every $j \leq (k+1)$. From Lemma 1 of [23], one can fix a sequence $(K_n)_n$ such that for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that for $n \geq n_k$, $K_n \in \operatorname{conv}\{S_n^{(k)}, S_{n+1}^{(k)}, \ldots\}$. From this, it is clear that $\lim_{n\to\infty} \|K_n(f_k) - f_k\| = 0$ for every $k \in \mathbb{N}$ and since $(f_k)_k$ is dense and $\sup_n \|K_n\| \leq 1$, $(K_n)_n$ verifies the requirements of the sublemma.

To complete the proof of the proposition, let $(K_n)_n$ be as in the above sublemma and consider C_n: $K_{w^*}(L^\infty[0,1], L^1[0,1]) \to I(L^\infty[0,1], L^1[0,1])$ $(T \to K_n \circ T)$. Since K_n is integral, the map C_n is well defined and is clearly continuous. Therefore $\omega \to K_n \circ \theta(\omega)$ is measurable for the integral norm. Since (K_n) converges to I for the strong operator topology and $\theta(\omega)$ is compact, then $\lim_{n\to\infty} \|K_n \circ \theta(\omega) - \theta(\omega)\| = 0$. Observe that $K_n \circ \theta(\omega) \leq \theta(\omega)$ for every $\omega \in \Omega$ and for every $n \in \mathbb{N}$. We conclude from Lemma 3 that $i(\theta(\omega) - K_n \circ \theta(\omega)) = \|\theta(\omega) - K_n \circ \theta(\omega)\|$ and hence for a.e. $\omega \in \Omega$,

$$\lim_{n\to\infty} i(\theta(\omega) - K_n \circ \theta(\omega)) = 0.$$

Since the $K_n \circ \theta(\cdot)$'s are measurable so is $\theta(\cdot)$, and the proposition is proved. $\square$

The following proposition is probably known but we do not know of any specific reference.

PROPOSITION 4. *Let X be a Banach space and S: $(\Omega, \lambda) \to \mathcal{L}(L^1[0,1], X)$ be a strongly measurable map with* $\sup_\omega \|S(\omega)\| \leq 1$. *Then the following assertions are equivalent*:

(a) *The operator H: $L^1(\Omega \times [0,1], \lambda \otimes m) \to X$ given by*

$$H(f) = \int_\Omega S(\omega)(f(\omega, \cdot))\, d\lambda(\omega)$$

is representable;

(b) *The operator K: $L^1[0,1] \to L^1(\lambda, X)$ given by $K(g) = S(\cdot)g$ is representable*;

(c) *$S(\omega)$ is representable for a.e. $\omega \in \Omega$.*

Proof. (a) $\Rightarrow$ (b) If H is representable, then we can find an essentially bounded measurable map ψ: $\Omega \times [0,1] \to X$ that represents H. The map ψ': $[0,1] \to L^1(\lambda, X)$ given by $t \to \psi(\cdot, t)$ belongs to $L^\infty([0,1], L^1(\lambda, X))$; in fact $\|\psi'(t)\| = \int_\Omega \|\psi(\omega, t)\|\, d\lambda(\omega)$ for every $t \in [0,1]$. Hence $\|\psi'\|_\infty \leq \|\psi\|_\infty$ and we claim that ψ' represents K. For each $g \in L^1[0,1]$, $\{\int \psi'(t)g(t)\, dt\}(\omega) = \int \psi(\omega, t)g(t)\, dt$ for a.e. ω. For every measurable subset A of Ω,

$$\begin{aligned} \int_A Kg(\omega)\, d\lambda(\omega) &= H(\chi_A \otimes g) \\ &= \int\int \psi(\omega, t)g(t)\chi_A(\omega)\, dt\, d\lambda(\omega) \\ &= \int_A \left\{\int \psi'(t)g(t)\, dt\right\}(\omega)\, d\lambda(\omega) \end{aligned}$$

which shows that $Kg = \int \psi'(t)g(t)\, dt$.

