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EIGENVALUES OF LAPLACIANS WITH MIXED BOUNDARY CONDITIONS, UNDER CONFORMAL MAPPING

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1. Introduction

The prototypical result of this paper says, roughly, that if $f(z) = \sum_{j\in\mathbf{Z}} a_j z^j$ is a conformal map of an annulus A onto a doubly connected plane domain Ω with $|a_1| = 1$, then

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j(\Omega)^s} \geq \sum_{j=1}^{\infty} \frac{1}{\lambda_j(A)^s} \qquad \text{for all } s > 1,$$

where $\lambda_j(\Omega)$ is the j-th eigenvalue of the Laplacian on Ω under Dirichlet boundary conditions on the outer boundary of Ω and Neumann conditions on the inner boundary, and similarly for $\lambda_j(A)$. That is, the zeta function of the Laplacian is at least as big for Ω as it is for the annulus A.

This introduction provides some historical context; then in Section 2 the results are all stated precisely. For similar results but under purely Dirichlet boundary conditions, see the earlier paper [13], written with C. Morpurgo. This present work draws heavily on the arguments and intuition in [13], and is best read in conjunction with that paper.

The eigenvalues of the Laplacian have many physical interpretations, for example as the frequencies of vibration of a membrane, as rates of decay for the heat (or mass diffusion) equation, and as cut-off frequencies for waveguides. However, the eigenvalues of doubly connected regions can be calculated exactly only for a few special regions, most notably for annuli, and while numerical methods are sophisticated and successful [11], they can only estimate finitely many of the eigenvalues. This paper will give sharp estimates involving *all* the eigenvalues. Incidentally, the mixed boundary conditions employed in this paper have drawn increasing attention in recent years (see for example [4], [17] and the references therein).

G. Pólya and G. Szegő [16] proved by conformal transplantation an *upper* bound on the first eigenvalue of a simply connected plane domain under Dirichlet boundary conditions: if $f(z)$ is a conformal map of the open unit disk D onto a bounded, simply connected plane domain Ω and if $|f'(0)| = 1$, then $\lambda_1(\Omega) \leq \lambda_1(D)$. In [13, Cor. 3], the author and C. Morpurgo proved a direct analogue of this for doubly connected

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domains: $\lambda_1(\Omega) \leq \lambda_1(A)$, where now $f(z) = \sum_{j\in\mathbf{Z}} a_j z^j$ is assumed to map the annulus A conformally onto Ω with $|a_1| = 1$. This paper proves the same result but for eigenvalues of the Laplacian under *mixed* boundary conditions: Dirichlet conditions on the outer boundaries of Ω and A and Neumann conditions on the inner boundaries, or vice versa. In fact Corollary 2 shows that for functions $\Phi(a)$ convex and increasing,

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega)}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A)}\right), \tag{1.1}$$

where n can be a positive integer or $+\infty$. Taking $\Phi(a) = a^s$ gives a zeta function inequality.

Also in this paper are eigenvalue inequalities and convexity results for doubly connected regions on cylinders, cones and the hyperbolic punctured disk, and on surfaces with curvature bounded above.

Section 3 contains open questions and conjectures relating to the trace of the heat kernel.

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2. Results

Take a bounded, doubly connected subdomain M of the complex plane and let w be a positive function on M. Write ∂M_D and ∂M_N for the two components of ∂M. Consider the following eigenvalue problem with mixed boundary conditions:

$$\frac{1}{w}\Delta\psi = -\lambda\psi \quad \text{in } M, \qquad \psi = 0 \quad \text{on } \partial M_D, \qquad \frac{\partial\psi}{\partial \mathrm{n}} = 0 \quad \text{on } \partial M_N, \tag{2.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

denotes the Laplacian in the plane and $\partial\psi/\partial\mathrm{n}$ is the normal derivative of ψ. We name the boundary components ∂M_D and ∂M_N to remind ourselves that they support Dirichlet and Neumann boundary conditions, respectively. Physically, one thinks of the domain M as representing an inhomogeneous membrane that is fixed on ∂M_D, free on ∂M_N, and has mass density w and total mass

$$|M|_w := \int_M w \, d\mu,$$

where μ is Lebesgue measure in the plane. The eigenvalues λ give the squares of the frequencies of this membrane's modes of vibration.

The following definitions describe the conditions we will want our domain M and function w to satisfy; we will show later that these conditions ensure the eigenvalues $\lambda = \lambda(w)$ actually exist and possess the properties we will need.

Definition ("acceptable"). Let Ω be a bounded, doubly connected plane domain. Name its two boundary components $\partial\Omega_D$ and $\partial\Omega_N$. Call Ω *acceptable* if $\partial\Omega_D$ contains more than one point and $\partial\Omega_N$ is a quasicircle.

Remember that a quasicircle is defined to be the image of a circle under a quasiconformal map of the complex plane, so that quasicircles are Jordan curves, in particular. See F. W. Gehring's lecture notes [6, Ch. 2] for many equivalent definitions of quasicircles. Readers unfamiliar with quasicircles might prefer to assume that $\partial\Omega_N$ is a Jordan curve that can be locally represented as the graph of a Lipschitz continuous function, for then it follows from a theorem of L. V. Ahlfors [6, p. 30] that $\partial\Omega_N$ is a quasicircle.

The preceding definition of "acceptable" implicitly depends on which boundary component of $\partial\Omega$ was chosen to be called $\partial\Omega_D$ and which one was chosen as $\partial\Omega_N$. For instance, if $\Omega = \{|z| < 2\} \setminus [-1, 1]$ then Ω is acceptable provided we choose $\partial\Omega_D = [-1, 1]$ and $\partial\Omega_N = \{|z| = 2\}$, but Ω is *not* acceptable if we choose $\partial\Omega_D = \{|z| = 2\}$ and $\partial\Omega_N = [-1, 1]$. This implicit dependence will not cause trouble, in practice.

Definition ("admissible"). Let M be an acceptable domain with boundary components ∂M_D and ∂M_N, and let w be a function on M. Call w *admissible* if two conditions hold:

- a conformal map $f(z)$ from M onto an acceptable domain Ω exists such that ∂M_D and $\partial\Omega_D$ correspond under f, and ∂M_N and $\partial\Omega_N$ correspond under f; and
- a function $h \in L^\infty(\Omega)$ exists with $h > 0$ a.e. and

$$w = (h \circ f)|f'|^2. \tag{2.2}$$

In particular, if M is acceptable and $w \in L^\infty(M)$ is positive a.e. then w is admissible. Observe also that if w is admissible then $w \in L^\infty_{loc}(M)$ and $w > 0$ a.e.

This paragraph describes some fundamental properties of the eigenvalues $\lambda_j(w)$; see Section 4 for the proofs. Let M be an acceptable, bounded, doubly connected plane domain with boundary components ∂M_D and ∂M_N. Assume w is an admissible function on M. Then the eigenvalue problem (2.1) has discrete spectrum $\{\lambda_j(w)\}$ with $0 < \lambda_1(w) < \lambda_2(w) \leq \lambda_3(w) \leq \cdots \to \infty$, and the eigenvalues are given by Poincaré's minimax principle in terms of the Rayleigh quotient:

$$\lambda_j(w) = \min_{L_j} \max_{\psi \in L_j \setminus \{0\}} \frac{\int_M |\nabla\psi|^2 \, d\mu}{\int_M \psi^2 w \, d\mu}, \tag{2.3}$$

where L_j ranges over all j-dimensional subspaces of the trial space $H^1_{mix}(M)$, with

$$H^1_{mix}(M) := \text{the closure in } H^1(M) \text{ of } \{\psi \in H^1(M) \cap C^\infty(M)\colon\ \psi = 0 \text{ on a neighborhood of } \partial M_D\}.$$

Here $H^1(M)$ denotes the usual Sobolev space, often called $W^{1,2}(M)$. It is important in this paper to observe that our trial space $H^1_{mix}(M)$ does *not* depend on w. Now, the eigenfunctions $\psi_j \in H^1_{mix}(M)$ are continuous and satisfy the eigenvalue equation

$$-\Delta\psi_j = \lambda_j(w) w \psi_j$$

weakly in $H^1_{mix}(M)$, which means that

$$\int_M \nabla\psi \cdot \nabla\psi_j \, d\mu = \lambda_j(w) \int_M \psi\psi_j w \, d\mu \qquad \text{for all } \psi \in H^1_{mix}(M).$$

