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# MONOTONIC TRIGONOMETRIC SUMS AND COEFFICIENTS OF BLOCH FUNCTIONS

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ABSTRACT. We establish a new class of monotonic trigonometric sums. Through a result of Andreev and Duren, our theorem provides information about the coefficients of certain Bloch functions.

## 1. Introduction

The class of Bloch functions consists of analytic functions $g$ in the unit disk $\mathbf{D}$ satisfying

$$\sup_{z\in\mathbf{D}}(1-|z|^2)\left|g'(z)\right|<\infty.$$

For $f(z)$ in the usual class S of analytic and univalent functions in $\mathbf{D}$, it is well known (cf. [10, p. 32]) that $\log f'(z)$ is a Bloch function. Now suppose that $f \in S$ and define the coefficients $\beta_n$ by

$$\log f'(z)=2\sum_{n=1}^{\infty}\beta_n z^n.$$

For the Koebe function $k(z)=\dfrac{z}{(1-z)^2}$ we have

$$\log k'(z)=2\sum_{n=1}^{\infty}\lambda_n z^n,$$

where

$$\lambda_n=\begin{cases}\frac{1}{n}, & \text{when } n \text{ is even,}\\ \frac{2}{n}, & \text{when } n \text{ is odd.}\end{cases} \tag{1.1}$$

In [1], V.V. Andreev and P.L. Duren considered the problem of maximizing the functional

$$\psi(f)=\sum_{k=1}^{n}\sigma_k\left|\beta_k\right|^2, \quad f\in S,$$

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where $\sigma_k$ is a sequence of nonnegative numbers. By using the method of boundary variation they derived the necessary condition for the weights $\sigma_k \geq 0$, for which the inequality

$$\sum_{k=1}^{n} \sigma_k \, |\beta_k|^2 \leq \sum_{k=1}^{n} \sigma_k \, |\lambda_k|^2 , \tag{1.2}$$

holds for all Bloch functions of the form $\log f'$ for some $f \in S$. Of course, inequality (1.2) is valid if and only if the Koebe function maximizes the left hand side among all Bloch functions of this form. In fact, Andreev and Duren [1] proved the following:

THEOREM. *Let $n \geq 1$ be a fixed integer and let the weights $\sigma_k \geq 0$ be given, $k = 1, 2, \ldots n$. If the inequality* (1.2) *holds for all functions $f \in S$, then*

$$\frac{d}{d\theta}\left\{\frac{\sin^4 \frac{\theta}{2}}{\sin\theta} \sum_{k=1}^{n} \sigma_k \lambda_k \sin k\theta\right\} \geq 0, \quad 0 < \theta < \pi, \tag{1.3}$$

*where $\lambda_k$ are defined by* (1.1).

Andreev and Duren [1] gave some applications of this theorem by showing that several instances of (1.2) are false for appropriate choices of the coefficients $\sigma_k$ because, for these $\sigma_k$, inequality (1.3) does not hold. However, they gave no example of trigonometric sum satisfying the condition (1.3) for all $n$.

In the present paper, our aim is to provide a wide class of trigonometric sums for which (1.3) is true for all $n$ and thus to give some information for the order of magnitude of the coefficients $\sigma_k$ for which (1.2) may be valid.

Our main result is the following:

THEOREM. *For every positive integer $n$, we have*

$$\frac{d}{d\theta}\left\{\frac{\sin^4 \frac{\theta}{2}}{\sin\theta} \sum_{k=1}^{n} \frac{\sin k\theta}{k^\alpha}\right\} > 0, \quad 0 < \theta < \pi, \tag{1.4}$$

*when $\alpha \geq 3$. This inequality is false for appropriate $n$ and $\theta$ when $\alpha < 3$.*

The first thing to be noted is that inequality (1.3) implies

$$\sum_{k=1}^{n} \sigma_k \lambda_k \sin k\theta \geq 0, \quad 0 < \theta < \pi, \tag{1.5}$$

because the function $\dfrac{\sin^4 \frac{\theta}{2}}{\sin\theta}$ is strictly increasing for this range of $\theta$. Thus, in order to obtain trigonometric sums for which an inequality like (1.4) is true, we should only consider sums with $\sigma_k \lambda_k$ satisfying the positivity condition (1.5).

