ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 2, Summer 1999

ON THE IMAGE OF THE GAUSS MAP OF AN IMMERSED SURFACE WITH CONSTANT MEAN CURVATURE IN $\mathbb{R}^3$

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ABSTRACT. We prove, generalizing a well known property of Delaunay surfaces, that if the Gauss image of a cmc surface in the Euclidean space is a compact surface with boundary, then any connected component of sphere minus the image is a strictly convex domain. We also obtain conditions under which the Gauss image has a regular boundary. These results relate to the question, raised by do Carmo, of whether the Gauss image of a complete cmc surface contains an equator of the sphere.

1. Introduction

A well known problem in differential geometry is to describe the Gauss image of a complete surface of constant mean curvature (cmc) in $\mathbb{R}^3$. When the mean curvature is zero, that is, in the minimal case, there is already a large literature on this problem. A striking result, due to H. Fujimoto [4] asserts that if the surface is not a plane then the Gauss image can omit at most four points of the sphere. As we know, in the nonzero mean curvature case, there are just the works of R. Osserman, R. Schoen and D. Hoffman [5] and W. Seaman [6]. This last paper contains a description of the Gauss image of a complete helicoidal surface of cmc and in [5] it is proved that if the Gauss image of a complete immersed surface with cmc in $\mathbb{R}^3$ is contained in an open hemisphere of the sphere then the surface is a plane and if it is contained in a closed hemisphere and is not a plane then it is a right circular cylinder. Connected to this problem, there is the question raised by Manfredo do Carmo asking if the image of a complete cmc surface always contain an equator of the sphere. The papers [5] and [6] and the known examples of complete surfaces of cmc show that there is a drastic difference between the zero (minimal) and nonzero mean curvature case.

In our work we extend to immersed cmc surfaces a peculiar property presented by the Gauss image of a Delaunay surface, namely, that any connected component of the complement in the sphere of this image, or the image of a sufficiently big compact piece of the surface, is a convex domain (in fact a disk in Delaunay case).

1.1 THEOREM. *Let M be an immersed surface (without boundary but not necessarily complete) with nonzero cmc in $\mathbb{R}^3$ and let $N\colon M \to S^2(1)$ be its Gauss map.*

Received December 2, 1996.
1991 Mathematics Subject Classification. Primary 53A10; Secondary 53C42.

If $N(M)$ is a closed surface in $S^2(1)$ with C^2 boundary, then any connected component of $S^2(1)\backslash N(M)$ is a convex domain in $S^2(1)$.

A crucial hypothesis in Theorem 1.1 is the regularity of $\partial N(M)$. Although we do not know if this hypothesis is satisfied on an arbitrary complete cmc surface in $\mathbb{R}^3$, it is possible to prove the regularity of $\partial N(M)$ assuming additional hypothesis on the surface. We obtain:

1.2 THEOREM. *Let M be a compact surface with boundary contained in an open subset of a cmc surface $\overline{M}$ immersed in $\mathbb{R}^3$, and such that*

(*a*) $\partial M = \{p \in M \mid K(p) = 0\}$ (K: *Gaussian curvature*),
(*b*) $N|_{\partial M}$ *is injective*,
(*c*) $N(\overline{M}) = N(M)$.

Then, either $N(M) = S^2$, or any connected component of $S^2\backslash N(M)$ is a convex domain with regular boundary.

Remark. It is also proved in Theorem 1.1 that the geodesic curvature k_g of $\partial N(M)$, with respect to the normal vector pointing to the exterior of $N(M)$, satisfies $k_g \geq \inf\{|\nabla K(p)| \mid p \in M,\ K(p) = 0\}$. In the case of a surface of Delaunay, one has $k_g = |\nabla K(p)|$, where p is any point of the surface with $K(p) = 0$.

As a consequence of the above theorems, one has the following result which answers do Carmo's question in special cases.

1.3 COROLLARY. *Let M be an immersed surface satisfying the hypothesis of Theorem 1.1 or 1.2. Let us assume further that $\partial N(M)$ has at most two connected components. Then $N(M)$ contains an equator of the sphere.*

2. Proof of the results

The following facts will be used in the course of our proofs.