(b) $\Leftrightarrow$ (c) Let $\mu_\omega \in M([0,1], X)$ be the representing measure for $S(\omega)$ (i.e., $S(\omega)(\chi_A) = \mu_\omega(A)$). It is well known that $S(\omega)$ is representable if and only if μ_ω has a Bochner density with respect to dt. Notice now that $K(g)(\omega) = S(\omega)(g) = \int g(t)\, d\mu_\omega(t)$. Hence, by the uniqueness of the representation of Theorem 1 (see [16], p. 316), the family $(\mu_\omega)_\omega$ represents K. Apply now Proposition 1 to conclude the equivalence.

(b) $\Rightarrow$ (a) If ψ': $[0,1] \to L^1(\lambda, X)$ represents K, then there is a map Γ: $\Omega \times [0,1] \to X$ so that $\Gamma \in L^1(\lambda \otimes m, X)$ and $\Gamma(\cdot, t) = \psi'(t)$ for a.e. $t \in [0,1]$ (see [10], p. 198). We claim that $\Gamma \in L^\infty(\lambda \otimes m, X)$ and represents H. To prove this claim, fix A a measurable subset of Ω and I a measurable subset of $[0,1]$. We have the following:

$$\begin{aligned} H(\chi_A \otimes \chi_I) &= \int_\Omega K(\chi_I)\chi_A \, d\lambda(\omega) \\ &= \int_A \Big(\int_I \psi'(t)\, dm(t)\Big)(\omega)\, d\lambda(\omega) \\ &= \iint_{A\times I} \Gamma(\omega, t)\, d(\lambda\otimes m)(\omega, t). \end{aligned}$$

This implies that $H(\chi_V) = \iint_V \Gamma(\omega, t)\, d(\lambda \otimes m)(\omega, t)$ for every Borel subset V of $\Omega \times [0, 1]$. Apply now Lemma 4-III of [7] to conclude that H is representable. □

3. Main result

THEOREM 2. *Let X be a Banach space and $(\Omega, \Sigma, \lambda)$ a finite measure space. Then $L^1(\lambda, X)$ has the NRNP if and only if X does.*

For the proof, let us assume without loss of generality that X is separable, Ω is a compact metric space and λ is a Radon measure in the Borel σ-algebra Σ of Ω. For what follows, J_X denotes the natural inclusion from $L^\infty(\lambda, X)$ into $L^1(\lambda, X)$.

We will begin with the proof of the following special case.

PROPOSITION 5. *Let X be a Banach space with the NRNP and let $T\colon L^1[0, 1] \to L^\infty(\lambda, X)$ be a bounded linear operator. Then $J_X \circ T$ is representable if and only if it is nearly representable.*

Proof. Let $T\colon L^1[0, 1] \to L^\infty(\lambda, X)$ be a bounded operator with $\|T\| \leq 1$. By Lemma 1 of [20], there exists a strongly measurable map $\omega \to T(\omega)$ $(\Omega \to \mathcal{L}(L^1[0, 1], X)_1)$ such that $Tf(\cdot) = T(\cdot)f$ for every $f \in L^1[0, 1]$.

Assume that $J_X \circ T$ is nearly representable but not representable. Proposition 4 asserts that there exists a measurable subset A of Ω with $\lambda(A) > 0$ and such that $T(\omega)$ is not representable for each $\omega \in A$. Since X has the NRNP, the operator $T(\omega)$ is not nearly representable for each $\omega \in A$. Using our selection result (Proposition 2), one can choose a strongly measurable map $\omega \to D(\omega)$ $(\Omega \to \mathcal{L}(L^1[0, 1], L^1[0, 1])_1)$ such that $D(\omega)$ is positive, Dunford-Pettis for every $\omega \in \Omega$ and $T(\omega) \circ D(\omega)$ is not representable for every $\omega \in A$. It should be noted that if $D \in \mathcal{L}(L^1[0, 1], L^1[0, 1])$ is a Dunford-Pettis operator, and since $J_X \circ T$ is nearly representable, $T(\omega) \circ D$ is representable for a.e. $\omega \in \Omega$ (see Proposition 4). However the exceptional set may depend on the operator D.