It is true that, for all positive integers $n$,

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k^{\alpha}} > 0, \quad 0 < \theta < \pi, \tag{1.6}$$

when $\alpha \geq 1$ and this follows by partial summation from the special case $\alpha = 1$, which is known as the Fejér-Jackson-Gronwall inequality. See [11], [12] and [13]. Inequality (1.6) fails to hold for $\alpha < 1$ and this has been shown in [8, Sec. 4]. Thus, we consider (1.4) for $\alpha \geq 1$.

Our theorem above, enables us to characterize the positive sine sums of (1.6) for which (1.4) is additionally satisfied.

Known results on monotonic trigonometric sums different from (1.4) are

$$\frac{d}{d\theta}\left\{\sum_{k=1}^{n} \frac{\sin k\theta}{k \sin \frac{\theta}{2}}\right\} < 0 \quad \text{for all } n, \quad 0 < \theta < \pi, \tag{1.7}$$

which has been obtained by R. Askey and J. Steinig in [4]. See also [2] and [6] for some more general inequalities.

The natural analogue of (1.7) for cosine sums has been established in [5]. This is

$$\frac{d}{d\theta}\left\{\cos\frac{\theta}{2}\left(1 + \sum_{k=1}^{n} \frac{\cos k\theta}{k^{\alpha}}\right)\right\} < 0 \quad \text{for all } n, \quad 0 < \theta < \pi, \tag{1.8}$$

if and only if $\alpha \geq 1$.

A straightforward differentiation shows that (1.4) is equivalent to

$$\frac{\sin^4 \frac{\theta}{2}}{\sin\theta}\left\{\left(4\cos^2\frac{\theta}{2} - \cos\theta\right)\sum_{k=1}^{n} \frac{\sin k\theta}{k^{\alpha}\sin\theta} + \sum_{k=1}^{n} \frac{\cos k\theta}{k^{\alpha-1}}\right\} > 0,$$

and, in turn,

$$(2 + \cos\theta)\sum_{k=1}^{n} \frac{\sin k\theta}{k^{\alpha}\sin\theta} + \sum_{k=1}^{n} \frac{\cos k\theta}{k^{\alpha-1}} > 0, \tag{1.9}$$

because $\dfrac{\sin^4 \frac{\theta}{2}}{\sin\theta} > 0$ for $0 < \theta < \pi$. Clearly, (1.9) for $\alpha \geq 3$, follows by partial summation from the special case $\alpha = 3$, which we prove in the next sections. It should be noted that inequalities (1.2), (1.3) and (1.4) are true for $n = 1$; thus from now on we assume that $n \geq 2$ in (1.9).

The difficulty of proving inequalities involving trigonometric polynomials, such as (1.3), (1.4) and (1.5) is acknowledged in Mathematical Reviews by Yuk Leung in his review of the paper [1] (cf. M.R. 90c:30026).

Our plan to achieve a proof of (1.9) is as follows. In Section 2, we determine the critical value $\alpha = 3$ for the validity of (1.9), that is, we show that this cannot hold for $1 \leq \alpha < 3$. In our proof of (1.9) for $\alpha = 3$ we consider separately the cases of even and odd $n$. We prove (1.9) for even $n$ in Section 3. In the final Section 4, we give the proof of (1.9) for all odd $n \geq 3$.