(The righthand integral does make sense, in view of (2.2) and Lemma 7 in Section 4.) Furthermore, the first eigenfunction ψ_1 is unique up to constant factors and is never zero in M. Finally, the eigenvalue problem is conformally invariant in the sense that if w is admissible (as in the above definition) with $h \in L^\infty(\Omega)$ and $w = (h \circ f)|f'|^2$, then $\lambda_j(w) = \lambda_j(h)$ and the eigenfunctions ψ_j on M and ϕ_j on Ω are related by $\psi_j = \phi_j \circ f$. This conformal invariance is best understood by verifying that if $-\Delta\phi = \lambda h \phi$ in Ω, then in M we have

$$-\Delta(\phi \circ f) = [-\Delta\phi \circ f]|f'|^2 = [(\lambda h \phi) \circ f]|f'|^2 = \lambda w(\phi \circ f). \tag{2.4}$$

We will not need these next remarks, but it is interesting that if $\partial\Omega_D$ is locally Lipschitz then ψ_j is continuous up to ∂M_D and it equals zero there, which is the classical Dirichlet boundary condition. Also, if γ is a C^1-smooth subarc of ∂M_N and $\psi_j \in C^2(M \cup \gamma)$ then the normal derivative of ψ_j vanishes on γ, which is the classical Neumann boundary condition.

Next, a lower bound of Weyl type,

$$\lambda_j(w) \geq \alpha j \qquad \text{for all } j \geq 1, \tag{2.5}$$

holds for some $\alpha \in (0, 1)$ that depends on w; this also will be justified in Section 4. (When w and ∂M are smooth, much more than (2.5) can be said [17] about the asymptotics of λ_j.) Thus the *zeta function* $\sum_{j=1}^{\infty} \lambda_j(w)^{-s}$ of the operator $w^{-1}\Delta$ on M is finite for $s > 1$. More general than the zeta function is the *Φ-functional*

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(w)}\right),$$

for convex increasing Φ and n either a positive integer or $+\infty$. Obviously this gives the zeta function when $n = +\infty$ and $\Phi(a) = a^s$ for fixed $s > 1$.

Next comes the basic extremal result for the Φ-functional, in which the domain M is actually an annulus A. The proof will be indicated in Section 5. Use the notation $\overline{w}(z) := \int_0^{2\pi} w(|z|e^{i\theta})\, d\theta/2\pi$ for the average of w over the circle of radius $|z|$.

THEOREM 1. *Let $A := \{z \in \mathbf{C}\colon 0 < R_0 < |z| < R < \infty\}$ and call the two boundary circles of this annulus ∂A_D and ∂A_N (in either order). Assume v and w are admissible functions on A and take n to be either a positive integer or $+\infty$. Let $\Phi(a)$ be convex and increasing for $a \geq 0$, with $\Phi(0) = 0$, $\Phi(\lambda_1(w)^{-1}) > 0$ and $\int_1^n \Phi(1/a)\,da$ finite.*

Assume v is radial and

$$\int_0^{2\pi} w(re^{i\theta})\,d\theta \geq 2\pi v(r) \qquad \text{for almost all } r \in (R_0, R). \tag{2.6}$$

Then

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(w)}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(v)}\right) \tag{2.7}$$

with strict inequality unless $\int_0^{2\pi} w(re^{i\theta})\,d\theta = 2\pi v(r)$ for almost all $r \in (R_0, R)$. If in addition $\Phi(a)$ is strictly convex, then (2.7) holds with strict inequality unless $w = v$ a.e.

In particular, averaging the conformal factor w over concentric circles decreases the Φ-functional, provided $\overline{w}$ is admissible also:

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(w)}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\overline{w})}\right).$$

Theorem 1 and its proof can be generalized to annuli $\{R_0 < |x| < R\}$ in $\mathbf{R}^N$, $N \geq 3$.

Flat surfaces Besides the euclidean metric g, doubly connected plane domains possess two other kinds of radial flat metric:

> the *cylinder* metric $|z|^{-2}g$, and the *cone* metric $\gamma^2|z|^{2\gamma-2}g$ for fixed $\gamma \in \mathbf{R}$, $\gamma \neq 0$.

See Section 2 of [13] for a discussion of the attributes of these metrics. Note that when $\gamma = 1$ the cone metric coincides with the euclidean metric.

In the corollary below,

$$\begin{aligned} \lambda_j(\Omega_{euclid}) &= j\text{-th eigenvalue of the euclidean Laplacian } \Delta \text{ on } \Omega, \\ \lambda_j(\Omega_{cone}) &= j\text{-th eigenvalue of the conical Laplacian } \gamma^{-2}|z|^{-2\gamma+2}\Delta \text{ on } \Omega, \\ \lambda_j(\Omega_{cylinder}) &= j\text{-th eigenvalue of the cylindrical Laplacian } |z|^2\Delta \text{ on } \Omega. \end{aligned}$$

That is, we take $M = \Omega$ and $w(z) = 1$, $w(z) = \gamma^2|z|^{2\gamma-2}$, or $w(z) = |z|^{-2}$, respectively, in (2.1), with Dirichlet boundary conditions on $\partial\Omega_D$ and Neumann conditions on $\partial\Omega_N$.

We shall see in Section 5 how to derive the following corollary, which gives extremal results for Φ-functionals of doubly connected domains, under mixed boundary conditions.

COROLLARY 2. *Let $0 < R_0 < R < \infty$ and suppose $f(z)$ is a conformal map of the annulus $A = A(R_0, R)$ onto the acceptable, bounded, doubly connected plane domain Ω. Suppose that the inner boundary components of A and Ω correspond under f, that ∂A_D and $\partial \Omega_D$ correspond under f, and that ∂A_N and $\partial \Omega_N$ correspond under f. Take n to be either a positive integer or $+\infty$. Let $\Phi(a)$ be convex and increasing for $a \geq 0$, with $\Phi(0) = 0$ and $\int_1^n \Phi(1/a)\,da$ finite.*

(*a*) [*Euclidean metrics.*] *If*

(*i*) *f has Laurent expansion $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$ with $|a_1| \geq 1$, or*

(*ii*) *$\int_0^{2\pi} \log|f'(re^{i\theta})|\,d\theta \geq 0$ for some $r \in (R_0, R)$, or*

(*iii*) *the euclidean area of the hole in Ω is greater than or equal to the area πR_0^2 of the hole in A,*

and if $\Phi(\lambda_1(\Omega_{euclid})^{-1}) > 0$, then

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega_{euclid})}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A_{euclid})}\right)$$

with strict inequality unless Ω is a translate of A.

(*b*) [*Cone metrics.*] *Fix $\gamma \in (0, 1)$. If $0 \notin \overline{\Omega}$ and the area (in the cone metric) of the hole in Ω is greater than or equal to the area $\gamma\pi R_0^{2\gamma}$ of the hole in A, and if $\Phi(\lambda_1(\Omega_{cone})^{-1}) > 0$, then*

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega_{cone})}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A_{cone})}\right)$$

with strict inequality unless $\Omega = A$.

(*c*) [*Cylinder metrics.*] *If $\overline{\Omega}$ separates the origin from the point at infinity and if we have $\Phi(\lambda_1(\Omega_{cylinder})^{-1}) > 0$, then*

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega_{cylinder})}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A_{cylinder})}\right)$$

with strict inequality unless Ω is a dilate of A.

For an annulus with the euclidean or cone metric, the eigenvalues can be computed in terms of zeros of Bessel functions. For example, when we impose Dirichlet conditions on the outer circle and Neumann conditions on the inner circle, the eigenfunctions have the form

$$re^{i\theta} \mapsto [Y_{\nu/\gamma}(\sqrt{\lambda}R^{\gamma})J_{\nu/\gamma}(\sqrt{\lambda}r^{\gamma}) - J_{\nu/\gamma}(\sqrt{\lambda}R^{\gamma})Y_{\nu/\gamma}(\sqrt{\lambda}r^{\gamma})][\sin(\nu\theta) \text{ or } \cos(\nu\theta)]$$

for $\nu = 0, 1, 2, \ldots$ and where the eigenvalues λ must be chosen to make the radial derivative of the eigenfunction vanish when $r = R_0$. (Remember here that $\gamma = 1$ for the euclidean metric and $\gamma \in (0, 1)$ for the cone metric.)