As before, let $\theta(\omega) = D(\omega)|_{L^\infty}$. We deduce from Proposition 3 that the map $\omega \to \theta(\omega)$ $(\Omega \to I(L^\infty[0, 1], L^1[0, 1]))$ is norm-measurable.

Let $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of finite measurable partition of Ω such that Π_{n+1} is finer than Π_n for every $n \in \mathbb{N}$ and Σ is generated by $\bigcup_{n\in\mathbb{N}}\{B\ ;\ B \in \Pi_n\}$.

For each $B \in \Sigma$, we denote by D_B the operator defined by

$$D_B(f) = \int_B D(\omega)(f)\, d\lambda(\omega) \quad \text{for every } f \in L^1[0,1]$$

and let

$$D_n(\omega) = \sum_{B\in\Pi_n} \frac{D_B}{\lambda(B)}\, \chi_B(\omega).$$

The operator D_B is a Dunford-Pettis operator for each $B \in \Sigma$ (see [25] Theorem 1.3) and therefore $D_n(\omega)$ is Dunford-Pettis for each $n \in \mathbb{N}$ and $\omega \in \Omega$.

Claim. The operator $T(\omega) \circ D_n(\omega)$ is representable for a.e. $\omega \in \Omega$.

To see this claim, notice that $T(\omega) \circ D_B$ is representable for a.e. $\omega \in \Omega$. Fix a set N_B with $\lambda(N_B) = 0$ and such that $T(\omega) \circ D_B$ is representable for $\omega \notin N_B$. Let $N = \bigcup_{n\in\mathbb{N}} \bigcup_{B\in\Pi_n} N_B$. Clearly $\lambda(N) = 0$ and for $\omega \notin N$, $T(\omega) \circ D_n(\omega) = \sum_{B\in\Pi_n} \frac{T(\omega)\circ D_B}{\lambda(B)}\, \chi_B(\omega)$ is representable.

Now if we denote by θ_n (resp. θ_B) the restriction on $L^\infty[0,1]$ of D_n (resp. D_B), we have

$$\theta_n(\omega) = \sum_{B\in\Pi_n} \frac{\theta_B}{\lambda(B)} \chi_B(\omega)$$

for each $\omega \in \Omega$, and since $\theta(\cdot)$ is measurable for the integral norm (see Proposition 3), we have

$$\theta_n(\omega) = \sum_{B\in\Pi_n} \frac{\text{Bochner} - \int_B \theta(s)\, d\lambda(s)}{\lambda(B)}\, \chi_B(\omega).$$

It is well known (for instance, see [7], Corollary V-2) that $\theta_n(\cdot)$ converges (for the integral norm) to $\theta(\cdot)$ a.e. Now since $T(\omega) \circ D_n(\omega)$ is representable for a.e. ω, the operator $T(\omega) \circ \theta_n(\omega)$ is nuclear for a.e. ω and since $\theta_n(\omega)$ converges a.e. to $\theta(\omega)$ for the integral norm, we have

$$\lim_{n\to\infty} i\,(T(\omega) \circ \theta_n(\omega) - T(\omega) \circ \theta(\omega)) = 0 \quad \text{for a.e. } \omega \in \Omega.$$

As a result, the representing measure of the operator $T(\omega) \circ \theta(\omega)$ is the limit for the total variation norm of a sequence of measures with Bochner integrable densities hence $T(\omega) \circ \theta(\omega)$ is nuclear for a.e. $\omega \in \Omega$ and this is equivalent to $T(\omega) \circ D(\omega)$ being representable for a.e. $\omega \in \Omega$. Contradiction. □

For the general case, let $T\colon L^1[0,1] \to L^1(\lambda, X)$ be a nearly representable operator and fix a strongly Borel measurable map $\omega \to T_\omega$ $(\Omega \to \Pi_1(C[0,1], X))$ as in

Theorem 1. Let us denote by μ_ω the representing measure of T_ω. Our goal is to show that for λ a.e. ω, μ_ω has a Bochner integrable density with respect to the Lebesgue measure m in $[0, 1]$. This will imply that T is representable by Proposition 1. To do that, we need to establish several steps:

LEMMA 4. *For λ a.e. ω in Ω, we have $|\mu_\omega| \ll m$.*

Proof. Note that for each $x^* \in X^*$, the map $\omega \to x^*\mu_\omega$ $(\Omega \to M[0, 1])$ is weak* measurable so it defines an operator T^{x^*}: $L^1[0, 1] \to L^1(\lambda)$. The operator T^{x^*} is nearly representable; in fact it is the composition of the nearly representable operator T with the operator V^{x^*}: $L^1(\lambda, X) \to L^1(\lambda)$ $(f \to x^*f)$. Since $L^1(\lambda)$ has the NRNP, the operator T^{x^*} is a representable operator and therefore $|x^*\mu_\omega| \ll m$ for λ a.e. ω (Proposition 1 of [12]).

Now, using the same argument as in Lemma 2 of [20], we have the conclusion of the lemma. □

As a consequence of Lemma 4, there exists a measurable subset, Ω', of Ω with $\lambda(\Omega \setminus \Omega') = 0$ and such that for each $\omega \in \Omega'$, $|\mu_\omega| \ll m$. Let $g_\omega \in L^1[0, 1]$ be the Radon-Nikodym density of $|\mu_\omega|$ with respect to m for $\omega \in \Omega'$ and $g_\omega = 0$ for $\omega \in \Omega \setminus \Omega'$. By (α) of Theorem 1, we have the following: for every I measurable subset of $[0, 1]$, the map $\omega \to |\mu_\omega|(I) = \int_I g_\omega(t)\, dt$ is measurable so one can deduce from the Pettis-measurability theorem that $\omega \to g_\omega$ $(\Omega \to L^1[0, 1])$ is norm-measurable. Moreover, $\int_\Omega \|g_\omega\|\, d\lambda(\omega) \le \|T\|$. From this, one can find a function $\Gamma \in L^1(\lambda \otimes m)$ with $\Gamma(\omega, \cdot) = g_\omega$ for λ a.e. $\omega \in \Omega$.

Let V_n be the measurable subset of $\Omega \times [0, 1]$ given by

$$V_n = \{(\omega, t);\ n - 1 \le \Gamma(\omega, t) < n\}.$$

The V_n 's are clearly disjoint and $\Omega \times [0, 1] = \bigcup_n V_n$.

Notice that for $\omega \in \Omega$, $\chi_{V_n}(\omega, \cdot)\Gamma(\omega, \cdot) \in L^\infty[0, 1]$ and therefore for every $h \in L^1[0, 1]$, $\chi_{V_n}(\omega, \cdot)h(\cdot)\Gamma(\omega, \cdot) \in L^1[0, 1]$; that is, $\chi_{V_n}(\omega, \cdot)h(\cdot) \in L^1(|\mu_\omega|)$. Hence the following map is well defined:

$$k_n\colon \Omega \longrightarrow \mathcal{L}(L^1[0, 1], X)$$

$$\omega \longrightarrow \begin{cases} k_n(\omega)(h) = \int \chi_{V_n}(\omega, t)h(t)d\mu_\omega(t) & \text{if } \omega \in \Omega' \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\|k_n(\omega)\| \le n$ for every ω.

Claim. The map $\omega \to k_n(\omega)$ is strongly measurable.

Since X is separable, it is enough to show that for every $f \in L^1[0, 1]$ and $x^* \in X^*$, the map $\omega \to \langle k_n(\omega)(f), x^*\rangle$ is measurable.