## 2. The critical value of $\alpha$

In this section, we shall show that inequality (1.9) fails to hold for appropriate $n$ and $\theta$ when $1 \leq \alpha < 3$. For this purpose, we let

$$S_n^\alpha(\theta) = (2+\cos\theta)\sum_{k=1}^{n}\frac{\sin k\theta}{k^\alpha \sin\theta} + \sum_{k=1}^{n}\frac{\cos k\theta}{k^{\alpha-1}}$$

and

$$\begin{aligned} g_n^\alpha(\theta) &= \frac{1}{\sin\theta}\frac{d}{d\theta}S_n^\alpha(\theta) = -\sum_{k=1}^{n}\frac{(k^2+1)\sin k\theta}{k^\alpha \sin\theta} \\ &+ \frac{2+\cos\theta}{\sin^2\theta}\left\{\sum_{k=1}^{n}\frac{\cos k\theta}{k^{\alpha-1}} - \cot\theta\sum_{k=1}^{n}\frac{\sin k\theta}{k^\alpha}\right\}. \end{aligned}$$

We examine the sign of $g_n^\alpha(\theta)$ in the vicinity of $\pi$. First, we observe that

$$\lim_{\theta\to\pi} g_n^\alpha(\theta) = \sum_{k=1}^{n}(-1)^k\left(\frac{1}{k^{\alpha-1}}+\frac{1}{k^{\alpha-3}}\right) + \sum_{k=1}^{n}\frac{1}{k^\alpha}M_k, \tag{2.1}$$

where

$$M_k = \lim_{\theta\to\pi}\left\{\frac{1}{\sin^2\theta}\left(k\cos k\theta - \cos\theta\frac{\sin k\theta}{\sin\theta}\right)\right\}.$$

A short calculation shows that

$$M_k = \frac{1}{3}(-1)^k(k-k^3).$$

Substituting in (2.1) we get

$$\lim_{\theta\to\pi} g_n^\alpha(\theta) = \frac{2}{3}\sum_{k=1}^{n}(-1)^k\left(\frac{2}{k^{\alpha-1}}+\frac{1}{k^{\alpha-3}}\right).$$

We next observe that

$$\sum_{k=1}^{n}(-1)^k\left(\frac{2}{k^{\alpha-1}}+\frac{1}{k^{\alpha-3}}\right) > 0,$$

for $\alpha < 3$ when $n$ is even and sufficiently large. In fact, it can be easily checked that

$$\lim_{N\to\infty}\frac{1}{(2N)^{3-\alpha}}\sum_{k=1}^{2N}(-1)^k k^{3-\alpha} = \frac{1}{2}.$$

That is to say that, in this case, the derivative of $S_n^\alpha(\theta)$ is positive sufficiently close to $\pi$, hence $S_n^\alpha(\theta)$ must assume negative values near $\pi$.

## 3. Proof of the theorem when $n$ is even

In the present section, we shall establish (1.9) when $\alpha = 3$, for all even $n$. Since

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k^3} > 0, \quad \text{for all } n, \ 0 < \theta < \pi,$$

we can obtain this result by showing that

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k^3 \sin\theta} + \sum_{k=1}^{n} \frac{\cos k\theta}{k^2} > 0, \tag{3.1}$$

for the same range of $\theta$. In order to prove this, we show that both sums on the left hand side are monotonically decreasing for $0 < \theta < \pi$. Since the left hand side of (3.1) vanishes for $\theta = \pi$, the desired result follows. In fact, in view of the Fejér-Jackson-Gronwall inequality, that is (1.6) for $\alpha = 1$, all the cosine sums

$$\sum_{k=1}^{n} \frac{\cos k\theta}{k^2}$$

are strictly decreasing for $0 < \theta < \pi$. For the sine sums in (3.1) we have the following:

LEMMA 1. *For all positive integers $N$, we have*

$$\frac{d}{d\theta} \sum_{k=1}^{2N} \frac{\sin k\theta}{k^3 \sin\theta} < 0 \quad \text{for } 0 < \theta < \pi. \tag{3.2}$$

*Proof.* This inequality can be considered as an inequality for ultraspherical polynomials $C_k^\lambda(x)$ defined, as usual, by the generating function

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(x) r^k, \quad \lambda > 0.$$

