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# STABILITY OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS

BERNARD BEAUZAMY

ABSTRACT. We introduce a new norm (derived from Bombieri's norm for polynomials) on a class of functions on the complex plane. This norm is hilbertian, and can be viewed as a weighted $L_2$ norm (or a weighted $l_2$ norm). It allows us to give quantitative results of the following sort: If we solve $P(D)u = f$ (with boundary conditions), and if we modify $f$, how is the solution $u$ modified?

Let $\mathcal{B}_2$ be the space of measurable functions $f$ in the plane, such that

$$\int_0^\infty \int_0^{2\pi} e^{-r^2} |f(re^{i\theta})|^2 r dr \frac{d\theta}{\pi} < +\infty,$$

equipped with the norm

$$\|f\| = \left( \int_0^\infty \int_0^{2\pi} e^{-r^2} |f(re^{i\theta})|^2 r dr \frac{d\theta}{\pi} \right)^{1/2}. \tag{1}$$

Let $\mathcal{P}_2$ be the closure of the analytic polynomials for this norm.

For instance, the function $e^z$ belongs to $\mathcal{P}_2$, since it can be approximated by analytic polynomials, and since

$$\int_0^\infty \int_0^{2\pi} e^{-r^2} |e^{re^{i\theta}}|^2 r dr \frac{d\theta}{\pi} = \int_0^\infty \int_0^{2\pi} e^{-r^2+2r\cos\theta} r dr \frac{d\theta}{\pi} < +\infty.$$

The space $\mathcal{B}_2$, and its subspace $\mathcal{P}_2$, are obviously Hilbert spaces, with the scalar product

$$\langle f, g \rangle = \int_0^\infty \int_0^{2\pi} e^{-r^2} f(re^{i\theta})\overline{g(re^{i\theta})} r dr \frac{d\theta}{\pi}. \tag{2}$$

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For a polynomial $P(z) = \sum_0^n a_j z^j$, the norm has an expression using its coefficients:

LEMMA 1. *For a polynomial $P$ in one complex variable, $P(z) = \sum_0^n a_j z^j$:*

$$\|P\| = \left(\sum_{j=0}^n j!\,|a_j|^2\right)^{1/2}. \tag{3}$$

The proof is obvious, and is left to the reader.

Therefore, the space $\mathcal{P}_2$ can be viewed as a space of analytic functions $f(z) = \sum_{j\geq 0} a_j z^j$ for which

$$\sum_{j\geq 0} j!|a_j|^2 < +\infty.$$

The expression (3) offers some similarity with Bombieri's norm, which, for a one-variable polynomial (see B. Beauzamy [2]), is defined by

$$[P] = \left(\sum_{j=0}^n \frac{j!(n-j)!}{n!}|a_j|^2\right)^{1/2}. \tag{4}$$

In this expression, the constant factor $n!$ is of course of no importance. The only difference between (4) and (3) is the presence of the reciprocal factor $(n-j)!$. The degree of $P$ appears in (4), not in (3): this allows us to define $\|f\|$ (as we did) for a class of entire functions, which was not possible with (4).

The similarity of the definitions will ensure that many properties of Bombieri's norm will pass to this new norm. We now explore these properties, and we will come back to the comparison between (3) and (4) at the end of the paper.

LEMMA 2 (TRANSPOSITION). *For any $f, g$ in $\mathcal{P}_2$,*

$$\left\langle \frac{\partial f}{\partial z}, g\right\rangle = \langle f, zg\rangle, \qquad \langle zf, g\rangle = \left\langle f, \frac{\partial g}{\partial z}\right\rangle. \tag{5}$$

*Proof of Lemma* 2. Take the norm under the form (3), with scalar product

$$\langle f, g\rangle = \sum_{j\geq 0} j!a_j\bar{b}_j; \tag{6}$$

if $f = \sum_0^\infty a_j z^j$, $g = \sum_0^\infty b_j z^j$. It is enough to prove the lemma for monomials, and (5) is obvious.

The next lemma is the equivalent of the "evaluation lemma" obtained by Bruce Reznick in [7] for Bombieri's norm:

LEMMA 3. *Let $f$ be in $\mathcal{P}_2$ and $w$ in $\mathbb{C}$. Then*

$$f(w) = \langle f, e^{\bar{w}z} \rangle. \tag{7}$$

*Proof of Lemma* 3. This is straightforward from the representation (6) of the scalar product.