Let h_ω: $[0, 1] \to X^{**}$ be a weak*-density of μ_ω with respect to m for $\omega \in \Omega'$ and 0 otherwise. The map $\omega \to \langle h_\omega(\cdot), x^*\rangle$ belongs to $L^1(\lambda, L^1[0, 1])$. Let $h^{x^*} \in L^1(\Omega \times [0, 1])$ so that for a.e. $\omega \in \Omega$, $h^{x^*}(\omega, \cdot) = \langle h_\omega(\cdot), x^*\rangle$. Now the map $(\omega, t) \to \chi_{V_n}(\omega, t)h^{x^*}(\omega, t)$ $(\Omega \times [0, 1] \to \mathbb{R})$ is measurable and for every $f \in L^1[0, 1]$,

$$\langle k_n(\omega)(f), x^*\rangle = \int_I \chi_{V_n}(\omega, t)\langle h_\omega(t), x^*\rangle\, dt = \int_I \chi_{V_n}(\omega, t)h^{x^*}(\omega, t)f(t)\, dt.$$

This shows that $\omega \to \langle k_n(\omega)(f), x^*\rangle$ is measurable.

Let us now define an operator $T^{(n)}$: $L^1[0, 1] \to L^\infty(\lambda, X)$ by $T^{(n)}(f) = k_n(\cdot)(f)$ for every $f \in L^1[0, 1]$.

LEMMA 5. *For every $n \in \mathbb{N}$, the operator $J_X \circ T^{(n)}$ is nearly representable.*

Proof. Fix a Dunford Pettis operator D and let $\gamma_k^{(n)} = \sum_{j=1}^{j_k} f_{j,k} \otimes h_{j,k}$ be an approximating sequence for χ_{V_n} in $L^1(\Omega \times L^1[0, 1])$ with $0 \le \gamma_k^{(n)} \le \chi_{V_n}$ for every $k \in \mathbb{N}$ (see [10], p. 198). Consider the sequence of operators $T_k^{(n)}$: $L^1[0, 1] \to L^1(\lambda, X)$ defined by

$$T_k^{(n)}(f)(\omega) = \int \gamma_k^{(n)}(\omega, t)f(t)\, d\mu_\omega(t).$$

We claim that the operator $T_k^{(n)}$ is nearly representable. Indeed, if we denote by $M_{f_{j,k}}$ and $M_{h_{j,k}}$ the multiplication by $f_{j,k}$ and $h_{j,k}$ respectively, we have $T_k^{(n)} = \sum_{j=1}^{j_k} M_{f_{j,k}} \circ T \circ M_{h_{j,k}}$. For that, let $f \in L^1[0, 1]$; for a.e. $\omega \in \Omega$,

$$\begin{aligned}\left(\sum_{j=1}^{j_k} M_{f_{j,k}} \circ T \circ M_{h_{j,k}}\right)(f)(\omega) &= \sum_{j=1}^{j_k} f_{j,k}(\omega)\, T(h_{j,k}.f)(\omega)\\ &= \sum_{j=1}^{j_k} f_{j,k}(\omega) \int h_{j,k}(t)f(t)\, d\mu_\omega(t)\\ &= \int \left(\sum_{j=1}^{j_k} f_{j,k}(\omega)h_{j,k}(t)f(t)\right) d\mu_\omega(t)\\ &= \int \gamma_k^{(n)}(\omega, t)f(t)\, d\mu_\omega(t).\end{aligned}$$

Now since for every $j \le j_k$, $M_{f_{j,k}} \circ T \circ M_{h_{j,k}} \circ D$ is representable, so is $T_k^{(n)} \circ D$. To conclude the proof of the lemma, let $\omega \to \nu_{k,\omega}^D$ and $\omega \to \nu_\omega^D$ be the representation given by Theorem 1 of $T_k^{(n)} \circ D$ and $J_X \circ T^{(n)} \circ D$ respectively. We have

$$\begin{aligned}
&\int |\nu_{k,\omega}^D - \nu_\omega^D| \, d\lambda(\omega) \\
&\quad = \int_\Omega \sup_{l \in \mathbb{N}} \sum_{m=1}^{2^l} \|\nu_{k,\omega}^D(I_{l,m}) - \nu_\omega^D(I_{l,m})\| \, d\lambda(\omega) \\
&\quad = \int_\Omega \sup_{l \in \mathbb{N}} \sum_{m=1}^{2^l} \| \int \left(\gamma_k^{(n)}(\omega, t) - \chi_{V_n}(\omega, t)\right) D(\chi_{I_{l,m}})(t) d\mu_\omega(t)\| \, d\lambda(\omega) \\
&\quad \leq \int\int |\gamma_k^{(n)}(\omega, t) - \chi_{V_n}(\omega, t)| \, |D|(\chi_{[0,1]})(t) \, \Gamma(\omega, t) \, dt \, d\lambda(\omega)
\end{aligned}$$