Setting $x = \cos\theta$ and recalling that

$$\frac{C_k^1(\cos\theta)}{C_k^1(1)} = \frac{\sin(k+1)\theta}{(k+1)\sin\theta},$$

we see that inequality (3.2) is equivalent to

$$\frac{d}{dx} \sum_{k=0}^{2N-1} \frac{1}{(k+1)^2} \frac{C_k^1(x)}{C_k^1(1)} > 0 \quad \text{for all } N, \quad -1 < x < 1, \tag{3.3}$$

which we proceed to prove. Using the differentiation formula

$$\frac{d}{dx} C_k^{\lambda}(x) = 2\lambda C_{k-1}^{\lambda+1}(x)$$

and the fact that

$$C_k^{\lambda}(1) = \frac{(2\lambda)_k}{k!} = \frac{\Gamma(k+2\lambda)}{k!\,\Gamma(2\lambda)},$$

(see [14], pp. 80-81), we find that (3.3), in turn, is equivalent to

$$\sum_{k=0}^{2N-2} a_k \frac{C_k^2(x)}{C_k^2(1)} > 0 \quad \text{for all } N, \quad -1 < x < 1, \tag{3.4}$$

where

$$a_k = \frac{(k+1)(k+3)}{(k+2)^2}. \qquad \square$$

We note, in passing, that since $a_k$ is a strictly increasing sequence, inequality (3.4) cannot hold for odd sums. The corresponding odd sums of (3.4) are negative for $x = -1$. However, (3.4) does hold for all even sums. Actually, we shall establish an inequality more general than this. Namely, for all positive $N$,

$$\sum_{k=0}^{2N-2} a_k \frac{C_k^{\lambda}(x)}{C_k^{\lambda}(1)} > 0, \quad -1 < x < 1, \quad \lambda \geq 1. \tag{3.5}$$

For the proof of (3.5) we need the following theorem, proved by R. Askey and G. Gasper in [2, Th. A].

THEOREM. *Let $\lambda > \nu > 0$. If*

$$\sum_{k=0}^{n} a_k \frac{C_k^{\nu}(x)}{C_k^{\nu}(1)} > 0, \quad -1 < x < 1,$$

*then*

$$\sum_{k=0}^{n} a_k \frac{C_k^{\lambda}(x)}{C_k^{\lambda}(1)} > 0, \quad -1 < x < 1.$$

See also [3].

According to this theorem, it is sufficient to prove (3.5) for $\lambda = 1$, which reduces to

$$\sum_{k=1}^{2N-1} \frac{\sin k\theta}{k+1} + \sum_{k=1}^{2N-1} \frac{\sin k\theta}{(k+1)^2} > 0, \quad 0 < \theta < \pi. \tag{3.6}$$

But this inequality holds true for all $N$. In fact, it is shown in [9, Th. A] that

$$\sum_{k=1}^{2N-1} \frac{\sin k\theta}{k+1} > 0, \quad N = 1, 2, \dots \quad 0 < \theta < \pi.$$

On the other hand, inequality

$$\sum_{k=1}^{2N-1} \frac{\sin k\theta}{(k+1)^2} > 0, \quad N = 1, 2, \dots \quad 0 < \theta < \pi,$$

has been proven in [7, Lemma 4]. The proof of Lemma 1 is now complete. Thus all our claims about the sums in (3.1) are established.

Unfortunately, the inequality of Lemma 1 is false for the corresponding odd sums. (In fact it fails near $\pi$.) So for the the case of odd $n$ we should follow a different argument to achieve a proof of (1.9) and this is given in the next section.

## 4. Proof of the theorem when $n$ is odd

In this section, we deal with (1.9), for $\alpha = 3$, in the case where $n$ is odd ($n \geq 3$). It is convenient to consider separately the intervals $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \theta < \pi$.

*Case* 1. The interval $0 < \theta < \frac{\pi}{2}$. We rewrite the left hand side of (1.9) as

$$\frac{2+\cos\theta}{2\cos\frac{\theta}{2}} \sum_{k=1}^{n} \frac{\sin k\theta}{k^3 \sin\frac{\theta}{2}} + \sum_{k=1}^{n} \frac{\cos k\theta}{k^2} = S_n(\theta). \tag{4.1}$$

A summation by parts shows that the Askey-Steinig inequality (1.7) yields

$$\frac{d}{d\theta} \sum_{k=1}^{n} \frac{\sin k\theta}{k^3 \sin\frac{\theta}{2}} < 0, \quad 0 < \theta < \pi.$$