For an annulus with the cylinder metric, the eigenvalues can be computed explicitly even more easily. By means of the map $re^{i\theta} \mapsto (\cos\theta, \sin\theta, \log r)$, we may regard the annulus as a cylinder of length $L = \log R - \log R_0$ and radius 1 sitting in $\mathbf{R}^3$, and so we may use coordinates $x_3 \in (0, L)$ and $\theta \in [0, 2\pi]$ on A. Impose Dirichlet conditions on $\{x_3 = 0\}$ and Neumann conditions on $\{x_3 = L\}$. The eigenfunctions of the cylindrical Laplacian on A are then $e^{i\nu\theta}\sin((2\ell - 1)\pi x_3/2L)$ for $\nu \in \mathbf{Z}, \ell \geq 1$, with eigenvalues $\nu^2 + (2\ell - 1)^2\pi^2/4L^2$.

An upper bound on $\lambda_1(\Omega_{euclid})$ of a kind somewhat different to that in Corollary 2 was found by L. E. Payne and H. F. Weinberger [14] (and see [2, p. 146] for further generalizations). They used not conformal mapping but rather a transplantation depending on the distance to the boundary, with their extremal domain being an annulus having the same area and outer perimeter length as Ω_{euclid}. It does not seem, however, that their method applies to eigenvalues higher than the first.

For *lower* bounds on λ_1, under mixed boundary conditions, see [2, p. 114 ff.].

Surfaces with curvature bounded above The next goal is to develop an extremal result for the Φ-functional on doubly connected surfaces with curvature bounded above, thus generalizing several of the results for *flat* doubly connected surfaces in Corollary 2.

Before approaching the next result, the reader should review the definitions of the hyperbolic/euclidean/spherical metric kg of constant curvature κ (as given before Corollary 4 in [13]), the constant curvature "cone" metric cg (described before Corollary 5 of [13]), and the total curvature ω_κ (discussed before [13, Cor. 5]). Finally, after reviewing all these definitions one observes that the next corollary applies in particular to doubly connected domains Ω in hyperbolic space, the plane or the sphere (putting $h = k$), or in cones formed from these spaces (putting $h = c$). The extremal domains in the corollary are geodesic annuli.

Write $\lambda_j(\Omega_{hg})$ for the j-th eigenvalue of $h^{-1}\Delta$ on Ω with Dirichlet boundary conditions on $\partial\Omega_D$ and Neumann conditions on $\partial\Omega_N$; that is, take $M = \Omega$ and $w = h$ in (2.1).

COROLLARY 3. *Fix $\kappa \in \mathbf{R}, \gamma \in (0, 1]$, and let $0 < R_0 < R < \infty$. Suppose $f(z)$ is a conformal map of the annulus $A = A(R_0, R)$ onto the acceptable, bounded, doubly connected plane domain Ω, and let $\mathcal{H}$ denote the hole in Ω. Suppose that the inner boundary components of A and Ω correspond under f, that ∂A_D and $\partial\Omega_D$ correspond under f, and that ∂A_N and $\partial\Omega_N$ correspond under f.*

Assume hg is a metric on $\Omega \cup \mathcal{H}$ of the form (1.38) in [2], with $h \in L^\infty(\Omega)$ and $h > 0$ a.e. in Ω, and suppose $\omega_\kappa^+(\Omega \cup \mathcal{H}, hg) \leq 2\pi(1 - \gamma)$. When $\kappa < 0$, assume $R < 1$, and when $\kappa > 0$, assume $|\Omega \cup \mathcal{H}|_h \leq 2\pi\gamma/\kappa$. Take n to be either *a positive integer or* $+\infty$. *Let* $\Phi(a)$ *be convex and increasing for* $a \geq 0$, *with* $\Phi(0) = 0$, $\Phi(\lambda_1(\Omega_{hg})^{-1}) > 0$ *and* $\int_1^n \Phi(1/a)\, da$ *finite.*

If the area of the hole $|\mathcal{H}|_h$ *is greater than or equal to the area* $|D(R_0)|_c$, *then*

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega_{hg})}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A_{cg})}\right)$$

with strict inequality unless Ω_{hg} *is isometric a.e. via* $f(z)$ *to* A_{cg}.

This corollary is established in Section 5. Notice that when $\kappa = 0$, the corollary implies the euclidean case (a)(iii) and the cone case (b) of Corollary 2.

The hyperbolic punctured disk Corollary 2(c) presents a result for the eigenvalues of the Laplacian on a cylinder. We now develop an analogue of this result for the hyperbolic punctured disk (which is the hyperbolic analogue of a cylinder, as discussed in Section 2 of [13]). Fix $\kappa < 0$ and define

$$\sigma(z) := \frac{1}{|\kappa|} \frac{1}{|z|^2(\log 1/|z|)^2}, \qquad 0 < |z| < 1.$$

Then σg describes a punctured disk of constant negative curvature κ (since $-\Delta \log \sigma = 2\kappa\sigma$).

In the following corollary, derived in Section 5, the notation Ω_{punct} refers to the domain Ω together with the metric σg of the punctured disk, and $\lambda_j(\Omega_{punct})$ means the j-th eigenvalue on Ω of the Laplacian $\sigma^{-1}\Delta$ of the punctured disk, with Dirichlet boundary conditions on $\partial\Omega_D$ and Neumann conditions on $\partial\Omega_N$; that is, take $M = \Omega$ and $w = \sigma$ in (2.1).

COROLLARY 4. *Fix* $\kappa < 0$ *and let* $0 < R_0 < R < 1$. *Suppose* $f(z)$ *is a conformal map of the annulus* $A = A(R_0, R)$ *onto the acceptable, doubly connected plane domain* Ω, *with the closure of* Ω *being contained in the punctured disk* $D(1) \setminus \{0\}$. *Suppose that the inner boundary components of* A *and* Ω *correspond under* f, *that* ∂A_D *and* $\partial\Omega_D$ *correspond under* f, *and that* ∂A_N *and* $\partial\Omega_N$ *correspond under* f. *Take* n *to be either a positive integer or* $+\infty$. *Let* $\Phi(a)$ *be convex and increasing for* $a \geq 0$, *with* $\Phi(0) = 0$, $\Phi(\lambda_1(\Omega_{punct})^{-1}) > 0$ *and* $\int_1^n \Phi(1/a)\, da$ *finite.*

If the area (in the punctured disk metric) of the hole in Ω *is greater than or equal to the area* $2\pi/(|\kappa| \log 1/R_0)$ *of the hole in* A, *then*

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(\Omega_{punct})}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(A_{punct})}\right)$$

with strict inequality unless $\Omega = A$.

Convexity of the Φ-functional of the eigenvalues The next theorem establishes convexity of the Φ-functional with respect to w, on a compact N-dimensional Riemannian manifold with boundary, for $N \geq 2$. For technical convenience, we assume the boundary is smooth.

Let M be a regular subdomain of an N-dimensional manifold $\widetilde{M}$, so that the boundary ∂M is smooth and the closure $\overline{M} := M \cup \partial M$ is compact. Divide ∂M into two disjoint pieces, ∂M_D and ∂M_N, with ∂M_D non-empty and open in ∂M. Let g be a Riemannian metric on $\widetilde{M}$ and take w to be a positive smooth function on $\overline{M}$. Under Dirichlet boundary conditions on ∂M_D and Neumann conditions on ∂M_N, the operator $w^{-1}\Delta_g$ on M is negative and has a discrete spectrum $\{-\lambda_j(w)\}$, with $0 < \lambda_1(w) < \lambda_2(w) \leq \lambda_3(w) \leq \cdots \to \infty$. The eigenvalues are given by the minimax principle

$$\lambda_j(w) = \min_{L_j} \max_{\psi \in L_j \setminus \{0\}} \frac{\int_M g(\nabla_g \psi, \nabla_g \psi)\, dV_g}{\int_M \psi^2 w\, dV_g},$$

where L_j ranges over all j-dimensional subspaces of

$$\begin{aligned} H^1_{mix}(M_g) \;:=\; & \text{the closure in } H^1(M_g) \text{ of } \{\psi \in H^1(M_g) \cap C^\infty(M)\colon\ \psi = 0 \\ & \text{on a neighborhood of } \partial M_D\}. \end{aligned}$$

