ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 4, Winter 1998

# ON THE STRONG TYPE MULTIPLIER NORMS OF RATIONAL FUNCTIONS IN SEVERAL VARIABLES

MICHAŁ WOJCIECHOWSKI

## 1. Introduction

Let $G$ be a locally compact abelian group, $\Gamma$ its dual. For $\phi \in L^\infty(\Gamma)$ denote by $T_\phi$ the $L^2(G)$ multiplier transform defined by $\phi$. If $T_\phi$ extends to a bounded operator on $L^p(G)$ we put $N_p(\phi) = \|T_\phi\colon L^p(G) \to L^p(G)\|$. Otherwise we put $N_p(\phi) = \infty$. Denote by $M(G)$ the space of regular complex-valued Borel measures on $G$ with the total variation (denoted $\|\cdot\|_{M(G)}$) as norm. We deal with the models $G = \mathbb{R}^d$ ($d$-dimensional Euclidean space) and $G = \mathbb{T}^d$ (the $d$-dimensional torus). In the present paper we study the dependence on $p$ of the function $p \mapsto N_p(\phi)$.

In Section 3 we show that if $\phi$ satisfies some regularity conditions and $\phi$ has no limit at infinity then $N_p(\phi) \geq C \cdot \max(p, \frac{p}{p-1})$ for some $C > 0$. In Section 4 we deal with rational multipliers $R = PQ^{-1}$ such that $Q$ is a somewhat elliptic polynomial in the sense of Definition 1 below.

Let $\mathbb{R}^d_+$ and $\mathbb{Z}^d_+$ denote respectively the subsets of elements of $\mathbb{R}^d$ and $\mathbb{Z}^d$ with non-negative coordinates. For $y = (y_\nu) \in \mathbb{R}^d$ and $z = (z_\nu) \in \mathbb{R}^d$ we write $y \leq z$ iff $y_\nu \leq z_\nu$ for $\nu = 1, 2, \ldots, d$. By $\mathcal{P}_d$ we denote the space of all polynomials in $d$ variables $x = (x_1, \ldots, x_d)$. If $Q \in \mathcal{P}_d$ then

$$Q(x) = \sum_\gamma a_\gamma x^\gamma$$

with all $\gamma$'s distinct, where $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \ldots x_d^{\gamma_d}$. In this framework, we put $\operatorname{sp} Q = \{\gamma \in \mathbb{Z}^d_+\colon a_\gamma \neq 0\}$ and we signify by $\operatorname{conv} Q$ the convex hull in $\mathbb{R}^d$ of the set $\bigcup_{\gamma \in \operatorname{sp} Q} \{\beta\colon 0 \leq \beta \leq \gamma\}$.

*Definition* 1. A polynomial $Q$ is called *somewhat elliptic* if there exists $C > 0$ such that

$$|Q(x)| > C \cdot |x^\gamma| \quad \text{whenever } \gamma \in \mathbb{Z}^d_+ \cap \operatorname{conv} Q \text{ and } x \in \mathbb{R}^d.$$

(*Here and in the sequel, the symbol "C" denotes a non-negative constant which can change in value from one occurrence to another.*)

---

Received May 29, 1997.
1991 Mathematics Subject Classification. Primary 42B15, 42B20, 60G46.
Supported in part by KBN grant 2 P301 004 06.

Examples of somewhat elliptic polynomials are the elliptic polynomials with no roots in $\mathbb{R}^d$ and fundamental polynomials of smoothnesses (cf. [8]).

*Remark.* The notion of somewhat elliptic polynomials is similar to but stronger than the notion of "strongly slightly elliptic polynomials" introduced in [8], p. 403.

The main result of Section 4 is Theorem 3 (stated in Section 2) which asserts the following dichotomy: for any rational function $R = PQ^{-1}$ where $Q$ is somewhat elliptic either $N_1(R) < \infty$ (that is, $R$ is the Fourier transform of a bounded measure) or $N_p(R) > C \max\left(p, \frac{p}{p-1}\right)$ for $1 < p < \infty$.

The origin of this paper was the study of special classes of rational multipliers which occur as entries of the multiplier matrix for the so-called canonical projection of the jet representation of a general anisotropic Sobolev space. This study has been initiated in [9] and [8] and developed further in the forthcoming memoir [1]. It turns out that fundamental polynomials of smoothnesses are special cases of somewhat elliptic polynomials. An application of the reasoning in Section 4 is the observation that the entries of the multiplier matrix of the canonical projection generated by non-maximal elements of a smoothness are the Fourier transforms of measures (cf. Corollary 5).

All the function spaces and measure spaces on $\mathbb{R}^d$ considered in this paper are embedded in the space of tempered distributions. The Fourier transform of a function $f$ or a measure $\mu$ (in symbols $\widehat{f}$, resp. $\widehat{\mu}$) is understood in the distributional sense (cf. [12], Chapt. 1, §3).

The author gratefully acknowledges many helpful suggestions made by Professor A. Pelczyński during the preparation of this paper.

## 2. Results

The main result of the reasoning in Section 3 is Theorem 1, which will be stated here. It concerns a wider class of multipliers than the rational ones, and gives a lower bound for the $L^p$-norms of multipliers as $p$ tends either to 1 or to $\infty$.

THEOREM 1. *Let $\phi\colon \mathbb{Z}^d \to \mathbb{C}$. Suppose that either*

(I) *there exist $a, b \in \mathbb{C}$ with $a \neq b$ and sequences $(k_j)_{j=1}^{\infty} \subset \mathbb{Z}^d$ and $(n_j)_{j=1}^{\infty} \subset \mathbb{Z}^d$ such that for every $n \in \mathbb{Z}^d$, we have, as $j \to \infty$,*

$$\phi(n + k_j) \to a, \quad \phi(n - k_j) \to a, \quad \phi(n + n_j) \to b, \quad \phi(n - n_j) \to b,$$

*or*

(II) *there exist $a, b \in \mathbb{C}$ with $a \neq b$ and a sequence $(k_j)_{j=1}^{\infty} \subset \mathbb{Z}^d$ such that for every $n \in \mathbb{Z}^d$, we have, as $j \to \infty$,*

$$\phi(n + k_j) \to a \quad , \quad \phi(n - k_j) \to b.$$

*Then there exists $C > 0$ such that for $1 < p < \infty$,*

$$(1) \qquad N_p(\phi) > C \cdot |a - b| \cdot \max\left(p, \frac{p}{p-1}\right),$$

*where $C > 0$ is a numerical constant independent of $\phi$.*

One can consider this result as a quantitative version of the Wiener theorem (cf. [8], Prop. 3.1) which under similar assumptions on $\phi$ asserts that $N_p(\phi) \to \infty$ as $p \to 1$. However in Wiener's theorem no information on the growth of $N_p(\phi)$ is given as $p \to 1$.

By the de Leeuw transference theorem (cf. [12], Chapt. VII, Th. 3.8) we immediately get:

COROLLARY 1. *Let the restriction to $\mathbb{Z}^d$ of a continuous function $\phi: \mathbb{R}^d \to \mathbb{C}$ satisfy either* (I) *or* (II). *Then $\phi$ satisfies* (1).

COROLLARY 2. *Let $\phi$: $\mathbb{Z}^d \to \mathbb{C}$ extend to a differentiable function, say $f$: $\mathbb{R}^d \to \mathbb{C}$ such that $\nabla f(x) \to 0$ as $|x| \to \infty$. Then both $N_p(f)$ and $N_p(\phi)$ satisfy* (1) *with some $a \neq b$ provided $f(x)$ has no limit at infinity.*

In the next theorem we apply the method used in the proof of Theorem 1 to estimate the growth of $N_p(\phi)$ as $p$ tends either to 1 or to infinity for discontinuous $\phi$.