Let M be a complete surface in $\mathbb{R}^3$, with cmc $H = 1$, which is not a cylinder. Let $p \in M$ be a nonumbilic point. It follows from [3] that there exists a local conformal parametrization $X\colon U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$, with $p \in X(U)$, whose coordinate curves are curvature lines, and a C^∞ function $w\colon U \to \mathbb{R}$ such that:

(i) the induced first fundamental form g in U is given by

$$g = \frac{e^{2w}}{4}(du^2 + dv^2),\ w \in U,$$

where u and v are local parameters of X;

(ii) the second fundamental form A is given by

$$A = \frac{e^{2w} - 1}{4} du^2 + \frac{e^{2w} + 1}{4} dv^2$$

(iii) w satisfies the sinh-Gordon equation

$$\Delta w + \sinh w \cosh w = 0. \tag{1}$$

We assume that $X(U)$ is a bounded open subset of M. We have

$$\begin{aligned} X_{uu} &= w_u X_u - w_v X_v + eN \\ X_{vv} &= -w_u X_u + w_v X_v + gN \\ X_{uv} &= w_v X_u + w_u X_v \end{aligned} \tag{2}$$

and

$$\begin{aligned} N_u &= -(1 - e^{-2w}) X_u \\ N_v &= -(1 + e^{-2w}) X_v \\ K &= 1 - e^{-4w} \end{aligned} \tag{3}$$

It follows from (3) that $K = 0$ if and only if $w = 0$. We then observe that the subset of U where K vanishes is the nodal set of the nonzero function w, which satisfies the linear elliptic equation $\Delta w + (\sinh w \cosh w/w)w = 0$. We can therefore apply Theorem 2.5 of [2] to conclude that the set $T := K^{-1}\{0\} \cap \{p \in U \mid \nabla K(p) \neq 0\}$ is an embedded one dimensional submanifold of U which has a finite number of connected components. Furthermore the set $K^{-1}(0) \cap \{p \in U \mid \nabla K(p) = 0\}$ is finite. Since $\Delta w + \sinh w \cosh w = 0$ is a quasi linear analytic elliptic equation, it follows from [1] that w is a real analytic function in U and hence that T is a real analytic submanifold of U.

The following lemma proves that a curve in $S^2(1)$ which is locally strictly convex is globally strictly convex. The equivalent statement for the Euclidean plane has a very simple proof that doesn't seem to be the case for the sphere.

2.1 LEMMA. *Let C be a Jordan curve C^1 embedded in $S^2(1)$ which is the boundary of an open connected subset Q of $S^2(1)$. Given any point $p \in C$, assume that there is a neighbourhood A_p of p in $S^2(1)$ such that $(A_p \cap g_p) - \{p\} \subset S^2(1)\backslash\overline{Q}$, where g_p is the geodesic of $S^2(1)$ passing through p and tangent to C. Then $g_p \cap \overline{Q} = \{p\}$, for all $p \in C$. In particular the region Q is convex; that is, given points $p, q \in Q$, the minimal geodesic segment connecting p and q is contained in Q.*

Proof. By way of contradiction, let us suppose the existence of $p \in C$ such that the geodesic g of $S^2(1)$ passing through p and tangent to C intersects C in another point. We can assume that g intersects C in some other point whose distance to p is smaller than π. In fact: otherwise g intersects C in only one point q with $d(p,q) = \pi$, where d denotes the spherical distance. We then consider the middle point r in a subarc of C comprehended between p and q. Let R be the connected component of $\mathrm{int}(H) \cap (S^2(1)\backslash\overline{Q})$ whose boundary contains r, where H is the closed hemisphere of $S^2(1)$ having g as boundary and such that $(A_p \cap C) \subset H$. Let h be the geodesic tangent to C at r. Since h is locally contained in $\overline{R}$, h has at least more two intersections with ∂R, and one point of this intersection necessarily belongs to C, say s. Therefore, the geodesic h intersects C in two points r and s with $d(r,s) < \pi$, since $r \in \mathrm{int}(H)$ and $s \in H$.