where $|D|$ is the modulus of D (see [18]). Notice that since $0 \leq \gamma_k^{(n)} \leq \chi_{V_n}$, we have

$$\begin{aligned}
|\gamma_k^{(n)}(\omega, t) - \chi_{V_n}(\omega, t)| \, |D|(\chi_{[0,1]})(t) \, \Gamma(\omega, t) &\leq 2 \, \chi_{V_n}(\omega, t) \, |D|(\chi_{[0,1]})(t) \, \Gamma(\omega, t) \\
&\leq 2n|D|(\chi_{[0,1]})(t).
\end{aligned}$$

And by the Lebesgue dominated convergence theorem, $\lim_{k\to\infty} \int |\nu_{k,\omega}^D - \nu_\omega^D| \, d\lambda(\omega) = 0$ and hence by passing to a subsequence (if necessary), we may assume that $\lim_{k\to\infty} |\nu_{k,\omega}^D - \nu_\omega^D| = 0$ for a.e. $\omega \in \Omega$.

Fix B_0 a subset of Ω with $\lambda(B_0) = 0$ and for every $\omega \notin B_0$, $\lim_{k\to\infty} |\nu_{k,\omega}^D - \nu_\omega^D| = 0$. Since $T_k^{(n)} \circ D$ is representable, one can find a subset B_k of Ω with $\lambda(B_k) = 0$ and such that for each $\omega \notin B_k$, $\nu_{k,\omega}^D$ has a Bochner integrable density. We can conclude that for $\omega \notin \bigcup_{k=0}^\infty B_k$, the measure ν_ω^D is the limit for the variation norm of a sequence of measures with Bochner integrable densities and therefore it has a Bochner integrable density. Now using Proposition 1, the operator $J_X \circ T^{(n)} \circ D$ is representable. The lemma is proved. □

We are now ready to complete the proof of the theorem. By Proposition 5, the operator $J_X \circ T^{(n)}$ is representable and therefore the operator $K_n \colon L^1(\Omega \times [0, 1]) \to X$ given by $K_n(f) = \int k_n(\omega)(f(\omega, \cdot)) \, d\lambda(\omega)$ is representable (see Proposition 4).

Let $\phi_n \colon \Omega \times [0, 1] \to X$ be a representation of K_n and consider $\varphi = \sum_{n=1}^\infty \phi_n \, \chi_{V_n}$. We claim that φ belongs to $L^1(\Omega \times [0, 1], X)$.

For that, fix $\alpha_\omega \colon [0, 1] \to X^{**}$ a weak*- density of μ_ω with respect to $|\mu_\omega|$(see [8] or [15]). The map α_ω satisfies:

(1) $\|\alpha_\omega(t)\| = 1$ $|\mu_\omega|$ a.e.;
(2) For every $x^* \in X^*$, $\langle x^*, \int f \, d\mu_\omega \rangle = \int \langle x^*, \alpha_\omega(t) \rangle f(t) \, d|\mu_\omega|(t)$.

It follows that for all $\lambda \otimes m$-measurable subsets V,

$$K_n(\chi_V) = \text{weak}^* - \int\int_V \chi_{V_n}(\omega, t) \, \alpha_\omega(t) \, \Gamma(\omega, t) \, dt \, d\lambda(\omega).$$

Since K_n is represented by ϕ_n, we have

$$K_n(\chi_V) = \int\int_V \phi_n(\omega, t)\, d\lambda(\omega)\, dt.$$

So if we denote by G_n the representing measure of the operator K_n, we have

$$\|\phi_n\| = |G_n|(\Omega \times [0,1]) = \int\int \chi_{V_n}(\omega, t)\Gamma(\omega, t) d\lambda \otimes m(\omega, t).$$

From this it follows that φ is Bochner integrable.