THEOREM 2. *Let $x$ be a limit point of an open set $\mathcal{U} \subset \mathbb{R}^d$ and let $\mathcal{U}$ be symmetric with respect to $x$. Suppose that $\phi$: $\mathbb{R}^d \to \mathbb{C}$ is a bounded function such that $\phi_{|\mathcal{U}}$ is a continuous function which has no continuous extension on $\mathcal{U} \cup \{x\}$. Then $N_p(\phi) > C \cdot \max\{p, \frac{p}{p-1}\}$ for $1 < p < \infty$.*

Let $h = (h_1, h_2, \ldots, h_d) \in \mathbb{R}^d$, $h \neq 0$. We define $\delta_h^t$: $\mathbb{R}^d \to \mathbb{R}^d$ for $h \in \mathbb{R}_+^d$ and $t > 0$ by letting

$$\delta_h^t x = (t^{h_1} x_1, t^{h_2} x_2, \ldots, t^{h_d} x_d)$$

for $x \in \mathbb{R}^d$. Let $h$ satisfy $h_j > 0$ for $j = 1, 2, \ldots, d$. A function $\phi$: $\mathbb{R}^d \to \mathbb{C}$ is called $h$-homogeneous of $h$-degree 0 if $\phi(x) = \phi(\delta_h^t x)$ for every $x \in \mathbb{R}^d$ and $t > 0$.

COROLLARY 3. *Let $\phi$: $\mathbb{R}^d \to \mathbb{C}$ be a bounded non-constant function, $h$-homogeneous of $h$-degree* 0 *which is continuous on $\mathbb{R}^d \setminus \{0\}$. Then $N_p(\phi) > C \cdot \max\{p, \frac{p}{p-1}\}$.*

Notice that a multiplier which satisfies the conclusion of Theorem 1 has to fulfill some regularity conditions. Indeed, let $\phi$ be the characteristic function of an infinite Sidon subset of $\mathbb{Z}$. Then $C_1 \cdot \sqrt{p} < N_p(\phi) < C_2 \cdot \sqrt{p}$ for $2 < p \leq \infty$ (cf. [10]).

In Section 4 we give a criterion (Proposition 1) for a rational multiplier in $\mathbb{R}^d$ with somewhat elliptic denominator to be the Fourier transform of a measure. A crucial point in our argument is an improvement of Boman's technique from [2]. Proposition 1 combined with Theorem 1 yields:

THEOREM 3. *Let $P, Q \in \mathcal{P}_d$. Assume that $Q$ is somewhat elliptic. Then either*

$$(P/Q)^\wedge \in M(\mathbb{R}^d),$$

*or, for some $C > 0$,*

$$N_p(P/Q) > C \cdot \max(p, \frac{p}{p-1}) \qquad (1 < p < \infty).$$

The next two corollaries concern multipliers related to smoothnesses For the definition of a smoothness $S$, its canonical projection $P_S$ and fundamental polynomial $Q_S$, see [8] and [1, Section 1]. Recall that $P_S$ is $p$-bounded if and only if all entries of the matrix $\big(i^{|\alpha|-|\beta|}\frac{x^{\alpha+\beta}}{Q_S(x)}\big)_{\alpha,\beta\in S}$ are $p$-bounded multipliers. As a consequence of Theorem 3 and the fact that the fundamental polynomials of smoothnesses are somewhat elliptic we get

COROLLARY 4. *Let $S \subset \mathbb{Z}_+^d$ be a smoothness. Then either the canonical projection $P_S$ is $L^1$-bounded or for some $\alpha, \beta \in S$ one has $N_p(\frac{x^{\alpha+\beta}}{Q_S(x)}) > C\max(p, \frac{p}{p-1})$ for $1 < p < \infty$.*

COROLLARY 5. *Let $S \subset \mathbb{Z}_+^d$ be a smoothness and let $\tau \in \mathbb{Z}_+^d \cap \operatorname{conv} 2S$. Assume that there exists $\gamma \in \operatorname{conv} 2S$ such that $\gamma_j > \tau_j$ for $j = 1, 2, \ldots, d$. Then $\big(x^\tau/Q_S(x)\big)^\wedge \in L^1(\mathbb{R}^d)$.*

## 3. A lower bound for strong type $(p, p)$ norms of multipliers

Fix a positive integer $n$. Let $\{\mathbb{T}_j\colon j = 1, 2, \ldots, d\}$ be a family of distinct copies of the circle group. For $m = 0, 1, 2, \ldots, n-1$ put $\mathbb{T}_m^n = \mathbb{T}_{m+1} \times \mathbb{T}_{m+2} \times \ldots \times \mathbb{T}_n$; let $t^{(m,n)} = (t_{m+1}, t_{m+2}, \ldots, t_n)$ denote a generic point of $\mathbb{T}_m^n$, and let $dt^{(m,n)}$ denote the normalized Haar measure of the group $\mathbb{T}_m^n$. For $m = 0$ put $\mathbb{T}^n = \mathbb{T}_0^n$, $t = t^{(0,n)}$ and $dt = dt^{(0,n)}$. Next define the functions $X_k\colon \mathbb{T}^n \to \mathbb{R}$ by $X_0 \equiv 1$ and $X_k(t) = (1 + \cos t_k)X_{k-1}$ for $k = 1, 2, \ldots, n$.

LEMMA 1. *Given $n = 1, 2, \ldots$, there exists a sequence $(\sigma_k)_{k=1}^n$ with terms $\pm 1$ such that*

$$\int_{\mathbb{T}^n} \left| \sum_{k=1}^n \sigma_k \cos t_k X_{k-1}(t) \right| dt > \frac{n}{142}. \tag{2}$$

*Proof.* For fixed $m \in \{1, 2, \dots, n-1\}$ define the non-negative martingale

$$\mathfrak{X}_m = (1, X^m_m, X^m_{m+1}, \dots, X^m_n)$$

by putting $X^m_k = \prod_{j=m}^k (1 + \cos t_j)$ for $t \in \mathbb{T}^n$, $m = 1, 2, \dots, n$ and $k = m, m+1, \dots, n$. Next put

$$Q_m = \begin{cases} ((1 - X^m_m)^2 + \sum_{k=m}^{n-1} (X^m_{k+1} - X^m_k)^2)^{\frac{1}{2}} & \text{if } 1 \le m < n \\ \cos^2 t_n & \text{if } m = n. \end{cases}$$

Notice that the functions $Q_m$ have the following properties:

(i) $Q^2_m = cos^2 t_m + (1 + \cos t_m)^2 \cdot Q^2_{m+1}$ for $m = 1, 2, \dots, n-1$.
(ii) $Q_m$ depends only on the variables $(t_m, t_{m+1}, \dots, t_n)$.
(iii) $(1 + Q^2_m)^{\frac{1}{2}}$ is the square function of $\mathfrak{X}_m$ for $m = 1, 2, \dots, n-1$.

It follows from (iii) by ([6], Prop. VIII-2-7) that the probability $P(\{(1 + Q^2_{m+1})^{\frac{1}{2}} \le 6\})$ is $\ge \frac{1}{2}$, and so *a fortiori* $P(\{Q_{m+1} \le 6\}) \ge \frac{1}{2}$. It follows from (ii) that $P(\{Q_{m+1} \le 6\}) = \int_{A_{m+1}} dt^{(m,n)}$ for $m = 1, 2, \dots, n-1$ where $A_{m+1}$ denotes the projection of the set $\{Q_{m+1} \le 6\}$ on $\mathbb{T}^n_m$. Put $B_{m+1} = \mathbb{T}^n_m \setminus A_{m+1}$. The condition (ii) also implies that $Q_m$ uniquely determines a function on $\mathbb{T}^n_{m-1}$ which we shall denote by $\widetilde{Q}_m$ for $m = 1, 2, \dots, n$.