Here $H^1(M_g)$ denotes the usual Sobolev space, often called $W^{1,2}(M_g)$. The eigenfunctions $\psi_j \in H^1_{mix}(M_g)$ are smooth on M, with the eigenvalue equation $-\Delta_g \psi_j = \lambda_j(w) w \psi_j$ holding pointwise in M and also weakly in $H^1_{mix}(M_g)$. Furthermore, the first eigenfunction ψ_1 is unique up to constant factors and is never zero in M. (For the preceding facts, argue as in [3, pp. 53–61, 71] and [7, pp. 213, 214].) For some $\alpha \in (0, 1)$ that depends on w, $\lambda_j(w) \geq \alpha j^{2/N}$ for all $j \geq 1$, since $\lambda_j(w) \geq \|w\|_\infty^{-1} \lambda_j(1)$ and $\lambda_j(1)$ is bounded below by the j-th eigenvalue of Δ_g on M with purely Neumann boundary conditions, which in turn is comparable to $j^{2/N}$ for large j by Weyl's asymptotic law [5, pp. 9, 172]; this implies $\lambda_j(w) \geq \alpha j^{2/N}$ for all j because we also know $\lambda_1(w) > 0$.

THEOREM 5. *Take m to be a positive integer and let $q \in (0, 1)$. Then*

$$\left(\sum_{j=1}^{m} \lambda_j(w)^{-1} \right)^q \qquad \textit{is convex as a functional of } w^q$$

for positive weight functions $w \in C^\infty(\overline{M})$, and

$$\log \left(\sum_{j=1}^{m} \lambda_j(w)^{-1} \right) \qquad \textit{is convex as a functional of } \log w.$$

In both cases, the convexity is strict except when applied to multiples of a fixed w.

When $q = 1$ we can say more. Take n to be either a positive integer or $+\infty$. Let $\Phi(a)$ be convex and increasing for $a \geq 0$, with $\Phi(0) = 0$ and $\int_1^n \Phi(1/a^{2/N})\,da$ finite. Then the Φ-functional

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j(w)}\right)$$

is convex as a functional of the weight function $w \in C^\infty(\overline{M})$, $w > 0$. If in addition $\Phi(a)$ is strictly convex, then the Φ-functional is strictly convex as a functional of w.

See Section 5 for the proof of the theorem. The requirement that $\int_1^n \Phi(1/a^{2/N})\,da$ be finite just serves to ensure that the Φ-functional is finite-valued.

In particular, the theorem shows that for fixed $s > N/2$, the zeta function $\sum_{j=1}^\infty \lambda_j(w)^{-s}$ is a strictly convex functional of w. (Requiring $s > N/2$ ensures the zeta function is finite, since $\lambda_j(w) \geq \alpha j^{2/N}$.) When $N = 2$ and M is two dimensional, the operator $w^{-1}\Delta_g$ equals the Laplace-Beltrami operator Δ_{wg} of the metric wg on M. Hence in two dimensions under mixed boundary conditions, the zeta function of the Laplace-Beltrami operator Δ_{wg} on M is a strictly convex functional of the conformal factor w.

The statements in Theorem 5 about the convexity of the Φ-functional ($q = 1$) were proved in the purely Dirichlet case (i.e., for ∂M_N empty) in [13, Th.8], and the sub-case where M is a plane domain and $\Phi(a) = a$ was proved much earlier by Pólya and Schiffer [15, p. 289].

Convexity with respect to w^q might be helpful for isoperimetric variational problems. For example, one might consider a two–dimensional surface M with metric g and then vary the conformal factor w in such a way that $w^{1/2}$ varies linearly, which implies that lengths of curves in M (such as ∂M) must vary linearly in response. Theorem 5 with $q = 1/2$ then tells us that the square root of the sum of the first m reciprocal eigenvalues must vary convexly, and so in particular in order to find a global minimum we need only find a critical point. Convexity with respect to $\log w$ might also be useful since $\log w$ enters into the definition of Gauss curvature.

Lastly, the convexity of $[\sum_j \lambda_j(w)^{-1}]^q$ with respect to w^q, proved in Theorem 5 for $q \in (0, 1]$, implies the convexity of $[\sum_j \lambda_j(|\alpha|^{1/q})^{-1}]^q$ with respect to the complex-valued function $\alpha \in C^\infty(\overline{M})$, $|\alpha| \neq 0$, simply because $|(\alpha + \beta)/2|^{1/q} \leq [(|\alpha| + |\beta|)/2]^{1/q}$. A natural setting for this result is when $N = 2, q = 1/2$, and $\alpha(z)$ is analytic. Pólya and Schiffer [15, p. 303] proved the cases $q = 1/2$ and $q = 1$ of this "analytic convexity", but their proof does not seem to generalize to arbitrary $q \in (0, 1]$. Note that one can similarly complexify the convexity of $\log(\sum_j \lambda_j^{-1})$ with respect to $\log w$.

Surfaces without mass constraints The next theorem concerns 2-dimensional surfaces, again, and differs in character from the other results in this paper: there is no constraint whatsoever on where the mass of the membrane may be concentrated, there is certainly no curvature constraint, and there is no analogous theorem for the case of purely Dirichlet boundary conditions. The key tool in the proof will be the rather special form of the eigenfunctions for the mixed eigenvalue problem on the homogeneous cylinder.

In the theorem, $\lambda_j^{DN}(\Omega_{hg})$ denotes the j-th eigenvalue of $h^{-1}\Delta$ on Ω under Dirichlet boundary conditions on one component Γ_1 of $\partial\Omega$ and Neumann conditions on the other component Γ_2. That is, we take $M = \Omega$, $\partial M_D = \Gamma_1$, $\partial M_N = \Gamma_2$ and $w = h$ in (2.1). Similarly, $\lambda_j^{ND}(\Omega_{hg})$ denotes the j-th eigenvalue of $h^{-1}\Delta$ under Neumann conditions on Γ_1 and Dirichlet conditions on Γ_2. Since both Γ_1 and Γ_2 are assumed in the theorem to be quasicircles, Ω is an acceptable domain for both the eigenvalue problems, and so indeed both $\lambda_j^{DN}(\Omega_{hg})$ and $\lambda_j^{ND}(\Omega_{hg})$ exist and have the properties described around (2.3).

THEOREM 6. *Let $0 < R_0 < R < \infty$ and suppose $f(z)$ is a conformal map of the annulus $A = A(R_0, R)$ onto a bounded, doubly connected plane domain Ω, with both components of $\partial\Omega$ being quasicircles. Take n to be either a positive integer or $+\infty$. Let $\Phi(a)$ be convex and increasing for $a \geq 0$, with $\Phi(0) = 0$ and $\int_1^n \Phi(1/a)\,da$ finite.*

Assume $h \in L^\infty(\Omega)$ with $h > 0$ a.e. Then

$$\frac{1}{2}\sum_{j=1}^{n}\Phi\left(\frac{1}{\lambda_j^{DN}(\Omega_{hg})}\right) + \frac{1}{2}\sum_{j=1}^{n}\Phi\left(\frac{1}{\lambda_j^{ND}(\Omega_{hg})}\right)$$

$$\geq \sum_{j=1}^{n}\Phi\left(\frac{1}{2}\left(\frac{1}{\lambda_j^{DN}(\Omega_{hg})} + \frac{1}{\lambda_j^{ND}(\Omega_{hg})}\right)\right) \tag{2.8}$$

$$\geq \sum_{j=1}^{n}\Phi\left(\frac{|\Omega|_h/|A|_{cylinder}}{\lambda_j^{DN}(A_{cylinder})}\right). \tag{2.9}$$

If also $\Phi(a)$ is strictly convex, then (2.9) *holds with strict inequality unless Ω_{hg} is isometric a.e. to the homogeneous cylinder in $\mathbf{R}^3$ of length* $\log(R/R_0)$, *radius* 1 *and total mass* $|\Omega|_h$.