For every $\lambda \otimes m$-measurable subset V, we get

$$\begin{aligned}
\int\int_V d\mu_\omega(t)\, d\lambda(\omega) &= \sum_{n=1}^{\infty} \int\int_V \chi_{V_n}(\omega, t)\, d\mu_\omega(t)\, d\lambda(\omega) \\
&= \sum_{n=1}^{\infty} K_n(\chi_V) = \sum_{n=1}^{\infty} K_n(\chi_V \cdot \chi_{V_n}) \\
&= \sum_{n=1}^{\infty} \int\int_V \phi_n(\omega, t)\chi_{V_n}(\omega, t)\, dt\, d\lambda(\omega) \\
&= \int\int_V \varphi(\omega, t)\, dt\, d\lambda(\omega).
\end{aligned}$$

In particular, for every $A \in \Sigma_m$ and $B \in \Sigma_\lambda$,

$$\int_B \mu_\omega(A)\, d\lambda(\omega) = \int_B \left\{ \int_A \varphi(\omega, t)\, dt \right\} d\lambda(\omega)$$

which shows that $\mu_\omega(A) = \int_A \varphi(\omega, t)\, dt$ for a.e. ω. The theorem is proved. □

Before stating the next extension, let us recall (as in [23]) that, if E is a Köthe function space on $(\Omega, \Sigma, \lambda)$ (in the sense of [18]) and X is a Banach space then $E(X)$ will be the space of all (classes of) measurable map $f\colon \Omega \to X$ so that $\omega \to \|f(\omega)\|$ belongs to E.

COROLLARY. *If E does not contain a copy of c_0 and X has the NRNP, then E(X) has the NRNP.*

Proof. Without loss of generality, we may assume that E is order continuous, $(\Omega, \Sigma, \lambda)$ is a separable probability space (see [18]) and the Banach space X is separable. By a result of Lotz, Peck and Porta [19], the inclusion map from E into $L^1(\lambda)$ is a semi-embedding. The same is true for the inclusion $J_X\colon E(X) \to L^1(\lambda, X)$ (see [21], Lemma 3). Now let $T\colon L^1[0,1] \to E(X)$ be a nearly representable operator. The operator $J_X \circ T$ is also nearly representable and hence representable (by Theorem 2). So the operator T must be representable (see [4]). □

4. Concluding remarks

If X and Y are Banach spaces with the NRNP, then $X\widehat{\otimes}_\pi Y$ ($\widehat{\otimes}_\pi$ is the projective tensor product) need not satisfy the NRNP. This can be seen from Pisier's famous example that $L^1/H_0^1\widehat{\otimes}_\pi L^1/H_0^1$ contains c_0 (hence failing the NRNP) while L^1/H_0^1 has the NRNP.

If X is a Banach space and (Ω, Σ) is a measure space, we denote by $M(\Omega, X^*)$ the space of X^*-valued σ-additive measures of bounded variation with the usual total variation norm. In light of Theorem 2, one can ask the following question: Does $M(\Omega, X^*)$ have the NRNP whenever X^* does? It should be noted that for non-dual space, the answer is negative: the space E constructed by Talagand in [22] is a Banach lattice that does not contain c_0 (so it has the NRNP) but $M(\Omega, E)$ contains c_0.

Finally, since L^1-spaces are the primary examples of Banach spaces with the NRNP, the following question arises: Do non-commutative L^1-spaces have the NRNP? Note that since C_1 (the trace class operators) has the RNP, it has the NRNP; however it is still unknown if C_E has the NRNP if E is a symmetric sequence space that does not contain c_0. We remark that non-commutative L^1-spaces have the ARNP [13].