Our first aim is to show the recursive inequality

$$\|Q_m\|_1 > \|Q_{m+1}\|_1 + \frac{1}{100} \qquad (m = 1, 2, \dots, n-1), \tag{3}$$

which, combined with the inequality $\|Q_n\|_1 = \int_{\mathbb{T}^n} |\cos t|\, dt \ge \frac{1}{100}$, implies

$$\|Q_1\|_1 \ge \frac{n}{100}. \tag{4}$$

To establish (3) notice that, by (ii),

$$\|Q_m\|_1 = \int_{\mathbb{T}^n} Q_m\, dt = \int_{\mathbb{T}^n_{m-1}} \widetilde{Q}_m\, dt^{(m-1,n)} = I_1 + I_2$$

where

$$I_1 = \int_{\mathbb{T}_m} \int_{A_{m+1}} \widetilde{Q}_m\, dt^{(m,n)}\, dt_m,$$
$$I_2 = \int_{\mathbb{T}_m} \int_{B_{m+1}} \widetilde{Q}_m\, dt^{(m,n)}\, dt_m.$$

Note that if $t^{(m,n)} \in A_{m+1}$ then $(1 + \cos t_m)\widetilde{Q}_{m+1} \le 12$. Thus combining (i) with the numerical inequality

$$(a^2 + b^2)^{\frac{1}{2}} \ge \frac{a^2}{25} + b \qquad \text{for } 0 \le a \le 1, 0 \le b \le 12,$$

we get

$$\begin{aligned} I_1 &\geq \int_{\mathbb{T}_m}\int_{A_{m+1}} \left(\frac{\cos^2 t_m}{25} + (1+\cos t_m)\widetilde{Q}_{m+1}\right) dt^{(m,n)}\, dt_m \\ &= \frac{1}{25}\int_{\mathbb{T}_m} \cos^2 t_m\, dt_m \int_{A_{m+1}} dt^{(m,n)} \\ &\quad + \int_{\mathbb{T}_m}\int_{A_{m+1}} (1+\cos t_m)\widetilde{Q}_{m+1} dt^{(m,n)}\, dt_m \\ &\geq \frac{1}{100} + \int_{\mathbb{T}_m}\int_{A_{m+1}} (1+\cos t_m)\widetilde{Q}_{m+1} dt^{(m,n)}\, dt_m. \end{aligned}$$

On the other hand, (i) yields

$$\begin{aligned} I_2 &= \int_{\mathbb{T}_m}\int_{B_{n+1}} \left(\cos^2 t_m + (1+\cos t_m)^2 \widetilde{Q}^2_{m+1}\right)^{\frac{1}{2}} dt^{(m,n)}\, dt_m \\ &\geq \int_{\mathbb{T}_m}\int_{B_{m+1}} (1+\cos t_m)\widetilde{Q}_{m+1} dt^{(m,n)}\, dt_m. \end{aligned}$$

Therefore, remembering that $\int_{\mathbb{T}_m}(1+\cos t_m)\, dt_m = 1$, we see that

$$\begin{aligned} \|Q_m\|_1 &= I_1 + I_2 \\ &\geq \frac{1}{100} + \int_{\mathbb{T}_m}\int_{\mathbb{T}^n_m} (1+\cos t_m)\widetilde{Q}_{m+1} dt^{(m,n)}\, dt_m \\ &\geq \frac{1}{100} + \int_{\mathbb{T}_m} (1+\cos t_m)\, dt_m \int_{\mathbb{T}^n_m} \widetilde{Q}_{m+1}\, dt^{(m,n)} \\ &= \frac{1}{100} + \|Q_{m+1}\|_1. \end{aligned}$$

Next observe that $X^1_1 = \cos t_1 = \cos t_1 \cdot X_0$ and $X^{k+1}_1 - X^k_1 = \cos t_{k+1} X_k$ for $k = 1, 2, \ldots, n-1$. Hence

$$\int_{\mathbb{T}^n} Q_1\, dt = \int_{\mathbb{T}^n} \left(\sum_{k=1}^{n} (\cos t_k \cdot X_{k-1})^2\right)^{\frac{1}{2}} dt. \tag{5}$$

Let $r_j\colon \Omega \to \mathbb{R}$ be the Bernoulli sequence of random variables (the Rademacher functions). Combining (4) and (5) with the Khinchine inequality (while using the latter's best constant—see [4], for example) we get

$$\begin{aligned} \mathbf{E}_\Omega \int_{\mathbb{T}^n} \left|\sum_{k=1}^{n} r_k(\omega)\cos t_k X_{k-1}(t)\right| dt &= \int_{\mathbb{T}^n} \mathbf{E}_\Omega \left|\sum_{k=1}^{n} r_k(\omega)\cos t_k X_{k-1}(t)\right| dt \\ &\geq \frac{1}{\sqrt{2}}\int_{\mathbb{T}^n} Q_1(t)\, dt \\ &\geq \frac{n}{142}. \end{aligned}$$

Hence there exists $\omega \in \Omega$ such that, upon letting $\sigma_k = r_k(\omega)$ for $k = 1, 2, \ldots, n$, we get (2). $\square$

*Remark.* As was observed by R. Latala (cf. [5]), inequality (2) holds (with another constant) with $\sigma_k = (-1)^k$ for $k = 1, 2, \ldots, n$.

LEMMA 2. *There exists $C > 0$ such that*

$$\int_{\mathbb{T}^n} \left| \sum_{k=1}^{n} e^{it_k} \cdot X_{k-1}(t) \right| dt > C \cdot n \qquad \textit{for } n = 1, 2, \ldots.$$

*Proof.* Let $S_k = \sum_{j=1}^{k} e^{it_j} X_{j-1}$ for $k = 1, 2, \ldots, n$. Then $(S_k)_{k=1}^n$ is an analytic martingale. Therefore, by Prop. 4.1 in [3], we can use (4) and (5) to obtain

$$\begin{aligned}
\int_{\mathbb{T}^n} \left| \sum_{k=1}^{n} e^{it_k} \cdot X_{k-1}(t) \right| dt &\geq C \cdot \int_{\mathbb{T}^n} \left( \sum_{k=1}^{n} X_{k-1}^2(t) \right)^{\frac{1}{2}} dt \\
&\geq C \cdot \int_{\mathbb{T}^n} \left( \sum_{k=1}^{n} \cos^2 t_k \cdot X_{k-1}^2(t) \right)^{\frac{1}{2}} dt \\
&\geq C \cdot \|Q_1\|_1 \\
&\geq C \frac{n}{142}.
\end{aligned}$$

$\square$

In the sequel $B(x, r)$ stands for the ball with center at $x \in \mathbb{R}^d$ and radius $r > 0$. The symbols $\langle \cdot, \cdot \rangle$ and $|\cdot|$ stand for the scalar product and the Euclidean norm respectively.