Thus, let us consider $p \in C$ such that the geodesic g tangent to C at p intersects C in a point q such that $d(p,q) < \pi$. We inductively define a sequence of points $p_n, q_n \in C$, a sequence of geodesics g_n, a sequence of closed hemispheres H_n and a sequence of subarcs C_n of C as follows.

We set $p_1 = p$, $g_1 = g$, $H_1 = H$ as above, R_1 the connected component of $\mathrm{int}(H_1) \cap (S^2(1)\backslash\overline{Q})$ with $p_1 \in \partial R_1$ such that $\partial R_1 \cap C$ has the smallest length. Set $q_1 = g_1 \cap (\partial R_1 - \{p_1\})$ and $C_1 = C \cap \partial R_1$.

Let p_2 be the point of C_1 such that $l(p_1, p_2) = l(p_2, q_1)$, where l is the arc length of C_1, and let g_2 be the geodesic tangent to C_1 at p_2. Since, by hypothesis, g_2 is locally contained in R_1, it follows that g_2 has at least more two intersection points with ∂R_1. Since $d(p_1, q_1) < \pi$, g_2 intersects $g_1 \cap \overline{R_1}$ at most in one point and it follows that $C_1 \cap g_2$ has more than one point. Let H_2 be the closed hemisphere having g_2 as boundary and such that $(A_{p_2} \cap C) \subset H_2$. Let R_2 be the connected component of $\mathrm{int}(H_2) \cap (S^2(1)\backslash\overline{Q})$ with $p_2 \in \partial R_2$ such that $\partial R_2 \cap C$ has the smallest length. Set $q_2 = g_2 \cap (\partial R_2 - \{p_2\})$ and $C_2 = C \cap \partial R_2$. We observe that $q_2 \in C_1\backslash\{p_1, q_1\}$ since $\partial R_2 \cap C$ has the smallest length and $d(p_1, q_1) < \pi$.

Using the same reasoning as above, we can inductively define the middle point p_{n+1} of C_n, obtaining a sequence of subarcs $C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots$ of C, $C_{n+1} \subset (C_n\backslash\{p_n, q_n\})$, and regions $R_1 \supset R_2 \supset \ldots$ such that each C_n has p_n and q_n as ending points, $p_n \neq q_n$, and each region R_n is bounded by C_n and by the minimizing geodesic subarc g_n connecting p_n and q_n. By compactness, we can take a subsequence $\{p_{n_k}\}$ converging to a point $p_0 \in C_1$. We claim that $p_0 \in C_n\backslash\{p_n, q_n\}$ for all n. In fact, choose $n_0 \geq 1$. According to the previous construction, $p_n \in C_n \subset C_{n_0+1} \subset (C_{n_0}\backslash\{p_{n_0}, q_{n_0}\})$, for all $n \geq n_0 + 1$. Again, from the compactness of C_{n_0+1}, $p_0 \in C_{n_0+1}$, proving our claim.

Let g be the geodesic tangent to C at p_0 and assume that g is oriented. Let $q_0', q_0'' \in g\backslash\{p_0\}$ be such that q_0' is the first point of $g \cap (C_1\backslash\{p_1, q_1\})$ on the right side of p_0 in g and q_0'' is the first point of $g \cap (C_1\backslash\{p_1, q_1\})$ on the left side of p_0 in g. It is clear that at least one of the points q_0' or q_0'' exists.

Since $p_0 \in \overline{R_n}$ and g is locally contained in R_n, g has at least two more intersections with ∂R_n. Moreover, g can not intersect $g_n \cap \mathrm{int}(H_1)$ in more than one point, so that $g \cap (C_n\backslash\{p_n, q_n, p_0\}) \neq \emptyset$, for all n. Thus, we can assume that $q_0' \in g \cap (C_n\backslash\{p_n, q_n, p_0\})$, for every n big enough. Since $l(C_n) \longrightarrow 0$, it follows that $q_0' = p_0$, which is a contradiction! This proves the first part of Lemma 2.1. The remaining part is easy to prove, so that the proof of Lemma 2.1 is concluded.