The conclusion (2.9) of the theorem can be made to look more symmetrical if one uses $\lambda_j^{DN}(A_{cylinder}) = \lambda_j^{ND}(A_{cylinder})$; these eigenvalues are known to have the form $m^2 + (2\ell - 1)^2\pi^2/4L^2$ (see the discussion after Corollary 2). Note also that by a simple re-scaling, the quantity $(|\Omega|_h/|A|_{cylinder})/\lambda_j^{DN}(A_{cylinder})$ appearing in (2.9) equals the reciprocal of the j-th eigenvalue of A with the metric $(|\Omega|_h/|A|_{cylinder})|z|^{-2}g$, and A with this metric represents the homogeneous cylinder imbedded in $\mathbf{R}^3$ of length $\log(R/R_0)$, radius 1 and total mass $|\Omega|_h$.

The case $\Phi(a) \equiv a$ of Theorem 6 was stated by J. Hersch [8, p. 32], and we follow his ideas when we prove the theorem in Section 6. The statement in the theorem about strict inequality is new.

Hersch's paper contains several other interesting results on eigenvalues (of simply connected regions) with mixed boundary conditions, and some but not all (cf. [8, §6]) of these results can be extended to the Φ-functional. This is left to the reader. Some additional results are in [9, §3].

3. Open questions

Recall that the functional $\sum_j e^{-\lambda_j t}$ is called the *trace of the heat kernel* in mathematics, and the *partition function* in physics, where it also has importance. Then ask: do the extremal results for the Φ-functional in Theorem 1 and Corollaries 2, 3 and 4 hold for the trace of the heat kernel? In particular, is it true that (in the notation of Corollary 2(a))

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega_{euclid})} \geq \sum_{j=1}^{\infty} e^{-t\lambda_j(A_{euclid})} \qquad \text{for all } t > 0? \tag{3.1}$$

Note that $\Phi(a) = e^{-t/a}$ is convex *only* for $a \leq t/2$, when $a > 0$, and thus Corollary 2(a) proves (3.1) only when $t \geq 2/\lambda_1(\Omega)$. The trace conjecture (3.1) does hold as $t \downarrow 0$, though, since either Ω is a translate of A or else (by the proof of Corollary 2(a)) Ω has greater euclidean area than A, while

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega_{euclid})} = \frac{|\Omega|_{euclid}}{4\pi t} + O(t^{-1/2}) \qquad \text{as } t \downarrow 0$$

(assuming $\partial\Omega$ is smooth). Similar statements hold for the analogues of Theorem 1 and Corollaries 2(b)(c), 3 and 4 for the trace of the heat kernel, for large t and as $t \downarrow 0$. For references to known extremal results for the zeta function and determinant of the Laplacian and the trace of the heat kernel (on various manifolds), see Section 3 of [12] or [13].

The analogue of the convexity result in Theorem 5 is *false* for the trace of the heat kernel, for small time, as we now show. Let M be a doubly connected plane domain with smooth boundary components ∂M_D and ∂M_N, let g be the euclidean metric and take w to be a positive smooth function on $\overline{M}$. Write $\lambda_j(w)$ for the j-th eigenvalue of $w^{-1}\Delta$, as in (2.3). Then by putting $f \equiv 1$ in [4, Th.7.2] we have the asymptotic expansion

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(w)} = \frac{|M|_w}{4\pi t} + \frac{|\partial M_N|_w - |\partial M_D|_w}{8\sqrt{\pi t}} + O(1) \qquad \text{as } t \downarrow 0, \tag{3.2}$$

where

$$|\partial M_D|_w := \int_{\partial M_D} \sqrt{w(z)}\,|dz| \qquad \text{and} \qquad |\partial M_N|_w := \int_{\partial M_N} \sqrt{w(z)}\,|dz|$$

denote the lengths of the boundary components of M in the metric wg. Plainly the first term of the asymptotic expansion is linear in w, since $|M|_w = \int_M w\,d\mu$, but the second term need be neither convex nor concave in w, since both $|\partial M_D|_w$ and $|\partial M_N|_w$ are concave in w. Thus it is false that the trace of the heat kernel must be convex in the weight function w, for small t. For large t the trace *is* convex in w as a consequence of Theorem 5; see above. Notice that for purely Dirichlet boundary conditions it remains reasonable to hope that the trace of the heat kernel is convex in w for all t, since $|\partial M_N|_w = 0$ in that case and the two leading terms of (3.2) are hence convex in w.

4. Fundamental properties

In this section we justify the claims made near the beginning of Section 2 about the fundamental properties of the eigenvalues $\lambda_j(w)$ and their eigenfunctions.

Assume M is an acceptable, bounded, doubly connected plane domain with boundary components ∂M_D and ∂M_N. Suppose w is an admissible function on M. We have that $f(z)$ maps M conformally onto an acceptable domain Ω, as in the definition of admissibility in Section 2, with $w = (h \circ f)|f'|^2$ where $h \in L^\infty(\Omega)$ and $h > 0$ a.e. Since Ω is doubly connected, its complement in the Riemann sphere has two components. Call these components Ω_D^c and Ω_N^c, where $\partial\Omega_D \subset \Omega_D^c$ and $\partial\Omega_N \subset \Omega_N^c$.

Since Ω is acceptable, $\partial\Omega_N$ is a quasicircle. Let U be a disk containing $\overline{\Omega}$ and let $\widetilde{\Omega} := U \cap (\Omega \cup \Omega_D^c)$. Then $\widetilde{\Omega}$ is a bounded plane domain with boundary consisting of one or two quasicircles, since if Ω_D^c is bounded then $\partial\widetilde{\Omega} = \partial\Omega_N$ and if Ω_D^c is unbounded then $\partial\widetilde{\Omega} = \partial U \cup \partial\Omega_N$. A theorem of P. W. Jones [10, Th.4] shows that $\widetilde{\Omega}$ is a Sobolev extension domain, which means the following. Given an open square Q containing the closure of $\widetilde{\Omega}$, a bounded linear operator E: $H^1(\widetilde{\Omega}) \to H_0^1(Q)$ exists with $E\phi = \phi$ pointwise in $\widetilde{\Omega}$ and with

$$\int_Q \left[(E\phi)^2 + |\nabla(E\phi)|^2\right] d\mu \le C \int_{\widetilde{\Omega}} \left[\phi^2 + |\nabla\phi|^2\right] d\mu \tag{4.1}$$

for all $\phi \in H^1(\widetilde{\Omega})$, where $C = C(\widetilde{\Omega}, Q) > 0$. Note that $H^1_{mix}(\Omega) \subset H^1(\widetilde{\Omega})$, just by extending functions in $H^1_{mix}(\Omega)$ to equal zero on Ω_D^c. Thus functions in $H^1_{mix}(\Omega)$ extend pointwise to functions in $H_0^1(Q)$, with the Sobolev norm increasing by only a bounded factor, and since $H_0^1(Q)$ imbeds compactly into $L^2(Q)$ (see Part IV (4) of [1, p. 144] with $j = 0, k = 2, m = 1, n = 2, p = 2, q = 2$), we conclude that $H^1_{mix}(\Omega)$ imbeds compactly into $L^2(\Omega)$.

To justify the claims about $\lambda_j(w)$ in the third paragraph of Section 2, one first considers the eigenvalue problem for $\lambda_j(h)$ on Ω, adapting the standard arguments (cf. [3, pp. 53, 55 ff.] and [7, pp. 212–214]) to apply to the Rayleigh quotient

$$\frac{\int_\Omega |\nabla\phi|^2\,d\mu}{\int_\Omega \phi^2 h\,d\mu}, \qquad \phi \in H^1_{mix}(\Omega).$$

This yields existence of the eigenvalues $\lambda_j(h)$ and eigenfunctions $\phi_j \in H^1_{mix}(\Omega) \cap C(\Omega)$, with $-\Delta\phi_j = \lambda_j(h)h\phi_j$ weakly in $H^1_{mix}(\Omega)$ and $0 \le \lambda_1(h) < \lambda_2(h) \le \lambda_3(h) \le \cdots \to \infty$. In particular the first eigenfunction never changes sign and the first eigenvalue is simple, that is, $\lambda_1(h) < \lambda_2(h)$. Hence the first eigenfunction ϕ_1 is unique up to constant multiples. Poincaré's minimax principle (2.3) holds for $\lambda_j(h)$ with trial space $H^1_{mix}(\Omega)$ by [2, p. 97] or [3, p. 61]. Note that these standard arguments are where we use the boundedness and positivity of h, for we want $\int_\Omega \phi^2 h\, d\mu$ to be positive and finite when $\phi \in L^2(\Omega)$, $\phi \not\equiv 0$. We also use in these arguments the fact that $H^1_{mix}(\Omega)$ imbeds compactly into $L^2(\Omega)$.