*Proof of Theorem* 1. First consider Case I. Without loss of generality we can assume that $a = 1$ and $b = -1$. Fix a positive integer $n$, and let $(\sigma_j)_{j=1}^n$ be the sequence of signs from Lemma 1. It follows easily from the assumption of case I that for every $\varepsilon > 0$ and $N \geq 3$, there exists a sequence $(m_j^N)_{j=1}^n \subset \mathbb{Z}^d$ such that

$$|m_{j+1}^N| > N \cdot |m_j^N| \tag{6}$$

and

$$|\phi(z) - \sigma_j| < \left(\frac{1}{6}\right)^j \varepsilon \qquad \text{for } z \in B\left(m_j^N, \sum_{i<j} |m_i^N|\right) \cap B\left(-m_j^N, \sum_{i<j} |m_i^N|\right). \tag{7}$$

Now, for $k = 1, 2, \ldots, n$, put

$$R_k^N(t) = \prod_{j \leq k} (1 + \cos\langle m_j^N, t \rangle) - 1$$

and

$$F_n^N(t) = \sum_{j=1}^n \sigma_j \cos\langle m_j^N, t\rangle \prod_{i<j}(1+\cos\langle m_i^N, t\rangle).$$

Clearly $\|F_n^N\|_1 \to \|\sum_{j=1}^n \sigma_j X_{j-1}\cos t_j\|_1$ for $N \to \infty$ (see [7] for more quantitative information). Hence, by Lemma 1, for $N$ chosen big enough,

$$\|F_n^N\|_1 > \frac{1}{142}\cdot n. \tag{8}$$

Since $R_n^N(t) = \sum_{j=1}^n \cos\langle m_j^N, t\rangle R_{j-1}^N(t)$ and

$$\{k\colon (\cos\langle m_j^N, \cdot\rangle R_{j-1}^N)^\wedge(k) \neq 0\} \subset B\left(m_j^N, \sum_{i<j}|m_i^N|\right) \cup B\left(-m_j^N, \sum_{i<j}|m_i^N|\right),$$

we infer by (6) and (7) that

$$|T_\phi R_n^N - F_n^N| < \varepsilon. \tag{9}$$

Choosing $\varepsilon$ small enough, by (8) and (9) we get

$$\|T_\phi R_n^N\|_1 > \frac{1}{142}\cdot n. \tag{10}$$

For the counterpart of (10) in Case II, we specify $a = 1$ and $b = 0$. By similar reasoning to that used in the preceding case, we define

$$R_k^N(t) = \prod_{j\le k}(1+\cos\langle m_j^N, t\rangle) - 1,$$

with the $m_j^N$'s chosen so as to insure that

$$\big(T_\phi R_n^N\big)(t) \simeq \sum_{j=1}^n e^{i\langle m_j^N, t\rangle} \prod_{i<j}(1+\cos\langle m_i^N, t\rangle).$$

Then (10) follows (with another constant) by the same argument as in Case I, with Lemma 1 now replaced by Lemma 2.

Fix $p \le 2$ and $q$ such that $\frac{1}{p}+\frac{1}{q} = 1$. By the well known properties of the Riesz products and the Hölder inequality,

$$\|R_n^N\|_p \le \|R_n^N\|_1^{1-\frac{2}{q}}\|R_n^N\|_2^{\frac{2}{q}} < 2\left(\frac{3}{2}\right)^{\frac{n}{q}}. \tag{11}$$

Therefore by (10) and (11),

$$N_p(\phi) \ge \frac{\|T_\phi R_n^N\|_p}{\|R_n^N\|_p} \ge \frac{\|T_\phi R_n^N\|_1}{\|R_n^N\|_p} \ge C\cdot n\cdot\left(\frac{2}{3}\right)^{\frac{n}{q}}.$$

Substituting for $n$ the integer closest to $\frac{q}{\log \frac{3}{2}}$ we get

$$N_p(\phi) > C \cdot \frac{p}{p-1}.$$

The case $p > 2$ follows by duality. □

*Proof of Theorem* 2. One can assume that $x = 0$. Accordingly, we see that there exist $a, b \in \mathbb{C}$, $a \neq b$ and an infinite sequence $(x_j)_{j=1}^{\infty} \subset \mathcal{U}$ such that $x_j \to 0$ and the sequence $\phi(x_j)$ does not converge. Moreover, one can assume that there exist sequences of real numbers $\varepsilon_j \to 0$ and $r_j \to 0$ satisfying $\sum_{k>j} r_k < r_j$ for $j = 1, 2, \ldots$, such that (passing to a subsequence if necessary) one of the following conditions holds: either

$$\begin{aligned} &|\phi(x) - a| < \varepsilon_j \qquad \text{for } j \text{ even and } x \in B(x_j, r_j) \cup B(-x_j, r_j) \\ &|\phi(x) - b| < \varepsilon_j \qquad \text{for } j \text{ odd and } x \in B(x_j, r_j) \cup B(-x_j, r_j), \end{aligned} \tag{12}$$

or

$$\begin{aligned} &|\phi(x) - a| < \varepsilon_j \qquad \text{for } x \in B(x_j, r_j) \\ &|\phi(x) - b| < \varepsilon_j \qquad \text{for } x \in B(-x_j, r_j). \end{aligned} \tag{13}$$

We shall show how (12) implies the assertion of Theorem 2. The argument in the case of (13) is similar. Obviously we can assume that $a = 1$ and $b = -1$. Then it follows that for every $\varepsilon > 0$ and every two integers $n$ and $N$ there exist a finite sequence $(\sigma_j)_{j=1}^n$ of signs from Lemma 1 and a finite sequence $(y_j^N)_{j=1}^n$ consisting of elements of the sequence $(x_\nu)_{\nu=1}^{\infty}$ such that for $j = 1, 2, \ldots, n$,

$$|y_{j+1}^N| > N \cdot |y_j^N| \tag{14}$$

$$|\phi(x) - \sigma_j| < \left(\frac{1}{6}\right)^j \varepsilon \qquad \text{for } \min\{|x + y_j^N|, |x - y_j^N|\} < \sum_{i<j} |y_i^N|. \tag{15}$$

Let $(\psi_t)_{t>0}$ be an approximate unit for $L^1(\mathbb{R}^d)$ such that each $\psi_t$ is a smooth function with bounded support. Then $\psi_t * \phi(x) \to \phi(x)$ uniformly in $x$ on every compact set. Hence one can choose $t > 0$ such that for $j = 1, 2, \ldots, n$,

$$|\psi_t * \phi(x) - \sigma_j| < \left(\frac{1}{6}\right)^j \varepsilon \qquad \text{for } \min\{|x + y_j^N|, |x - y_j^N|\} < \sum_{i<j} |y_i^N|. \tag{16}$$

On the other hand,

$$N_p(\psi_t * \phi) < \|\psi_t\|_1 \cdot N_p(\phi), \tag{17}$$

Since $\psi_t * \phi$ is a continuous function, one can choose $\lambda > 0$ such that (14) and (16) hold for $(y_j^N)_{j=1}^n$ replaced by some sequence $(k_j^N)_{j=1}^n \subset \lambda\mathbb{Z}^d$. Put $\widetilde{\phi}(x) = \psi_t * \phi(\lambda^{-1}x)$. By [12], Chapt. VII, §3, we have

$$N_p(\widetilde{\phi}) = N_p(\psi_t * \phi). \tag{18}$$

Now the de Leeuw transference theorem (cf. [12], Th. 3.8) yields

$$N_p(\widetilde{\phi}_{|\mathbb{Z}^d}) \le N_p(\widetilde{\phi}). \tag{19}$$

By (14) and (16), the sequence $(m_j^N)_{j=1}^n$ defined by $m_j^N = \lambda^{-1}k_j^N \in \mathbb{Z}^d$ satisfies (6) and (7), with $\widetilde{\phi}_{|\mathbb{Z}^d}$ playing the rôle of $\phi$. Hence the same procedure as in the proof of Theorem 1 (with suitable choices for $\varepsilon$, $n$ and $N$) shows that $N_p(\widetilde{\phi}_{|\mathbb{Z}^d}) > C \cdot \max\{p, \frac{p}{p-1}\}$. So the desired conclusion follows from (17), (18) and (19). □

## 4. Rational multipliers

In the sequel we shall need the following property of somewhat elliptic polynomials:

PROPOSITION 1. *Let $Q \in \mathcal{P}_d$ be somewhat elliptic, $\rho_s = (s, s, \dots, s) \in \mathbb{R}^d$ where $0 < s < 1$, and $\alpha \in \mathbb{Z}_+^d \cap \operatorname{conv} Q$. Assume that $\alpha + \rho_s \in \operatorname{conv} Q$ for some $0 < s \le 1$. Then for every $p$ such that $1 \le p < (1-s)^{-1}$, the Fourier transform of the function*

$$f(x) = x^\alpha / Q(x)$$

*belongs to $L^1 \cap L^p$.*

To prove Proposition 1 we need a couple of lemmas.