In the next lemma, the point p_0 is a *fold* point for the Gauss map which, under the hypothesis of the lemma, is an *excellent map*, as defined by H. Whitney in [7]. Although it is a result that follows from Theorem 15A of [7], we present another proof specific to our case.

2.2 LEMMA. *Let M be an immersed cmc surface in $\mathbb{R}^3$ and let $\gamma \subset M$ be a regular curve of M such that $K(p) = 0$ for all $p \in \gamma$. Let us suppose that there is $p_0 \in \gamma$ such that $\nabla K(p_0) \neq 0$ and $N|_\gamma \colon \gamma \to S^2(1)$ is an immersion. Then, there is an open set V of M, $p_0 \in V$, such that $N(\gamma \cap V) \subset \partial N(V)$.*

Proof. Since p_0 is not umbilic, we can find a local parametrization $X\colon U \to M$ as in the beginning of Section 2. Let us assume that γ is parametrized (regularly) in a neighbourhood of p_0 by $\gamma(t) = (u(t), v(t))$, $t \in (-\epsilon, \epsilon)$, $\gamma(0) = p_0$. Since $N|_\gamma\colon \gamma \to S^2(1)$ is an immersion, we have $v'(t) \neq 0$, $t \in (-\epsilon, \epsilon)$. Moreover, since K cannot have the same sign in both sides of γ (otherwise ∇K would vanish in a open set of γ), we may suppose without loss of generality, that the orientation of γ is such that the points which are locally in the right side of γ have $K > 0$.

For a fixed t, and for s small enough, we define a curve $\beta_t(s) = (s+u(t), v(t))$ and set $\phi_t(s) = N(\beta_t(s))$, $f_t(s) = \langle \phi_t(s) - \phi_t(0), n(t)\rangle$, where $n(t) = v'(t)X_u(\gamma(t))$. Note that $n(t)$ is tangent to the sphere and orthogonal to $(N\circ\gamma)'(t) = -2v'(t)X_v(\gamma(t))$. We then have $f_t(0) = 0$ and $f_t'(0) = 0$. Moreover, since

$$f_t'(s) = -(1 - e^{-2w}) \langle X_u(\beta_t(s)), n(t)\rangle ,$$

we obtain

$$f_t''(0) = -2w_u v'(t) \langle X_u, X_u\rangle = -\frac{1}{2} w_u v'(t).$$

If $v'(t) > 0$, since the points on the right side of γ have $K > 0$ and ∇K points to the region with $K > 0$, we must have $w_u > 0$ (note that $\nabla K = 4\nabla w$ at γ).

If $v'(t) < 0$, the same reasoning shows that $w_u < 0$. In both cases, we must have $f_t''(0) < 0$, for all $t \in (-\epsilon, \epsilon)$, this implying that $f_t(s) < 0$, for some $\epsilon_t > 0$ and for all $s \in (-\epsilon_t, \epsilon_t)$. Taking the Taylor series of $f_t(s)$ with respect to s at $s = 0$, we have

$$\lim_{s\to 0} \frac{f_t(s)}{s^2} = -\frac{1}{4} w_u(\gamma(t))v'(t) \leq M < 0$$

for some $\delta_0 > 0$ and for all $t \in (-\delta_0, \delta_0)$. Hence, there is $\epsilon_0 > 0$ such that $f_t(s) < 0$, for all $t \in (-\delta_0, \delta_0)$ and for all $s \in (-\epsilon_0, \epsilon_0)$, $s \neq 0$. It follows that, for all these values of s and t, $\phi_t(s)$ is contained in the hemisphere of $S^2(1)$ having as boundary the geodesic of $S^2(1)$ passing through $N(\gamma(t))$ orthogonal to $n(t)$ and whose normal exterior vector is $n(t)$. Therefore,

$$V = \bigcup_{t\in(-\delta_0,\delta_0),\ s\in(-\epsilon_0,\epsilon_0)} \beta_t(s) = \bigcup_{t\in(-\delta_0,\delta_0),\ s\in(-\epsilon_0,\epsilon_0)} (s+u(t), v(t))$$

is an open set containing $\gamma(-\delta_0,\delta_0)$ such that $N(\gamma(-\delta_0,\delta_0)) \subset \partial N(V)$.