On the other hand, as mentioned earlier, the Fejér-Jackson-Gronwall inequality implies that the cosine sums in (4.1) decrease for $0 < \theta < \pi$, as well. Observe now that the function $h(\theta) = \dfrac{2+\cos\theta}{2\cos\frac{\theta}{2}}$ is positive and strictly decreasing for $0 < \theta < \frac{\pi}{2}$. Therefore, $S_n(\theta)$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$ for all $n$. Thus it suffices to prove the positivity of $S_n(\theta)$ for $\frac{\pi}{2} \leq \theta < \pi$.

*Case* 2. The interval $\frac{\pi}{2} \leq \theta < \pi$. Evidently, in this case the positivity of $S_n(\theta)$ is equivalent to

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k^3} + \frac{\sin\theta}{2+\cos\theta} \sum_{k=1}^{n} \frac{\cos k\theta}{k^2} > 0. \tag{4.2}$$

To prove this inequality in the interval under consideration we need the following elementary lemma.

LEMMA 2. *For every $n \geq 2$, we have*

$$\sum_{k=1}^{n} \frac{\cos k\theta}{k^2} < 0, \quad \frac{\pi}{2} \leq \theta < \pi.$$

*Proof.* Once more we take into account the fact that these cosine sums are monotonically decreasing on the interval in question and this is deduced from the Fejér-Jackson-Gronwall inequality. Thus it suffices to prove that the above cosine sums are negative for $\theta = \frac{\pi}{2}$. Let

$$A_n = \sum_{k=1}^{n} \frac{\cos k\frac{\pi}{2}}{k^2}.$$

It is clear that

$$A_{2k} = A_{2k+1} \quad \text{for} \quad k = 1, 2, \ldots.$$

Hence we need only to consider the case where $n$ is even. Let $n = 2N$, then

$$A_n = \frac{1}{4} \sum_{k=1}^{N} (-1)^k \frac{1}{k^2} < 0, \quad \text{for all } N.$$

The proof of Lemma 2 is complete. □

We now turn to (4.2). It follows readily that the left hand side of (4.2) exceeds

$$T_n(\theta) = \sum_{k=1}^{n} \frac{\sin k\theta}{k^3} + \frac{\sin\theta}{2+\cos\theta}\rho_n \quad \text{for} \quad \frac{\pi}{2} \leq \theta < \pi,$$

where

$$\rho_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k^2}.$$

Thus we seek to prove positivity of $T_n(\theta)$ for $\frac{\pi}{2} \leq \theta < \pi$. We show that $T_n(\theta)$ is decreasing on this interval. Since clearly $T_n(\pi) = 0$, positivity follows. We see that inequality

$$\frac{d}{d\theta} T_n(\theta) < 0,$$

is equivalent to

$$(2+\cos\theta)^2 \sum_{k=1}^{n} \frac{\cos k\theta}{k^2} + \rho_n + 2\rho_n \cos\theta < 0, \quad \text{for} \quad \frac{\pi}{2} \leq \theta < \pi. \tag{4.3}$$

In view of Lemma 2, this inequality is an immediate consequence of

$$\sum_{k=1}^{n} \frac{\cos k\theta}{k^2} + \rho_n + 2\rho_n \cos\theta < 0, \tag{4.4}$$

for the same range of $\theta$. We now observe that the left hand side of (4.4) vanishes for $\theta = \pi$. We shall show that it is also strictly increasing for this range of $\theta$, hence (4.4) follows. Thus we need to prove that

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} + 2\rho_n \sin\theta < 0 \quad \text{for} \quad \frac{\pi}{2} \leq \theta < \pi. \tag{4.5}$$

For the proof of this inequality we shall use techniques similar to those of [6] in estimating the Fejér-Jackson-Gronwall sum appearing in it.