The following argument shows that $\lambda_1(h) > 0$. For suppose instead that $\lambda_1(h) = 0$. Then $\nabla\phi_1 = 0$ a.e. and so $\phi_1 \in H^1_{mix}(\Omega)$ is a non-zero constant, say $\phi_1 \equiv 1$. (This *ought* to be impossible, since we think that ϕ_1 equals zero on $\partial\Omega_D$, and the succeeding argument simply makes this precise without assuming any smoothness of $\partial\Omega_D$.) Since the constant function 1 belongs to $H^1_{mix}(\Omega)$, a sequence $\{\eta_j\}$ of functions in $H^1(\Omega) \cap C^\infty(\Omega)$ exists with each η_j equalling zero near $\partial\Omega_D$ and with η_j converging to 1 in $H^1(\Omega)$. Extend η_j by defining $\eta_j := 0$ on Ω^c_D. We can assume that $\eta_1 \in C^\infty(\mathbf{R}^2)$ and that η_1 equals 1 on a neighborhood of Ω^c_N. After replacing η_j by $\eta_1 + (1-\eta_1)\eta_j$ for $j \ge 2$, we can also assume that $\eta_j \in C^\infty(\mathbf{R}^2)$ and that a fixed neighborhood of Ω^c_N exists on which every η_j equals 1. Next, since $\Omega \cup \Omega^c_N$ is simply connected and $\partial(\Omega\cup\Omega^c_N) = \partial\Omega_D$ contains more than one point by hypothesis, the Riemann mapping theorem provides a conformal map F of the unit disk D onto $\Omega \cup \Omega^c_N$. Then $\eta_j \circ F$ is a smooth function with compact support in D and so it can be used as a trial function for $\lambda_1^{Dir}(D)$, the first eigenvalue of the euclidean Laplacian on D with purely Dirichlet boundary conditions. Also, $\eta_j \circ F = 1$ on $F^{-1}(\Omega^c_N)$. Thus

$$\begin{aligned} 0 < \lambda_1^{Dir}(D) &\le \frac{\int_D |\nabla(\eta_j \circ F)|^2\, d\mu}{\int_D (\eta_j \circ F)^2\, d\mu} \\ &\le \frac{\int_D |\nabla(\eta_j \circ F)|^2\, d\mu}{\mu(F^{-1}(\Omega^c_N))} \\ &= \frac{\int_\Omega |\nabla\eta_j|^2\, d\mu}{\mu(F^{-1}(\Omega^c_N))} \to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

since $\eta_j \to 1$ in $H^1_{mix}(\Omega)$. This contradiction implies that $\lambda_1(h) > 0$, as we wished to show.

Finally, the claims about $\lambda_j(w)$ in the third paragraph of Section 2 hold simply by conformally transplanting from Ω to M; that is, one actually defines $\lambda_j(w) := \lambda_j(h)$ and $\psi_j := \phi_j \circ f$. Certainly ψ_j is continuous, and since $\psi_j \in H^1_{mix}(\Omega)\circ f = H^1_{mix}(M)$ by Lemma 7 below, a simple change of variable shows that $-\Delta\psi_j = \lambda_j(w)w\psi_j$ weakly in $H^1_{mix}(M)$. Further, the minimax principle (2.3) holds just by changing variable with f in the minimax principle for $\lambda_j(h)$. Lastly, (2.3) shows that $\lambda_j(w)$ is independent of the conformal map f by which the admissibility of w is established.

We must still establish the lemma required in the preceding paragraph, which says that conformal maps between acceptable domains leave the trial space H^1_{mix} invariant.

LEMMA 7. *Suppose M and Ω are acceptable, bounded, doubly connected plane domains. If $f(z)$ maps M conformally onto Ω with ∂M_D and $\partial\Omega_D$ corresponding under f and ∂M_N and $\partial\Omega_N$ corresponding under f, then $H^1_{mix}(\Omega)\circ f = H^1_{mix}(M)$.*

Proof of Lemma 7. It suffices to prove the inclusion $H^1_{mix}(M) \subset H^1_{mix}(\Omega)\circ f$, since the roles of M and Ω can then be interchanged.

Let $u \in H^1_{mix}(M)$, with u equalling the limit in $H^1(M)$ of some sequence of test functions $\eta_j \in H^1(M)\cap C^\infty(M)$, each of which equals zero on some neighborhood of ∂M_D. By an approximation argument, we can assume each η_j is bounded. Then $\eta_j \circ f^{-1} \in L^2(\Omega)\cap C^\infty(\Omega)$ and this function equals zero on a neighborhood of $\partial\Omega_D$. Further, $\eta_j\circ f^{-1} \in H^1_{mix}(\Omega)$ since $\int_\Omega |\nabla(\eta_j\circ f^{-1})|^2\,d\mu = \int_M |\nabla\eta_j|^2\,d\mu < \infty$, and so

$$\begin{aligned}\int_\Omega (\eta_j\circ f^{-1} - \eta_\ell\circ f^{-1})^2\,d\mu &\leq \frac{1}{\lambda_1(\Omega)}\int_\Omega |\nabla(\eta_j\circ f^{-1} - \eta_\ell\circ f^{-1})|^2\,d\mu\\ &= \frac{1}{\lambda_1(\Omega)}\int_M |\nabla(\eta_j - \eta_\ell)|^2\,d\mu \to 0\end{aligned}$$

as $j,\ell\to\infty$, where $\lambda_1(\Omega) > 0$ denotes the eigenvalue $\lambda_1(h)$ that was considered in the first part of this section, with $h\equiv 1$. Thus $\{\eta_j\circ f^{-1}\}$ is a Cauchy sequence in $H^1(\Omega)$ consisting of smooth functions that equal zero on some neighborhood of $\partial\Omega_D$, and so $\eta_j\circ f^{-1}$ converges in $H^1(\Omega)$ to some function $\tilde{u}\in H^1_{mix}(\Omega)$. By passing to subsequences, we can assume further that $\eta_j\to u$ a.e. in M and $\eta_j\circ f^{-1}\to\tilde{u}$ a.e. in Ω. Thus $u = \tilde{u}\circ f$ a.e. in M, and so $H^1_{mix}(M)\subset H^1_{mix}(\Omega)\circ f$, which proves the lemma. □

To conclude this section, we justify the lower bound $\lambda_j(w)\geq \alpha j$ of Weyl type, in (2.5). We need only establish this for all large j, since $\lambda_1(w) > 0$. From the Sobolev extension property (4.1) we see that if $\phi\in H^1(\widetilde{\Omega})$ then

$$\frac{\int_{\widetilde{\Omega}}|\nabla\phi|^2\,d\mu}{\int_{\widetilde{\Omega}}\phi^2\,d\mu} \geq \frac{1}{C}\,\frac{\int_Q |\nabla(E\phi)|^2\,d\mu}{\int_Q (E\phi)^2\,d\mu} - 1.$$

Recalling the minimax principle (2.3), we therefore deduce that

$$\lambda_j(w) = \lambda_j(h) = \min_{L_j}\max_{\phi\in L_j\setminus\{0\}} \frac{\int_\Omega|\nabla\phi|^2\,d\mu}{\int_\Omega \phi^2 h\,d\mu}, \qquad L_j\subset H^1_{mix}(\Omega),$$

$$
\begin{aligned}
&\geq \frac{1}{\|h\|_\infty} \min_{L_j} \max_{\phi \in L_j \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 \, d\mu}{\int_\Omega \phi^2 \, d\mu}, \qquad L_j \subset H^1_{mix}(\Omega), \\
&\geq \frac{1}{\|h\|_\infty} \min_{L_j} \max_{\phi \in L_j \setminus \{0\}} \left(\frac{1}{C} \frac{\int_Q |\nabla \phi|^2 \, d\mu}{\int_Q \phi^2 \, d\mu} - 1 \right), \qquad L_j \subset H^1_0(Q), \\
&= \frac{1}{\|h\|_\infty} \left(\frac{1}{C} \lambda_j^{Dir}(Q) - 1 \right),
\end{aligned}
$$

where $\lambda_j^{Dir}(Q)$ denotes the j-th eigenvalue of the euclidean Laplacian on Q under purely Dirichlet boundary conditions. Since $\lambda_j^{Dir}(Q)$ is comparable to j, for large j, we deduce that $\lambda_j(w) \geq \alpha j$ for large j, which was our goal.