Remark. According to the notation of Lemma 2.2, if we parametrize γ in such a way that the region with $K > 0$ is locally at the right side of γ, then $v'(0)X_u(\gamma(0))$ points to the exterior of $N(V)$.

Proof of Theorem 1.1. Let us assume that M has mean curvature $H = 1$. We will prove that through any point of $\partial N(M)$ passes a geodesic of $S^2(1)$ whose intersection with some neighbourhood of this point in $S^2(1)$ is contained in $N(M)$. The theorem follows then from Lemma 2.1. Set

$$\Lambda = \{p \in M \mid \nabla K(p) \text{ is a multiple of a principal direction having nonzero principal curvature and } N(p) \in \partial N(M)\}.$$

Since $K(p) \neq 0$ implies that N is a diffeomorphism in a neighbourhood of p, we must have $\Lambda \subset \{K = 0\}$. Clearly, Λ is a closed subset of M. We will prove that Λ is discrete. Let $p \in \Lambda$. Since p is not umbilic, we can get a parametrization $X\colon U \to M$ as before, with $p \in X(U)$. We use the same notation as in beginning of Section 2.

Since $\nabla K|_{\{K=0\}} = 4(w_u, w_v)|_{\{K=0\}}$ is a multiple of the principal direction having nonzero principal curvature if and only if $w_u = 0$, it follows, by analyticity, that p is either isolated or there is an open subset E of $K = 0$ with $p \in E$ such that $w_u|_E = 0$. In the second case, we can further assume that $\nabla K(p') \neq 0$ if $p' \neq p$, $p' \in E$. We will prove that if the second alternative holds then $N(E\backslash\{p\}) \subset \operatorname{int}(M)$ if $\nabla K(p) = 0$, and $N(E) \subset \operatorname{int}(N(M))$ if $\nabla K(p) \neq 0$. This proves that Λ is discrete.

Let $p_0 = (u_0, v_0) \in E$ with $\nabla K(p_0) \neq 0$ and let $\gamma(t) = (u(t), v(t))$, $t \in (-\epsilon, \epsilon)$, be a regular parametrization of a neighbourhood of p_0 in E with $\gamma(0) = p_0$ and such that $\nabla K(\gamma(t)) \neq 0$, $t \in (-\epsilon, \epsilon)$. Since $w_u(\gamma(t)) = 0$ and $w(\gamma(t)) = 0$ we obtain $v'(t) = 0$, for all $t \in (-\epsilon, \epsilon)$. Given t, the curve $\beta_t(s) = N(u(t), v_0 + s)$ satisfies $\beta_t(0) = q_0 = N(p_0)$ and $\beta_t'(0) = -2X_v(u(t), v_0)$. If $q_0 \in \partial N(M)$, then $\beta_t'(0) = \lambda(t)\beta_0'(0)$, for some scalar $\lambda(t)$ (observe that the rank of dN_{p_0} is 1). But since $|\beta_t'(0)| = 2|X_v(u(t), v_0)| = 1$ it follows that $\lambda(t) = \text{const}$ which shows that $X_v(u(t), v_0)$ does not depend on t. Taking the derivative with respect to t and using (2), we obtain $w_v(u(t), v_0) = 0$, for all $t \in (-\epsilon, \epsilon)$, so that ∇K vanishes in an open set of E, contradiction! Therefore, it is proved that Λ is a closed discrete subset of U and $\partial N(M)\backslash N(\Lambda)$ is dense in $\partial N(M)$.