LEMMA 3. *Let $Q \in \mathcal{P}_d$ be somewhat elliptic. Let $\alpha \in \mathbb{Z}_+^d$, $\rho \in \mathbb{R}_+^d$, $\alpha + \rho \in \operatorname{conv} Q$. Then for every $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ there exist a somewhat elliptic polynomial $P$, a non-empty finite set $S \subset \mathbb{Z}_+^d$ and a sequence of coefficients $(a_\gamma)_{\gamma \in S}$, such that*

$$\frac{\partial^{|n|}}{\partial x^n}\left(\frac{x^\alpha}{Q(x)}\right) = \sum_{\gamma \in S} \frac{a_\gamma x^\gamma}{P(x)}$$

*and for every $m \in \mathbb{Z}_+^d$, $m \le n$,*

$$\gamma + \rho + m \in \operatorname{conv} P \qquad \textit{for every } \gamma \in S.$$

*Proof.* It is enough to prove the lemma for derivatives of order 1. Let $e_k$ denote the $k$-th coordinate unit vector. We can assume that $\alpha \ge e_k$ (if not, the proof is still similar), and deduce that

$$\frac{\partial}{\partial x_k}\left(\frac{x^\alpha}{Q(x)}\right) = \left(\alpha_k x^{\alpha - e_k} Q(x) - x^\alpha \frac{\partial}{\partial x_k} Q(x)\right) \cdot (Q(x))^{-2}.$$

Thus, putting $P = Q^2$ and $S = \alpha + (\mathrm{sp}\, Q - e_k) \cap \mathbb{Z}_+^d$, we get

$$S + \rho \subset \alpha + \rho + (\mathrm{sp}\, Q - e_k) \cap \mathbb{Z}_+^d \subset \mathrm{conv}\, Q + \mathrm{conv}\, Q = \mathrm{conv}\, P.$$

Similarly we get $S + \rho + e_k \subset \mathrm{conv}\, P$. □

The next lemma is a modified version of Theorem 5.1 in [8].

LEMMA 4. *Let $P, Q \in \mathcal{P}_d$, $Q$ be somewhat elliptic and* $\mathrm{sp}\, P \subset \mathrm{conv}\, Q$. *Then $\frac{P}{Q}$ is a bounded $L^p$ multiplier for $1 < p < \infty$.*

*Proof.* Lemma 3 yields

$$\left| \frac{\partial^{|n|}}{\partial x^n} \left( \frac{P(x)}{Q(x)} \right) \right| \leq C \cdot |x^{-n}|$$

for every $x \in \mathbb{R}^d$ with non-zero coordinates and $n \in \mathbb{Z}_+^d$; in particular for $n$ with $n_j \in \{0, 1\}$ for $j = 1, 2, \ldots, d$. Hence the lemma follows by the Marcinkiewicz multidimensional multiplier theorem (cf. [11], Chapt. VI, §6, Theorem 6'). □

The next lemma is a modified version of a result due to Boman (cf. [2], Lemma 1).

LEMMA 5. *Let $S$ be a finite subset of $\mathbb{Z}_+^d$, $0 < s < 1$, and $\beta + \rho_s \in$* conv *$S$. Then for every $p$ satisfying $1 \leq p < (1-s)^{-1}$ there exist functions $h_\alpha$ ($\alpha \in S$) such that $\widehat{h_\alpha} \in L^p$ and*

$$x^\beta = \sum_{\alpha \in S} x^\alpha \cdot h_\alpha(x).$$

*Proof.* Take $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(y) = \psi(-y)$, $\psi(y) = 0$ in a neighborhood of 0, and

$$\int_{-\infty}^{\infty} \psi(e^{-y})\, dy = 1. \tag{20}$$

For $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$, set

$$\Psi_t(x) = \prod_{i=1}^{d} \psi(x_i e^{-t_i}).$$

Then in view of (20),

$$\int \Psi_t(x)\, dt = 1$$

if $x_i \neq 0$ for each $i = 1, 2, \ldots, d$. For fixed $t_i$ the function $\psi(x_i e^{-t_i})$ is equal to zero in a neighborhood of $x_i = 0$. Hence for any $\alpha \in \mathbb{Z}^d$, $\beta \in \mathbb{Z}^d$ and $t \in \mathbb{R}^d$, the function $x^{\beta-\alpha}\Psi_t(x)$ belongs to $C_0^\infty(\mathbb{R}^d)$, and hence

$$\left(x^{\beta-\alpha}\Psi_t(x)\right)^\wedge \in L^1 \cap L^p.$$

We now study the $t$-dependence of the $L^p$-norm of this function. Since for an arbitrary function $\theta(x_i)$ such that $\widehat{\theta}(x_i) \in L^p(\mathbb{R})$, we have, upon setting $s = 1 - \frac{1}{p}$,

$$\|\left(\theta(x_i e^{-t_i})\right)^\wedge\|_{L^p(\mathbb{R}, dx_i)} = e^{st_i}\|\left(\theta(x_i)\right)^\wedge\|_{L^p(\mathbb{R})},$$

it follows that

$$\begin{aligned}
(21) \qquad \|\left(x^{\beta-\alpha}\Psi_t(x)\right)^\wedge\|_{L_p(\mathbb{R}^d)} &= \prod_{i=1}^d \|\left(x_i^{\beta_i-\alpha_i}\psi(x_i e^{-t_i})\right)^\wedge\|_{L_p(\mathbb{R})} \\
&= C \cdot \prod_{i=1}^d e^{st_i} e^{t_i(\beta_i-\alpha_i)} \\
&= C \cdot e^{\langle t, \beta+\rho_s-\alpha\rangle}.
\end{aligned}$$

Next we prove that if $\beta + \rho_s \in \operatorname{int}\operatorname{conv} S$, then

$$(22) \qquad \int_{\mathbb{R}^d} \inf_{\alpha \in \operatorname{sp} Q} e^{\langle t, \beta+\rho_s-\alpha\rangle}\, dt < \infty.$$

In fact

$$\begin{aligned}
\inf_{\alpha\in S} \exp\langle t, \beta + \rho_s - \alpha\rangle &= \exp(-\sup_{\alpha\in S}\langle t, \alpha - \beta - \rho_s\rangle) \\
&= \exp(-H_E(t)),
\end{aligned}$$

where $H_E(t)$ is the supporting function for the convex set

$$E = \operatorname{conv} S - (\beta + \rho_s).$$

But the assumption $\beta + \rho_s \in \operatorname{int}\operatorname{conv} Q$ is equivalent to

$$0 \in \operatorname{int} E,$$

and hence implies that

$$H_E(t) > c|t|$$

for some $c > 0$. This proves (22). Now put

$$A_\alpha = \left\{t \in \mathbb{R}^d : e^{\langle t, \beta+\rho_s-\alpha\rangle} = \inf_{\gamma\in S} e^{\langle t, \beta+\rho_s-\gamma\rangle}\right\}$$

and take $B_\alpha \subset A_\alpha$ such that

$$\bigcup_{\alpha\in S} B_\alpha = \mathbb{R}^d \quad \text{and} \quad B_\alpha \cap B_{\alpha'} = \emptyset \quad \text{for} \quad \alpha \neq \alpha'.$$