LEMMA 4. *Let $Q$ be in $\mathcal{P}_2$. Then, for any complex $\alpha$,*

$$\|(z-\alpha)Q\|^2 = \|Q\|^2 + \|\bar{\alpha}Q - Q'\|^2. \tag{8}$$

*In particular, $\|(z-\alpha)Q\| \geq \|Q\|$, with equality if and only if $Q(z) = c\exp\{\bar{\alpha}z\}$, for some complex number $c$.*

*Proof of Lemma* 4. Using Lemma 2, we write

$$\begin{aligned}
&\langle (z-\alpha)Q, (z-\alpha)Q \rangle \\
&\quad = \langle zQ, zQ \rangle - \langle \alpha Q, zQ \rangle - \langle zQ, \alpha Q \rangle + \langle \alpha Q, \alpha Q \rangle \\
&\quad = \langle Q', Q' \rangle + \langle Q, Q \rangle - \alpha \langle Q', Q \rangle - \bar{\alpha} \langle Q, Q' \rangle + |\alpha|^2 \langle Q, Q \rangle \\
&\quad = \|Q\|^2 + \|Q' - \bar{\alpha}Q\|^2,
\end{aligned}$$

which proves Lemma 4.

As a consequence of formula (8), we see that if $f$ is analytic and $(z-\alpha)f$ is in $\mathcal{P}_2$, then $f$ is in $\mathcal{P}_2$.

COROLLARY 5. *For any polynomial $P$ written as*

$$P = a(z-\alpha_1)\cdots(z-\alpha_n),$$

*and any polynomial $Q$, we have*

$$\|PQ\| \geq |a|\,\|Q\|.$$

This is clear from Lemma 4.

Now, we want to solve equations of the form

$$P(D)u = f, \tag{12}$$

where $D = \dfrac{\partial}{\partial z}$, $f$ is in $\mathcal{P}_2$, and we look for $u$, with $u \in \mathcal{P}_2$, $P(D)u \in \mathcal{P}_2$.

LEMMA 6. *Let $f$ in $\mathcal{P}_2$. The equation $P(D)u = f$ is satisfied with $u$ in $\mathcal{P}_2$, such that $P(D)u$ is in $\mathcal{P}_2$, if and only if, for any polynomial $Q$,*

$$\langle u, P^*Q \rangle = \langle f, Q \rangle, \tag{13}$$

*where $P^*$ is the conjugate polynomial $\sum \bar{a}_j z^j$.*

*Proof.* If $u$ is a solution, (13) must hold for any polynomial, by the transposition Lemma. On the other hand, assume there is $u$, in $\mathcal{P}_2$ with $P(D)u$ in $\mathcal{P}_2$, satisfying (13). Then we have $\langle P(D)u, Q\rangle = \langle f, Q\rangle$ for all $Q$, which implies $P(D)u = f$, since polynomials are dense in $\mathcal{P}_2$.

Let $n$ be (as before) the degree of the polynomial $P$.

LEMMA 7. *The set $\{P^*g;\, g \in \mathcal{P}_2\}$ is a closed subspace of $\mathcal{P}_2$, with codimension $n$.*

*Proof of Lemma* 7. The fact that this is a closed subspace is obvious; let us turn to its codimension. In order to prove our claim, all we have to do is to show that $(z+a)\mathcal{P}_2$ is of codimension 1 in $\mathcal{P}_2$; we then iterate the procedure and show that $(z+a)(z+b)\mathcal{P}_2$ is of codimension 1 in $(z+a)\mathcal{P}_2$, and so on.

So let us consider $(z+1)\mathcal{P}_2 = E$. We claim that this space is exactly the subspace of $\mathcal{P}_2$ made of functions which vanish at $-1$. One inclusion is clear. On the other hand, if $f$ is in $\mathcal{P}_2$ and vanishes at $-1$, since $f$ is analytic, $f = (z+1)g$ with $g$ analytic, and Lemma 4 shows that $g$ is also in $\mathcal{P}_2$, which proves our claim.

By the evaluation lemma (Lemma 3), we have

$$E = \{f;\, \langle f, e^{-z}\rangle = 0\},$$

which is of codimension 1 in $\mathcal{P}_2$. This proves Lemma 7.