5. Proof of Theorems 1, 5 and Corollaries 2, 3, 4

Proof of Theorem 1 The proof of Theorem 1 goes exactly like the proof of [13, Th. 1] for the case of purely Dirichlet boundary conditions, with just the following two changes. Instead of M we use A, and instead of the space $H^1_0(M)$ of trial functions, we use the space $H^1_{mix}(A)$.

Proofs of Corollaries 2, 3 *and* 4 Corollaries 2, 3, 4 are proved exactly like Corollaries 3, 5, 6 (respectively) in [13], where the author and C. Morpurgo dealt with purely Dirichlet boundary conditions. Of course, during the proofs one should apply Theorem 1 of *this* paper instead of Theorem 1 of [13], and one should invoke the definition of "admissible" from *this* paper rather than the (different) definition in [13].

Note that in Corollaries 2, 3 and 4, it is a hypothesis that the circle of radius R_0 corresponds under f to the inner boundary component of Ω. Thus one need not arrange this correspondence during the proof, which was done in [13] by means of the self-map $z \mapsto R_0 R/z$ of the annulus. (This self-map does not preserve mixed boundary conditions on the annulus, and so we must avoid using it in this paper.)

Proof of Theorem 5 The manifold M and the metric g are fixed in this theorem. Let m be a positive integer. We write

$$
S(w) := \sum_{j=1}^{m} \frac{1}{\lambda_j(w)}
$$

for the sum of reciprocal eigenvalues, and we also write $\nabla := \nabla_g$ for the gradient, $\Delta := \Delta_g$ for the Laplace-Beltrami operator, and $dV := dV_g$ for the volume element, in the metric g.

The variational characterization [2, pp. 99–100] of the sum of reciprocal eigenvalues for $-\Delta\psi = \lambda w \psi$ is that

$$S(w) = \sup_{\{\psi_1,\dots,\psi_m\}} \sum_{j=1}^{m} \int_M \psi_j^2 w \, dV, \tag{5.1}$$

where $\{\psi_1, \dots, \psi_m\}$, is required to be a collection of m linearly independent functions in the Sobolev space $H^1_{mix}(M_g)$ with $\int_M g(\nabla\psi_i, \nabla\psi_j)\, dV = \delta_{ij}$.

To prove the first part of the theorem, we take $q \in (0, 1)$ and let u and v be positive smooth functions on $\overline{M}$. Fix $t \in (0, 1)$ and put $w := [tu^q + (1-t)v^q]^{1/q}$, so that $w^q = tu^q + (1-t)v^q$. Let $\psi_1, \dots, \psi_m \in H^1_{mix}(M_g)$ be linearly independent eigenfunctions of $w^{-1}\Delta$ on M that satisfy

$$\lambda_j(w) = \frac{\int_M g(\nabla\psi_j, \nabla\psi_j)\, dV}{\int_M \psi_j^2 w \, dV} \quad \text{and} \quad \int_M g(\nabla\psi_i, \nabla\psi_j)\, dV = \delta_{ij},$$

so that $S(w) = \sum_{j=1}^m \int_M \psi_j^2 w \, dV$. Writing $\psi := \sum_{j=1}^m \psi_j^2 \geq \psi_1^2 > 0$, we have

$$\begin{aligned} S(w)^q = \left(\sum_{j=1}^{m} \int_M \psi_j^2 w \, dV \right)^q &= \left(\int_M \psi [tu^q + (1-t)v^q]^{1/q} \, dV \right)^q \\ &= \left\| t\psi^q u^q + (1-t)\psi^q v^q \right\|_{1/q} \\ &\leq \left\| t\psi^q u^q \right\|_{1/q} + \left\| (1-t)\psi^q v^q \right\|_{1/q} \qquad (5.2) \\ &= t \left(\int_M \psi u \, dV \right)^q + (1-t) \left(\int_M \psi v \, dV \right)^q \\ &\leq tS(u)^q + (1-t)S(v)^q \end{aligned}$$

by the variational characterization (5.1); since $1/q > 1$, Minkowski's inequality at (5.2) is strict unless u is a positive multiple of v. This proves that $S(w)^q$ is a convex functional of w^q, with the convexity being strict except when applied to multiples of a fixed w.

Next, redefine $w := u^t v^{1-t}$, so that $\log w = t \log u + (1-t) \log v$. With ψ_j and ψ defined as before, with respect to this new w we have

$$\begin{aligned} \log S(w) = \log \left(\sum_{j=1}^{m} \int_M \psi_j^2 w \, dV \right) &= \log \int_M (\psi u)^t (\psi v)^{1-t} \, dV \\ &\leq \log \left\{ \left(\int_M \psi u \, dV \right)^t \left(\int_M \psi v \, dV \right)^{1-t} \right\} \qquad (5.3) \\ &= t \log \int_M \psi u \, dV + (1-t) \log \int_M \psi v \, dV \\ &\leq t \log S(u) + (1-t) \log S(v) \end{aligned}$$

by the variational characterization (5.1), and Hölder's inequality at (5.3) is strict unless u is a positive multiple of v. This proves that $\log S(w)$ is a convex functional of $\log w$, with the convexity being strict except when applied to multiples of a fixed w. In fact the convexity of $\log S(w)$ with respect to $\log w$ is the limiting case as $q \to 0$ of the convexity of $S(w)^q$ with respect to w^q, because letting $q \to 0$ in

$$S([tu^q + (1-t)v^q)]^{1/q}) \le \left[tS(u)^q + (1-t)S(v)^q\right]^{1/q}$$

gives

$$S(u^t v^{1-t}) \le S(u)^t S(v)^{1-t},$$

which is equivalent to

$$\log S(\exp[t \log u + (1-t)\log v]) \le t \log S(u) + (1-t)\log S(v).$$

To prove the claims in Theorem 5 about convexity of the Φ-functional, proceed exactly as in the proof of [13, Th.8] for the case of purely Dirichlet boundary conditions, except instead of using the space $H_0^1(M_g)$ of trial functions, use the space $H^1_{mix}(M_g)$.

6. Proof of Theorem 6

Let $L = \log(R/R_0)$, so that $|A|_{cylinder} = 2\pi L$. We begin by collecting facts about the eigenvalues and eigenfunctions of the annulus. Observe to start with that the eigenvalues $\{\lambda_j^{DN}(A_{cylinder})\colon j = 1,2,3,\ldots\}$ can be computed by separation of variables to be $\{\lambda_{\nu\ell} = \nu^2 + (2\ell-1)^2\pi^2/4L^2\colon \nu \in \mathbf{Z}, \ell \ge 1\}$, with corresponding normalized eigenfunctions

$$\psi_{\nu\ell}^{DN}(re^{i\theta}) := \sqrt{2/(\pi L\lambda_{\nu\ell})}\sin[(2\ell-1)\pi(\log r/R_0)/2L] \times \left\{ \begin{array}{ll} \sin\nu\theta, & \text{if } \nu > 0 \\ 1/\sqrt{2}, & \text{if } \nu = 0 \\ \cos\nu\theta, & \text{if } \nu < 0 \end{array} \right\}.$$

Then $\{\psi_{\nu\ell}^{DN}\}$ is a linearly independent set in the trial space

$$\begin{aligned} H^1_{DN}(A) := {} & \text{the closure in } H^1(A) \text{ of } \{\psi \in H^1(A) \cap C^\infty(A)\colon \psi = 0 \\ & \text{on a neighborhood of } \Gamma_1\} \end{aligned}$$

and it satisfies the orthonormality condition

$$\int_A \nabla\psi_{\nu\ell}^{DN} \cdot \nabla\psi_{\nu'\ell'}^{DN}\, d\mu = \delta_{\nu\nu'}\delta_{\ell\ell'}.$$

We proceed similarly for the eigenvalue problem with the boundary conditions swapped: since $\{\lambda_j^{ND}(A_{cylinder})\} = \{\lambda_{\nu\ell}\}$, we have normalized eigenfunctions

$$\psi_{\nu\ell}^{ND}(re^{i\theta}) := \sqrt{2/(\pi L\lambda_{\nu\ell})}\cos[(2\ell-1)\pi(\log r/R_0)/2L]\times\left\{\begin{array}{ll}\sin\nu\theta, & \text{if } \nu>0\\ 1/\sqrt{2}, & \text{if } \nu=0\\ \cos\nu\theta, & \text{if } \nu<0\end{array}\right\}$$

with corresponding eigenvalues $\lambda_{\nu\ell}$, and $\{\psi_{\nu\ell}^{ND}\}$ is a linearly independent set in the trial space $H^1_{ND}(A)$ with the orthonormality condition

$$\int_A \nabla\psi_{\nu\ell}^{ND}\cdot\nabla\psi_{\nu'\ell'}^{ND}\,d\mu = \delta_{\nu\nu'}\delta_{\ell\ell'}.$$

For each $m\geq 1$, let $I(m)$ be a set of m distinct elements from $\{(\nu,\ell)\colon\ \nu\in\mathbf{Z},\ \ell\geq 1\}$ with the property that the numbers $\lambda_{\nu\ell}$ for $(\nu,\ell)\in I(m)$ are a permutation of the eigenvalues $\lambda_j^{DN}(A_{cylinder})$ for $j=1,\dots,m$.