Clearly, by (22),

$$\int_{B_\alpha} e^{\langle t,\beta+\rho_s-\alpha\rangle}\,dt < \infty$$

for each $\alpha \in S$. Define $h_\alpha(x)$ for $x_i \neq 0$ by

$$h_\alpha(x) = \int_{B_\alpha} x^{\beta-\alpha}\Psi_t(x)\,dt.$$

According to (21) we have

$$\|\widehat{h}_\alpha\|_{L^p} \leq C\int_{B_\alpha} e^{\langle t,\beta+\rho_s-\alpha\rangle}\,dt < \infty,$$

i.e., $\widehat{h}_\alpha \in L^p$. Finally,

$$\begin{aligned}\sum_{\alpha\in S} x^\alpha\cdot h_\alpha(x) &= \sum_{\alpha\in S}\int_{B_\alpha} x^\beta\Psi_t(x)\,dt\\ &= x^\beta\int_{\mathbb{R}^d}\Psi_t(x)\,dt\\ &= x^\beta.\end{aligned}$$

□

*Remark.* Lemma 5 can be generalized to $\rho = \rho_s + n$ where $0 < s < 1$ and $n \in \mathbb{Z}_+^d$. Specifically, we can obtain the following result.

LEMMA 5′. *Let $S$ be a finite subset of $\mathbb{Z}_+^d$, $0 < s < 1$, $n \in \mathbb{Z}_+^d$ and $\beta + \rho_s + n \in \operatorname{conv} S$. Then for every $p$ such that $1 \leq p < (1-s)^{-1}$ there exist functions $h_\alpha$, ($\alpha \in S$), such that $\frac{\partial^{|n|}}{\partial x^n}\widehat{h}_\alpha \in L^p(\mathbb{R}^d)$ and*

$$x^\beta = \sum_{\alpha\in S} x^\alpha\cdot h_\alpha(x).$$

*Proof of Proposition* 1. Let $n \in \mathbb{Z}_+^d$. By Lemma 3, $\frac{\partial^{|n|}}{\partial x^n} f(x) = \sum_{\gamma\in S}\frac{x^\gamma}{P(x)}$. Now by Lemmas 3 and 5, for every $\gamma \in S$ we have

$$x^\gamma = \sum_{\alpha\in \operatorname{sp} P} x^\alpha\cdot h_\alpha(x)$$

with $\widehat{h}_\alpha \in L^p$ for $\alpha \in \operatorname{sp} P$. Since $P$ is a power of $Q$ we infer that $P(x) \neq 0$ for $x \in \mathbb{R}^d$. Dividing both sides by $P$ we get

$$\frac{x^\gamma}{P(x)} = \sum_{\alpha\in \operatorname{sp} P}\frac{x^\alpha}{P(x)}\cdot h_\alpha(x).$$

By Lemma 4, for every $\alpha \in \operatorname{sp} P$ the function $\frac{x^\alpha}{P(x)}$ is a bounded $L^p$ multiplier for $p > 1$. Hence $\frac{x^\alpha}{P(x)} \cdot h_\alpha(x)$ is the Fourier transform of an $L^p$ function for every $\alpha \in \operatorname{sp} P$. Therefore

$$\big(x^\gamma / P(x)\big)^\wedge \in L^p(\mathbb{R}^d)$$

for every $\gamma \in S$, and consequently

$$\left(\frac{\partial^{|n|}}{\partial x^n} f\right)^\wedge \in L^p(\mathbb{R}^d)$$

for every $n \in \mathbb{Z}_+^d$ and $1 < p < (1-s)^{-1}$ . This means that

$$\xi^n \widehat{f}(\xi) \in L^p(\mathbb{R}^d)$$

for every $n \in \mathbb{Z}_+^d$ and $1 < p < (1-s)^{-1}$. In particular,

$$(1 + |\xi|)^d \widehat{f}(\xi) \in L^p(\mathbb{R}^d).$$

Thus, by the Hölder inequality (with $p' = \frac{p}{p-1}$),

$$\begin{aligned} \|\widehat{f}\|_1 &= \int (1+|\xi|)^d |\widehat{f}(\xi)| \cdot (1+|\xi|)^{-d} \, d\xi \\ &\le \left(\int (1+|\xi|)^{pd} |\widehat{f}(\xi)|^p \, d\xi\right)^{1/p} \cdot \left(\int (1+|\xi|)^{-p'd} \, d\xi\right)^{1/p'} \\ &< \infty. \end{aligned}$$

□

*Remark.* In fact, our proof of Proposition 1 shows that $\widehat{f}$ multiplied by any polynomial belongs to $L^1 \cap L^p$.

An intersection of a convex polyhedron $W$ with a supporting hyperplane is called a *face* of $W$. The family of all faces of a convex polyhedron $W$ is denoted $\Upsilon(W)$.

A polyhedron $W \subset \mathbb{R}_+^d$ is called *solid* if $x \in W$, $y \in \mathbb{R}_+^d$ and $y \le x$ imply $y \in W$.

For a polynomial $P(x) = \sum_{\gamma \in \operatorname{sp} P} b_\gamma x^\gamma$ and $A \in \Upsilon(\operatorname{conv} Q)$ we put

$$P_A(x) = \sum_{\gamma \in \operatorname{sp} P \cap A} b_\gamma x^\gamma .$$

PROPOSITION 2. *Let $Q \in \mathcal{P}_d$ be somewhat elliptic. Let $P \in \mathcal{P}_d$ satisfy* $\operatorname{sp} P \subset \operatorname{conv} Q$ *and let $(P_A/Q_A)^\wedge \in M(\mathbb{R}^d)$ for every $A \in \Upsilon(\operatorname{conv} Q)$. Then $(P/Q)^\wedge \in M(\mathbb{R}^d)$.*

*Proof.* Define $f \colon \mathbb{R}^d \to \mathbb{C}$ by taking

$$f(x) = \frac{P(x)}{Q(x)} + \sum_{A \in \Upsilon(\operatorname{conv} Q)} \frac{P_A(x)}{Q_A(x)} (-1)^{d - \dim A}.$$

It is enough to prove that $\widehat{f} \in L^1(\mathbb{R}^d)$. To this end we first show that for every $A \in \Upsilon(\operatorname{conv} Q)$ there exist $\widetilde{P}_A$ and $\widetilde{Q}_A$ in $\mathcal{P}_d$, with $\widetilde{Q}_A$ somewhat elliptic, such that

$$\frac{\widetilde{P}_A}{\widetilde{Q}_A} = \frac{P_A}{Q_A}. \tag{23}$$

We begin with the case when $A \in \Upsilon(\operatorname{conv} Q)$ satisfies the following:

(∗) The linear manifold spanned by $A$ is a coordinate subspace i.e. linear subspace, say $K$, spanned by some coordinate vectors of $\mathbb{R}^d$.