We make the transformation $\phi = \pi - \theta$ and define

$$I_n(\phi) = \int_0^{\phi} \frac{\cos(n+\frac{1}{2})t}{2\cos\frac{t}{2}}\, dt,$$

$$f_n(\phi) = -\frac{\phi}{2} - 2\rho_n \sin\phi.$$

Suppose that $n$ is odd. It can be easily checked that (4.5) is equivalent to

$$f_n(\phi) - I_n(\phi) > 0, \quad 0 < \phi \leq \frac{\pi}{2}, \tag{4.6}$$

which we prove next.

An elementary calculation shows that $f_n(\phi)$ is a positive, concave function of $\phi$ in $\left[0, \frac{\pi}{2}\right]$ for all $n$.

In what follows, we fix the notation $\gamma = \dfrac{\pi}{n+\frac{1}{2}}$.

In order to establish (4.6) we now consider the following cases.

*Case* 2*a*. The interval $0 < \phi \leq \frac{\gamma}{2}$. For $\phi$ lying in this interval, we show that the left hand side of (4.6) is strictly increasing from $f_n(0) - I_n(0) = 0$. In fact, differentiating we get

$$-\frac{1}{2} - 2\rho_n \cos\phi - \frac{\cos(n+\frac{1}{2})\phi}{2\cos\frac{\phi}{2}},$$

whose positivity follows from

$$-\cos\frac{\phi}{2} - 4\rho_n \cos\phi\cos\frac{\phi}{2} - \cos(n+\frac{1}{2})\phi \geq -\cos\frac{\phi}{2} - 4\rho_n \cos\phi\cos\frac{\phi}{2} - 1 > 0$$

and the last inequality follows by an elementary calculation.

*Case* 2*b*. The interval $\frac{\gamma}{2} < \phi \leq 3\frac{\gamma}{2}$. Here we observe that the left hand side of (4.6) increases from $f_n\left(\frac{\gamma}{2}\right) - I_n\left(\frac{\gamma}{2}\right) > 0$.

*Case* 2*c*. The interval $3\frac{\gamma}{2} \leq \phi \leq \frac{\pi}{2}$. Let us suppose first that $n = 4N+3$. Then, we have

$$
\begin{aligned}
I_n(\phi) &\leq \int_0^{(n-2)\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt \\
&= \int_0^{\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt + L_n,
\end{aligned} \tag{4.7}
$$

where

$$
\begin{aligned}
L_n &= \int_{\frac{\gamma}{2}}^{(n-2)\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt \\
&= \sum_{k=1}^{N} \int_{(4k-3)\frac{\gamma}{2}}^{(4k+1)\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt.
\end{aligned} \tag{4.8}
$$

It is not hard to see that

$$
\begin{aligned}
&\int_{(4k-3)\frac{\gamma}{2}}^{(4k+1)\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt \\
&= \frac{\gamma}{2\pi}\int_{(4k-3)\frac{\pi}{2}}^{(4k+1)\frac{\pi}{2}} \frac{\cos t}{\cos\frac{\gamma t}{2\pi}}\,dt \\
&= \frac{\gamma}{2\pi}\left\{\int_{(4k-3)\frac{\pi}{2}}^{(4k-1)\frac{\pi}{2}} \left(\frac{1}{\cos\frac{\gamma t}{2\pi}} - \frac{1}{\cos(\frac{\gamma t}{2\pi}+\frac{\gamma}{2})}\right)\cos t\,dt\right\}.
\end{aligned} \tag{4.9}
$$

From this it follows easily that

$$
\int_{(4k-3)\frac{\gamma}{2}}^{(4k+1)\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt \leq \frac{\gamma}{\pi}\left(\frac{1}{\cos(4k+1)\frac{\gamma}{4}} - \frac{1}{\cos(4k-1)\frac{\gamma}{4}}\right).
$$

Thus, from this and (4.8) we deduce that

$$
\begin{aligned}
L_n &\leq \frac{\gamma}{\pi}\sum_{k=1}^{N}\left(\frac{1}{\cos(4k+1)\frac{\gamma}{4}} - \frac{1}{\cos(4k-1)\frac{\gamma}{4}}\right) \\
&= \frac{\gamma}{\pi}\sum_{k=1}^{2N}(-1)^k\frac{1}{\cos(2k+1)\frac{\gamma}{4}}
\end{aligned}
$$

$$= \frac{\gamma}{\pi} \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{\sin k\frac{\gamma}{2}}$$

$$< \frac{2}{\pi} \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{k}. \tag{4.10}$$