We complete the preliminaries by putting $w=(h\circ f)|f'|^2$ and observing that w is admissible for the eigenvalue problems defining $\lambda_j^{DN}(A_{wg})$ and $\lambda_j^{ND}(A_{wg})$. Then, by the observations about conformal invariance before (2.4),

$$\lambda_j^{DN}(\Omega_{hg}) = \lambda_j^{DN}(A_{wg}) \qquad\text{and}\qquad \lambda_j^{ND}(\Omega_{hg}) = \lambda_j^{ND}(A_{wg}).$$

We will need the following variational characterization for the sum of the first m reciprocal eigenvalues. The characterization follows from the minimax principle (2.3) (as is proved in [2, pp. 99–100]), and it says that

$$\sum_{j=1}^m \frac{1}{\lambda_j^{DN}(A_{wg})} = \sup_{\{\psi_1,\dots,\psi_m\}}\sum_{j=1}^m\int_A \psi_j^2 w\,d\mu,$$

where $\{\psi_1,\dots,\psi_m\}$ is required to be a collection of m linearly independent functions in $H^1_{DN}(A)$, with $\int_A\nabla\psi_i\cdot\nabla\psi_j\,d\mu=\delta_{ij}$. By using the functions $\psi_{\nu\ell}^{DN}$ as trial functions in this variational characterization and by arguing similarly with the boundary conditions interchanged, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^m\frac{1}{\lambda_j^{DN}(A_{wg})}+\frac{1}{2}\sum_{j=1}^m\frac{1}{\lambda_j^{ND}(A_{wg})}\\
&\geq \frac{1}{2}\sum_{(\nu,\ell)\in I(m)}\int_A|\psi_{\nu\ell}^{DN}|^2 w\,d\mu+\frac{1}{2}\sum_{(\nu,\ell)\in I(m)}\int_A|\psi_{\nu\ell}^{ND}|^2 w\,d\mu \qquad (6.1)\\
&= \sum_{(\nu,\ell)\in I(m)}\frac{1}{\pi L\lambda_{\nu\ell}}\int_A\Big[\sin^2[(2\ell-1)\pi(\log r/R_0)/2L]\\
&\qquad\qquad + \cos^2[(2\ell-1)\pi(\log r/R_0)/2L]\Big]\\
&\quad\times\left\{\begin{array}{ll}\sin^2\nu\theta, & \text{if } \nu>0\\ 1/2, & \text{if } \nu=0\\ \cos^2\nu\theta, & \text{if } \nu<0\end{array}\right\} w(re^{i\theta})\,r\,dr\,d\theta
\end{aligned}$$

$$= \sum_{(\nu,\ell)\in I(m)} \frac{2}{|A|_{cylinder}\lambda_{\nu\ell}} \int_A \left\{ \begin{array}{ll} \sin^2 \nu\theta, & \text{if } \nu > 0 \\ 1/2, & \text{if } \nu = 0 \\ \cos^2 \nu\theta, & \text{if } \nu < 0 \end{array} \right\} w(re^{i\theta})\, r dr d\theta.$$

Now repeat this computation, except replacing $I(m)$ by $\{(-\nu, \ell)\colon (\nu, \ell) \in I(m)\}$; adding the two inequalities gives

$$\begin{aligned} &\sum_{j=1}^{m} \frac{1}{2}\left(\frac{1}{\lambda_j^{DN}(A_{wg})} + \frac{1}{\lambda_j^{ND}(A_{wg})}\right) \\ &\geq \sum_{(\nu,\ell)\in I(m)} \frac{1}{|A|_{cylinder}\lambda_{\nu\ell}} \int_A \{\sin^2 \nu\theta + \cos^2 \nu\theta\} w(re^{i\theta})\, r dr d\theta \\ &= \sum_{(\nu,\ell)\in I(m)} \frac{|A|_w/|A|_{cylinder}}{\lambda_{\nu\ell}} \\ &= \sum_{j=1}^{m} \frac{|\Omega|_h/|A|_{cylinder}}{\lambda_j^{DN}(A_{cylinder})} \end{aligned} \tag{6.2}$$

for each $m = 1, 2, 3, \ldots$. Incidentally, the proof up to this point follows the lines indicated (though not spelled out) by Hersch [8, pp. 27, 32].

Now inequality (2.9) follows from (6.2) by the majorization method of Hardy, Littlewood and Pólya [13, Prop.10]. Inequality (2.8), of course, relies only on the convexity of Φ.

Assume for the rest of this proof that Φ is strictly convex, and suppose that (2.9) holds with equality. Then the strict convexity of Φ allows us to invoke a result due to Schur, a result given as equality case (v) of [13, Prop.10]. This gives

$$\frac{1}{2}\left(\frac{1}{\lambda_1^{DN}(A_{wg})} + \frac{1}{\lambda_1^{ND}(A_{wg})}\right) = \frac{|\Omega|_h/|A|_{cylinder}}{\lambda_1^{DN}(A_{cylinder})} = \frac{|\Omega|_h/|A|_{cylinder}}{\lambda_{01}}. \tag{6.3}$$

Hence equality holds at (6.1) with $m = 1$, meaning that

$$\lambda_1^{DN}(A_{wg}) = \frac{\int_A |\nabla\psi_{01}^{DN}|^2\, d\mu}{\int_A |\psi_{01}^{DN}|^2 w\, d\mu} \quad \text{and} \quad \lambda_1^{ND}(A_{wg}) = \frac{\int_A |\nabla\psi_{01}^{ND}|^2\, d\mu}{\int_A |\psi_{01}^{ND}|^2 w\, d\mu}.$$

Thus ψ_{01}^{DN} is a $\lambda_1^{DN}(A_{wg})$-eigenfunction, so that

$$w\psi_{01}^{DN} = \frac{-\Delta\psi_{01}^{DN}}{\lambda_1^{DN}(A_{wg})} = \frac{\lambda_{01}|z|^{-2}}{\lambda_1^{DN}(A_{wg})}\psi_{01}^{DN}$$

weakly in $H_{DN}^1(A)$, and hence

$$w = \frac{\lambda_{01}|z|^{-2}}{\lambda_1^{DN}(A_{wg})} \quad \text{a.e.; similarly} \quad w = \frac{\lambda_{01}|z|^{-2}}{\lambda_1^{ND}(A_{wg})} \quad \text{a.e.}$$

Adding the last two equalities and then using (6.3) yields that for almost all z,

$$2w(z) = \lambda_{01}|z|^{-2}\left(\frac{1}{\lambda_1^{DN}(A_{wg})} + \frac{1}{\lambda_1^{ND}(A_{wg})}\right) = 2|z|^{-2}|\Omega|_h/|A|_{\text{cylinder}}.$$

That is, $w(z) = |z|^{-2}|\Omega|_h/|A|_{cylinder}$ a.e., which is the mass density function on A representing the homogeneous cylinder imbedded in $\mathbf{R}^3$ of length L, radius 1 and total mass $|\Omega|_h$. Since Ω_{hg} is isometric via $f(z)$ to A_{wg}, we deduce that Ω_{hg} is isometric a.e. to the cylinder.

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