COROLLARY 8. *The problem $P(D)u = f$, where $f \in \mathcal{P}_2$ and the solution $u$ satisfies $u \in \mathcal{P}_2$, $P(D)u \in \mathcal{P}_2$, is well posed (that is admits a unique solution) if and only if we add to it $n$ independent conditions of the form*

$$\left\{\begin{array}{lcl} \langle u, g_1\rangle & = & c_1 \\ & \vdots & \\ \langle u, g_n\rangle & = & c_n \end{array}\right. \tag{14}$$

*for given $g_1, \dots, g_n \in \mathcal{P}_2$ and complex scalars $c_1, \dots, c_n$. The subspace generated by $g_1, \dots, g_n$ must not intersect (except at 0) the subspace $P^*\mathcal{P}_2$.*

LEMMA 9. *The sequence of monomials $z^k/\sqrt{k!}$ is a Hilbertian basis of $\mathcal{P}_2$.*

*Proof of Lemma* 9. Clearly from (3) these monomials are normalized; from (6) one sees that they are orthogonal. Assume $\langle f, z^k\rangle = 0$ for all $k$. Then $\langle f, Q\rangle = 0$ for any polynomial $Q$, and therefore $f = 0$ since polynomials are dense in $\mathcal{P}_2$. This proves the lemma.

So any $f$ in $\mathcal{P}_2$ has a unique decomposition

$$f = \sum_{k\geq 0} \alpha_k \frac{z^k}{\sqrt{k!}}, \tag{15}$$

where $\sum |\alpha_k|^2 < +\infty$, and

$$\|f\| = \left(\sum |\alpha_k|^2\right)^{1/2}. \tag{16}$$

From now on, we assume that $n$ conditions of the type (14) have been added to our problem, which is now well posed.

The solution $u$ will be decomposed into two orthogonal pieces

$$u = u_1 + u_2,$$

where $u_2$ belongs to the orthogonal complement of the subspace $\{P^*g;\ g \in \mathcal{P}_2\}$ (thus meaning that $P(D)u_2 = 0$), and satisfies conditions (14). The application $f \to u_1$ is well-defined, and is linear.

If we take on $u_1$ the norm

$$\|u_1\|_2 = (\|u_1\|^2 + \|P(D)u_1\|^2)^{1/2}, \tag{17}$$

the application $u_1 \to f$ is continuous. Therefore, by the closed graph theorem, the inverse application is also continuous. The stability result we are looking for is a consequence of this continuity: we are looking for a constant $C$ such that, for all $f$,

$$\|u_1\| \leq C\|f\|. \tag{18}$$

The computation of this constant will be done through the following lemma:

LEMMA 10. *The transpose of the application* $P \times \sum_k \alpha_k \frac{z^k}{\sqrt{k!}} \to (\alpha_k)_{k\geq 0}$, *from* $P\,\mathcal{P}_2$ *into* $l_2$, *is the application* $f \to u_1$.

*Proof of Lemma* 10. We denote by $A$ the application

$$A : P \times \sum_{k\geq 0} \beta_k \frac{z^k}{\sqrt{k!}} \to (\beta_k)_{k\geq 0}. \tag{19}$$

Its transpose, ${}^tA$, is defined, for $\beta = (\beta_k) \in l_2$, by the formula

$$\begin{aligned}\left\langle {}^tA\beta, P\sum_{k\geq 0} \alpha_k \frac{z^k}{\sqrt{k!}} \right\rangle &= \left\langle \beta, A\left(P\sum_{k\geq 0} \alpha_k \frac{z^k}{\sqrt{k!}}\right)\right\rangle_{l_2} \\ &= \sum_{k\geq 0} \beta_k \overline{\alpha_k}.\end{aligned}$$

Take $\alpha_j = 1$, the others 0; for all $j$ we get

$$\left\langle {}^tA\beta, P\frac{z^j}{\sqrt{j!}} \right\rangle = \beta_j. \tag{20}$$

Now set $f = \sum_{k\geq 0} \beta_k \frac{z^k}{\sqrt{k!}}$, and let $u_1$ be the solution of the associated problem. Then, for all $j$,

$$\left\langle u, \frac{P^* z^j}{\sqrt{j!}} \right\rangle = \left\langle f, \frac{z^j}{\sqrt{j!}} \right\rangle = \beta_j,$$

which is the same as (20) and proves that ${}^t A\beta = u_1$, which is our claim.