References

1. S. Argyros and M. Petrakis, *A property of non-strongly regular operators—geometry of Banach spaces*, London Math. Soc. Lecture Note Series, no. 158, Cambridge Univ. Press, Cambridge, 1990, pp. 5–23.
2. J. Bourgain, *A characterization of non-Dunford-Pettis operators on L_1*, Israel J. Math. **37** (1980), 48–53.
3. ——, *Dunford-Pettis operators on L_1 and the Radon-Nikodym property*, Israel J. Math. **37** (1980), 34–47.
4. J. Bourgain and H. P. Rosenthal, *Applications of the theory of semi-embeddings to Banach space theory*, J. Funct. Anal. **52** (1983), 149–188.
5. D. L. Cohn, *Measure theory*. Birkhäuser, Basel, 1980.
6. J. Diestel, *Sequences and series in Banach spaces*, first edition, Graduate Text in Mathematics, vol. 92, Springer-Verlag, New York, 1984.
7. J. Diestel and J. J. Uhl, Jr., *Vector measures*, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
8. N. Dinculeanu, *Vector measures*. Pergamon Press, New York, 1967.
9. P. Dowling, *The analytic Radon-Nikodym property in Lebesgue Bochner function spaces*. Proc. Amer. Math. Soc. **99** (1987), 119–121.
10. N. Dunford and J. T. Schwartz, *Linear operators. Part I, General theory*. Interscience, New York, 1958.
11. G. Emmanuele, *Some more Banach spaces with the* (*NRNP*), Matematiche (Catania) **48** (1993), 213–218.
12. H. Fakhoury, *Représentations d'operateurs à valeur dans $L^1(X, \Sigma, \mu)$*. Math. Ann. **240** (1979), 203–212.
13. U. Haagerup and G. Pisier, *Factorization of analytic functions with values in non-commutative L_1-spaces*. Canad. J. Math. **41** (1989), 882–906.
14. W. Hensgen, *Some properties of vector-valued Banach ideal space $E(X)$ derived from those of E and X*, Collect. Math. **43** (1992), 1–13.
15. A. Ionescu-Tulsea and C. Ionescu-Tulsea, *Topics in the theory of lifting*, first edition, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 48, Springer-Verlag, New York, 1969.
16. N. Kalton, *Isomorphisms between L_p-function spaces when $p < 1$*, J. Funct. Anal. **42** (1981), 299–337.

17. R. Kaufman, M. Petrakis, L. H. Riddle, and J. J. Uhl, Jr., *Nearly representable operators*, Trans. Amer. Math. Soc. **312** (1989), 315–333.
18. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, first edition, Modern Survey in Mathematics, vol. 97, Springer-Verlag, New York, 1979.
19. H. P. Lotz, N. T. Peck, and H. Porta, *Semi-embeddings of Banach spaces*, Proc. Edinburg Math. Soc. **22** (1979), 233–240.
20. N. Randrianantoanina and E. Saab, *Complete continuity property in Bochner function spaces*, Proc. Amer. Math. Soc. **117** (1993), 1109–1114.
21. ———, *Stability of some types of Radon-Nikodym properties*. Illinois J. Math. **39** (1995), 416–430.
22. M. Talagrand, *Quand l'espace des mesures a variation borneé est-il faiblement sequentiellement complet*, Proc. Amer. Math. Soc. **99** (1984), 285–288.
23. ———, *Weak Cauchy sequences in* $L^1(E)$, Amer. J. Math. **106** (1984), 703–724.
24. J. B. Turett and J. J. Uhl, Jr., $L_p(\mu, X)$ $(1 < p < \infty)$ *has the Radon-Nikodym property if X does by martingales*, Proc. Amer. Math. Soc. **61** (1976), 347–350.
25. J. Voigt, *The convex compactness property for the strong operator topology*. Note Mat. **12** (1992), 259–269.
26. P. Wojtaszczyk, *Banach spaces for analysts*, first edition, Cambridge University Press, 1991.

Narcisse Randrianantoanina, Department of Mathematics, University of Texas, Austin, TX 78712

Current address: Department of Mathematics and Statistics, Miami University, Oxford, OH 45056
randrin@muohio.edu

Elias Saab, Department of Mathematics, University of Missouri, Columbia, MO 65211
elias@esaab.math.missouri.edu