Then $\alpha \in \operatorname{sp} Q_A$ implies $\alpha \in K$. Hence $Q_A(x) = Q(pr_K x)$ where $pr_K$ denotes the orthogonal projection from $\mathbb{R}^d$ onto $K$. Since $x^\gamma = (pr_K x)^\gamma$ for $\gamma \in K \cap \mathbb{Z}^d_+$, $Q_A$ is somewhat elliptic because the somewhat ellipticity of $Q$ implies the existence of $C > 0$ such that for every $\gamma \in \operatorname{conv} Q_A \cap \mathbb{Z}^d_+$ and $x \in \mathbb{R}^d$ we have $|Q(x)| > C|x^\gamma|$. Hence

$$|Q_A(x)| = |Q(pr_K x)| > C \cdot |(pr_K x)^\gamma| = C \cdot |x^\gamma|.$$

We put $\widetilde{Q}_A = Q_A$, $\widetilde{P}_A = P_A$.

It remains to consider the case when $A \in \Upsilon(\operatorname{conv} Q)$ fails (∗) for every coordinate subspace $K$. Let $L \subset \mathbb{R}^d$ be the smallest coordinate subspace containing $A$. Let $B = L \cap \operatorname{conv} Q$. Clearly $B$ is a face of $\operatorname{conv} Q$ satisfying (∗). Hence, as we have already proved, $Q_B$ is somewhat elliptic. Since $Q_A = (Q_B)_A$ and $P_A = (P_B)_A$, without loss of generality we can assume that $L = \mathbb{R}^d$.

We represent $\mathbb{R}^d$ as the product $\mathbb{R}^k \times \mathbb{R}^{d-k}$ where $A$ is parallel to the coordinate vectors $e_1, e_2, \dots, e_k$ of $\mathbb{R}^d$ which span $\mathbb{R}^k$, and $A$ is not parallel to the remaining coordinate vectors $e_{k+1}, e_{k+2}, \dots, e_d$ spanning $\mathbb{R}^{d-k}$. We also represent $\mathbb{Z}^d$ as $\mathbb{Z}^k \times \mathbb{Z}^{d-k}$. Since $\operatorname{conv} Q$ is solid, there exists a hyperplane $H$ supporting $\operatorname{conv} Q$ and satisfying $H \cap \operatorname{conv} Q = A$, with normal vector $(0, h) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ such that $h \in \mathbb{R}^{d-k}$ has all coordinates strictly positive. For every fixed $z \in \mathbb{R}^k$ the function $g\colon \mathbb{R}^{d-k} \to \mathbb{C}$ given by

$$g(y) = \frac{P_A(z, y)}{Q_A(z, y)}$$

is $h$-homogeneous. Therefore, by Corollary 3 , the assumption $\widehat{g} \in M(\mathbb{R}^d)$ enables us to infer that $g$ is constant (in fact we do not need to use here the full strength of this corollary, but only its weaker version which follows from Wiener's theorem, see [8], Prop. 3.1). Thus

$$\frac{P_A(z, y)}{Q_A(z, y)} = w(z) \qquad \text{for } y \in \mathbb{R}^{d-k}. \tag{24}$$

Let $e = (1, 1, \dots, 1) \in \mathbb{R}^{d-k}$. We define $\widetilde{P}_A(z, y) = P_A(z, e)$ and $\widetilde{Q}_A(z, y) = Q_A(z, e)$. Then (24) yields (23). We will show that $\widetilde{Q}_A$ is somewhat elliptic. Indeed, let $(\tau, 0) \in \mathbb{Z}_+^d \cap \operatorname{conv} \widetilde{Q}_A$ be an extremal point of $\operatorname{conv} \widetilde{Q}_A$ (clearly it is enough to check the inequality from Definition 1 for the extremal points). Since $A$ is parallel to $\mathbb{R}^k$ we get $\operatorname{conv} \widetilde{Q}_A = pr_{\mathbb{R}^{d-k}}(A)$ (this follows from the property that if $(\alpha, \beta) \in A$ and $\gamma \leq \alpha$ then $(\gamma, \beta) \in A$). Hence there exists an extremal point $(\mu, \nu) \in A \cap \mathbb{Z}_+^d$ of $A$ such that $pr_{\mathbb{R}^{d-k}}(\mu, \nu) = (\tau, 0)$, i.e., $\mu = \tau$. If $(\alpha, \beta) \in A \cap \mathbb{Z}_+^d$ then $\langle(\alpha, \beta), (0, h)\rangle = \langle\beta, h\rangle = 1$; in particular $\langle(\tau, \nu), (0, h)\rangle = 1$. If $(\alpha, \beta) \notin A \cap \operatorname{sp} Q$ then $\langle\beta, h\rangle < 1$. Thus, by somewhat ellipticity of $Q$, if $z^\tau \neq 0$ then

$$
\begin{aligned}
C &< \left| \frac{Q(z, \delta_h^t e)}{z^\tau (\delta_h^t e)^\nu} \right| \\
&= \left| \frac{\sum_{(\alpha,\beta) \in A \cap \operatorname{sp} Q} a_{(\alpha,\beta)} z^\alpha e^\beta t^{\langle h, \beta\rangle} + \sum_{(\alpha,\beta) \notin A \cap \operatorname{sp} Q} a_{(\alpha,\beta)} z^\alpha e^\beta t^{\langle h, \beta\rangle}}{z^\tau e^\nu t^{\langle h, \nu\rangle}} \right| \\
&\leq \frac{|Q_A(z, e)|}{|z^\tau|} + \left| \frac{\sum_{(\alpha,\beta) \notin A \cap \operatorname{sp} Q} a_{(\alpha,\beta)} z^\alpha t^{\langle h, \beta\rangle - 1}}{z^\tau} \right| .
\end{aligned}
$$

Upon letting $t$ tend to infinity, we get $C|z^\tau| < |Q_A(z, e)| = |\widetilde{Q}_A(x)|$. Thus

$$
|\widetilde{Q}_A(x)| = |Q_A(z, e)| > C \cdot |z^\tau| = C \cdot |x^{(\tau,0)}| \qquad \text{for } x = (z, y) \in \mathbb{R}^d.
$$

Hence $\widetilde{Q}_A$ is somewhat elliptic.

By (23) we obtain

$$
f(x) = \frac{P(x)}{Q(x)} + \sum_{A \in \Upsilon(\operatorname{conv} Q)} \frac{\widetilde{P}_A(x)}{\widetilde{Q}_A(x)} (-1)^{d - \dim A}.
$$

Hence

$$
f(x) = \frac{S(x)}{R(x)}
$$

where $R(x) = Q(x) \prod_{A \in \Upsilon(\operatorname{conv} Q)} \widetilde{Q}_A(x)$ is somewhat elliptic (as a product of polynomials with this property).

To complete the proof of Proposition 2 it is enough to show that for $h \in \mathbb{R}_+^d$, $h \neq 0$, and $x \in \mathbb{R}^d$ satisfying $\delta_h^t x \to \infty$ and $Q_A(x) \neq 0$ whenever $A \in \Upsilon(\operatorname{conv} Q)$, we have

$$
f(\delta_h^t x) \to 0 \qquad \text{for} \quad t \to \infty. \tag{25}
$$

Indeed, assuming (25) we infer that $\operatorname{sp} S$ does not contain maximal points of $\operatorname{conv} R$ and the desired conclusion follows from Proposition 1.

The identity (25) follows from the next two lemmas applied with $S = P$ and $R = Q$.