LEMMA 11. *Assume we have an inequality of the sort*

$$\left\| P \sum_{k\geq 0} \alpha_k \frac{z^k}{\sqrt{k!}} \right\| \geq \delta \left( \sum_{k\geq 0} |\alpha_k|^2 \right)^{1/2}, \tag{21}$$

*where $\delta$ is independent of the $\alpha_k$'s (but may depend on $P$). Then*

$$\|u_1\| \leq \frac{1}{\delta} \|f\|. \tag{22}$$

*Proof of Lemma* 11. Inequality (21) expresses the fact that $A$, defined in (19), is continuous and satisfies

$$\|A\|_{op} \leq 1/\delta.$$

But then its transpose is also continuous and has same norm (see for instance B. Beauzamy [1]). Since this transpose is precisely the application $f \to u_1$, inequality (22) follows.

But Corollary 5 provides us with an inequality of type (21); namely, we know that if $a$ is the leading coefficient of $P$,

$$\|PQ\| \geq |a| \, \|Q\|$$

for any polynomial $Q$.

Summarizing, we have obtained:

THEOREM. *Let $P$ be any complex polynomial, and let $a$ be its leading coefficient, $n$ its degree. Consider a problem*

$$P(D)u = f$$

*with $f$ in $\mathcal{P}_2$, $u$ in $\mathcal{P}_2$. Add $n$ independent conditions of the form* (14) *so that the problem is well posed. Decompose $u$ into two orthogonal pieces $u_1 + u_2$, where $P(D)u_2 = 0$ and $u_2$ satisfies the $n$ conditions* (14). *Then we have the estimate*

$$\|u_1\| \leq \frac{1}{|a|} \|f\|. \tag{23}$$

This estimate is indeed a stability result: if $f$ and $g$ satisfy $\|f - g\| < \varepsilon$ and if $u$, $v$ are the solutions of $P(D)u = f$, $P(D)v = g$, with same conditions (14), then $\|u - v\| < \varepsilon/|a|$.

There are not too many results concerning this type of stability in general, that is, without special assumptions on $P$. Of course, in special cases, there are classical theories: for instance when $P$ is elliptic (see H. Brezis [6]).

Let us now come back to the comparison between this new norm and Bombieri's norm.

The usual Bombieri's norm can be defined with a kernel similar to the one we used in definition (1). Indeed, take a polynomial $P(z) = \sum_0^n a_j z^j$. Consider the associated homogeneous two-variable polynomial

$$\widetilde{P}(x_1, x_2) = \sum_0^n a_j x_1^j x_2^{n-j}.$$

Then one proves easily that

$$n![P]^2 = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} e^{-r_1^2 - r_2^2} |\tilde{P}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^2 r_1 dr_1 \frac{d\theta_1}{\pi} r_2 dr_2 \frac{d\theta_2}{\pi}.$$

As observed by Bruce Reznick [7], in the definition of Bombieri's norm one should replace $[P]$ by $\sqrt{n!}\,[P]$: this would be more consistent. Assume this modification to be done in our earlier work (Beauzamy-Bombieri-Enflo-Montgomery [3], Beauzamy-Dégot [4]). Then one sees that Bombieri's norm is exactly the same kernel as (1), except that the kernel is applied to a homogeneous polynomial in 2 variables.

Bombieri's inequality (see [3], [4]) reads

$$[PQ] \geq [P][Q],$$

and the corresponding inequality for $\|.\|$ is false (take $P = 1 + z$, $Q = 1 - z$). Lemma 4 gives a kind of replacement, which can be written in symmetric form:

$$\|PQ\| \geq \max\{a\|Q\|, b\|P\|\}$$

where $a, b$ are the leading coefficients of $P$, $Q$ respectively.

Finally, we observe that the definitions and results can be extended to functions in several complex variables, using the norm

$$\|f\|^2 = \int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-(r_1^2 + \cdots + r_n^2)} \times |f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})|^2 r_1 dr_1 \frac{d\theta_1}{\pi} \cdots r_n dr_n \frac{d\theta_n}{\pi}.$$

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Société de Calcul Mathématique, S.A. 111, Faubourg Saint Honoré, 75008 Paris
beauzamy@aol.com