We also have

$$\int_0^{\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt \leq \frac{\gamma}{2\pi\cos\frac{\gamma}{4}}. \tag{4.11}$$

Combining this with (4.7) and (4.10) we obtain

$$I_n(\phi) \leq \frac{\gamma}{2\pi\cos\frac{\gamma}{4}} + \frac{2}{\pi} \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{k}. \tag{4.12}$$

On the other hand, since the functions $f_n(\phi)$ are concave in the interval under consideration, we have

$$f_n(\phi) \geq \min\left\{f_n\left(3\frac{\gamma}{2}\right), f_n\left(\frac{\pi}{2}\right)\right\} = f_n\left(3\frac{\gamma}{2}\right). \tag{4.13}$$

The validity of (4.6) in this interval follows from the inequality

$$\frac{1}{\gamma} f_n\left(3\frac{\gamma}{2}\right) - \frac{1}{2\pi\cos\frac{\gamma}{4}} - \frac{2}{\pi\gamma} \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{k} > 0, \tag{4.14}$$

which we shall prove using the estimates obtained above. In fact, it is easily seen that

$$\frac{1}{\gamma} \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{k} < \frac{1}{\pi} \lim_{N\to\infty} (4N + \frac{7}{2}) \sum_{k=2N+3}^{4N+2} (-1)^{k+1} \frac{1}{k} = \frac{1}{2\pi}. \tag{4.15}$$

On the other hand, it is readily shown that, for $n \geq 3$,

$$\frac{1}{\gamma} f_n\left(3\frac{\gamma}{2}\right) > 1.12$$

and

$$\frac{1}{2\pi\cos\frac{\gamma}{4}} < 0.164.$$

A combination of the above with (4.15) establishes (4.14).

In a similar way we can estimate the integral $I_n(\phi)$, in the case where $n = 4N + 1$. It is clear that now we have

$$I_n(\phi) \leq \int_0^{n\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt$$

$$= \int_0^{\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt + R_n, \tag{4.16}$$

where

$$R_n = \int_{\frac{\gamma}{2}}^{n\frac{\gamma}{2}} \frac{\cos\frac{\pi}{\gamma}t}{2\cos\frac{t}{2}}\,dt.$$

As above, we find that

$$R_n < \frac{2}{\pi}\sum_{k=2N+1}^{4N}(-1)^{k+1}\frac{1}{k}.$$

Now, using this, (4.11) and (4.16) we obtain

$$I_n(\phi) \le \frac{\gamma}{2\pi\cos\frac{\gamma}{4}} + \frac{2}{\pi}\sum_{k=2N+1}^{4N}(-1)^{k+1}\frac{1}{k}. \tag{4.17}$$

Thus, in view of (4.13) and (4.17) the desired inequality (4.6) is deduced from

$$\frac{1}{\gamma}f_n\left(3\frac{\gamma}{2}\right) - \frac{1}{2\pi\cos\frac{\gamma}{4}} - \frac{2}{\pi\gamma}\sum_{k=2N+1}^{4N}(-1)^{k+1}\frac{1}{k} > 0. \tag{4.18}$$

To see that (4.18) is valid, we first observe that

$$\frac{1}{\gamma}\sum_{k=2N+1}^{4N}(-1)^{k+1}\frac{1}{k} < \frac{1}{\pi}\lim_{N\to\infty}(4N+1)\sum_{k=2N+1}^{4N}(-1)^{k+1}\frac{1}{k} = \frac{1}{2\pi}. \tag{4.19}$$

Then, we can easily verify that, for $n \ge 5$, we have

$$\frac{1}{\gamma}f_n\left(3\frac{\gamma}{2}\right) > 1.46$$

and

$$\frac{1}{2\pi\cos\frac{\gamma}{4}} < 0.161.$$

A combination of these inequalities with (4.19) yields (4.18).

The proof of our main result is now complete.

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