LEMMA 6. *Let $P, Q \in \mathcal{P}_d$, $Q$ somewhat elliptic. Let $0 \neq h = (h_j) \in \mathbb{R}^d_+$ and let $H$ be a supporting hyperplane of $W = \operatorname{conv} Q$ perpendicular to $h$. Let $B = W \cap H$. Then for every $A \in \Upsilon(W) \cup \{W\}$ and for every $x \in \mathbb{R}^d$ such that $Q_{A\cap B}(x) \neq 0$,*

$$\lim_{t\to\infty} \frac{P_A(\delta^t_h x)}{Q_A(\delta^t_h x)} = \frac{P_{A\cap B}(x)}{Q_{A\cap B}(x)}. \tag{26}$$

*Proof.* Our hypotheses on $H$ and $h$ imply the existence of $c > 0$ such that $\langle h, x\rangle = c$ for $x \in B$, and $\langle h, x\rangle < c$ for $x \in W \setminus B$. Setting $c_\gamma = \langle \gamma, h\rangle$, we have

$$\begin{aligned}\frac{P_A(\delta^t_h x)}{Q_A(\delta^t_h x)} &= \frac{\sum\limits_{\gamma\in A} b_\gamma x^\gamma t^{\langle\gamma,h\rangle}}{\sum\limits_{\gamma\in A} a_\gamma x^\gamma t^{\langle\gamma,h\rangle}} \\ &= \frac{\sum\limits_{\gamma\in A\cap B} b_\gamma x^\gamma t^c + \sum\limits_{\gamma\in A\setminus B} b_\gamma x^\gamma t^{c_\gamma}}{\sum\limits_{\gamma\in A\cap B} a_\gamma x^\gamma t^c + \sum\limits_{\gamma\in A\setminus B} a_\gamma x^\gamma t^{c_\gamma}} \\ &= \frac{\sum\limits_{\gamma\in A\cap B} b_\gamma x^\gamma + \sum\limits_{\gamma\in A\setminus B} b_\gamma x^\gamma t^{c_\gamma - c}}{\sum\limits_{\gamma\in A\cap B} a_\gamma x^\gamma + \sum\limits_{\gamma\in A\setminus B} a_\gamma x^\gamma t^{c_\gamma - c}}.\end{aligned}$$

Since $c_\gamma - c < 0$ for $\gamma \in A \setminus B$, (26) now follows. □

LEMMA 7. *Let $H$ be a supporting hyperplane of $W = \operatorname{conv} Q$ and $Q_{H\cap A}(x) \neq 0$ for every $A \in \Upsilon(W)$. Then*

$$\frac{P_{H\cap W}(x)}{Q_{H\cap W}(x)} + \sum_{A\in\Upsilon(W)} \frac{P_{H\cap A}(x)}{Q_{H\cap A}(x)}(-1)^{d-\dim A} = 0. \tag{27}$$

*Proof.* It is enough to show that for every $C, B \in \Upsilon(W)$ such that $B \subset C$,

$$\sum_{\substack{A\in\Upsilon(W)\cup\{W\}\\ A\cap C = B}} (-1)^{\dim A} = 0. \tag{28}$$

Indeed, multiplying both side of (28) by $\frac{P_B(x)}{Q_B(x)}$ and summing over all $B \in \Upsilon(W)$ we get (27). Formula (28) follows from the fact that the Euler - Poincaré characteristic of a convex polyhedron equals 1. □

*Proof of Theorem* 3. If $(\frac{P_A}{Q_A})^\wedge \in M(\mathbb{R}^d)$ for every $A \in \Upsilon(\operatorname{conv} Q)$, then by Proposition 2, $(\frac{P}{Q})^\wedge \in M(\mathbb{R}^d)$. Otherwise there is $A \in \Upsilon(\operatorname{conv} Q)$ such that $(\frac{P_A}{Q_A})^\wedge \notin M(\mathbb{R}^d)$. Then by reasoning as in the proof of Proposition 2 we see that, after relabeling the coordinates and writing $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$, the polynomial $z \mapsto Q_A(z,0)$ has no roots in $\mathbb{R}^k$, and, moreover, if $g\colon \mathbb{R}^k \to \mathbb{C}$ is defined by

$$g(z) = \frac{P_A(z,0)}{Q_A(z,0)},$$

then $g$ is a non-constant and $h$-homogeneous function for some vector $h \in \mathbb{R}^k$ with all coordinates positive. Thus, by Corollary 3, for some $C > 0$,

$$N_p(g) > C \cdot \max\left(p, \frac{p}{p-1}\right). \tag{29}$$

Let $\widetilde{Q}(z) = Q(z,0)$, $\widetilde{P}(z) = P(z,0)$ and let $H \subset \mathbb{R}^k$ be a subspace supporting conv $\widetilde{Q}$ such that $H \cap \operatorname{conv} \widetilde{Q} = A$. Let $h \in \mathbb{R}^k_+$ be the vector normal to $H$. Then, by Lemma 5,

$$\lim_{t\to\infty} \frac{\widetilde{P}(\delta_h^t z)}{\widetilde{Q}(\delta_h^t z)} = \frac{\widetilde{P}_A(z)}{\widetilde{Q}_A(z)} = g(z). \tag{30}$$

Since the norm of an $L^p$ multiplier remains unchanged after a non-singular linear change of variables, and the class of $L^p$ multipliers is closed under pointwise convergence by sequences which are uniformly bounded in multiplier norm, (30) implies

$$N_p(\widetilde{P}/\widetilde{Q}) \geq N_p(g). \tag{31}$$

Clearly, since $\widetilde{P}/\widetilde{Q}$ is the restriction of a continuous function $P/Q$ to the subspace $\mathbb{R}^k \subset \mathbb{R}^d$, it follows by a well-known version of de Leeuw's restriction theorem that

$$N_p(P/Q) \geq N_p(\widetilde{P}/\widetilde{Q}). \tag{32}$$

Finally (29), (31) and (32) give

$$N_p(P/Q) \geq C \cdot \max\left(p, \frac{p}{p-1}\right).$$ □

## References


[1] E. Berkson, J. Bourgain, A. Pelczyński and M. Wojciechowski, *Characterizations of the n-dimensional second order smoothnesses whose canonical projection is of weak type* (1,1), in preparation.

[2] J. Boman, *Supremum norm estimates for partial derivatives of functions of several real variables* Illinois J. Math. **16** (1972), 203–216.

[3] J. Bourgain and W. J. Davis, *Martingale transforms ond complex uniform convexity*, Trans. Amer. Math. Soc. **294** (1986), 501–515.

[4] S. Kwapien, R. Latala and K. Oleszkiewicz, *Comparision of moments of sums of independent random variables and differential inequalities*, J. Funct. Anal. **136(1)** (1996), 258–268.

[5] R. Latala, *personal communication.*

[6] J. Neveu, *Martingales a temps discret*, Masson, Paris, 1972.
[7] Y. Meyer, *Endomorphismes des idéaux fermé de $L^1(G)$, classes de Hardy et séries de Fourier lacunaires*, Ann. Sci. École Norm. Sup (4) (1968), 499–580.
[8] A. Pelczynski, *Boundedness of the canonical projection for Sobolev spaces generated by finite families of linear differential operators*, Analysis at Urbana, vol. I, London Math. Soc. Lecture Note Ser. 137, Cambridge Univ. Press, 1989, pp. 395–415.
[9] A. Pelczyński and K. Senator, *On isomorphisms of anisotropic Sobolev spaces with "classical Banach spaces" and a Sobolev type embedding theorem*, Studia Math. **84** (1986), 169–215.
[10] W. Rudin, *Trigonometric series with gaps*, J. Math. and Mech. **9** (1960), 203–228.
[11] E. M. Stein, *Singular integrals and differential properties of functions*, Princeton Univ. Press, Princeton, 1970.
[12] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, 1971.

Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, I p., 00-950 Warszawa, Poland
miwoj@impan.gov.pl