Let $q \in \partial N(M)\backslash N(\Lambda)$ and $p \in M\backslash\Lambda$ be such that $N(p) = q$. We can get a parametrization $X\colon U \to M$ as before with $p \in X(U)$ and a regular curve $\gamma(t) = (u(t), v(t))$, $t \in (-\epsilon, \epsilon)$, which is assumed to be parametrized by arc length (in the metric induced by X), such that $K(\gamma(t)) = 0$ for all $t \in (-\epsilon, \epsilon)$ and $\gamma(0) = p$. By choosing ϵ small enough, we can assume, since $v'(0) \neq 0$, that $v'(t) \neq 0$. Since $(N \circ \gamma)'(t) = -2v'(t)X_v(\gamma(t))$, $\alpha(t) := N(\gamma(t))$ is a regular curve.

In order to compute the geodesic curvature of α, we introduce a convention. Given an oriented 2-dimensional Riemannian manifold M, we denote by J the operator on the tangent bundle of M which rotates any vector of TM of an angle of $\pi/2$ in the positive orientation.

The geodesic curvature of $\alpha(t)$ in $S^2(1)$ in the $J(\alpha'(t))/\|J(\alpha'(t))\|$ direction is given by

$$k_g(\alpha(t)) = \frac{1}{\|\alpha'(t)\|^3}\left\langle\alpha''(t), J(\alpha'(t))\right\rangle. \tag{4}$$

Since $\alpha'(t) = -2v'(t)X_v$ and since $\{X_u, X_v\}$ is an oriented base of M we have $J(\alpha'(t)) = 2v'X_u$. Using (2), we have,

$$\begin{aligned}\alpha'' &= -2v''X_v - 2v'(u'X_{uv} + v'X_{vv})\\ &= -2v''X_v - 2v'(u'w_vX_u + u'w_uX_v - v'w_uX_u + v'w_vX_v + v'gN)\end{aligned}$$

so that

$$\left\langle\alpha'', J(\alpha')\right\rangle = 4(v')^2\left((u',v')\cdot(-w_v, w_u)\right)\langle X_u, X_u\rangle = (v')^2\left(\gamma'\cdot J(\nabla w)\right), \tag{5}$$

where '$\cdot$' denotes the usual inner product in the plane.

Now, let us analyse the sign of this last inner product. We first observe that since ∇K is not zero in $\gamma(0)$, there is a neighbourhood V of $\gamma(0)$ such that $V\backslash\gamma$ has a connected component, say V^+, in which $K > 0$. We choose the orientation of γ such that the points of V^+ are in the right side of γ for increasing values of the parameter t. Since ∇w points towards increasing values of K, it points to V^+.

Therefore, by definition of J, γ' and $J(\nabla w)$ are colinear and point to the same direction so that

$$\gamma'\cdot J(\nabla w) = \|\gamma'(t)\|\|J(\nabla w)\| = |\nabla K|$$

since $\|\gamma'\| = 2$ and $\|J(\nabla w)\| = (1/2)|\nabla K|$, where $\|\ \|$ is the Euclidean norm of the plane and $|\ |$ is the norm in the surface. From (4) and (5),

$$k_g(\alpha(t)) \geq |\nabla K(\gamma(t))| > 0. \tag{6}$$

It follows from the remark after Lemma 2.2, that $J(\alpha'(0)) = 2v'(0)X_u(\gamma(0))$ points to the exterior of $N(V)$ at $\alpha(0)$. Hence we have proved that given a point q of $\partial N(M)\backslash N(\Lambda)$, $q = N(p)$, the geodesic curvature of $\partial N(M)$ at q with respect to the exterior unit normal is positive and greater than or equal to $|\nabla K(p)| > 0$. By continuity, since $\partial N(M)$ is of class C^2, the geodesic curvature k_g of $\partial N(M)$ with respect to the unit exterior normal vector is everywhere nonnegative and satisfies

$$\begin{aligned} k_g &\geq \inf\{|\nabla K(p)| | p \in M, N(p) \in \partial N(M)\} \\ &\geq \inf\{|\nabla K(p)| | p \in M, K(p) = 0\}. \end{aligned}$$

To conclude the proof of Theorem 1.1, in view of Lemma 2.1, it is enough to prove that in each point q of $\partial N(M)$ there exists a neighbourhood W of q in $S^2(1)$ such that $W \cap g \subset N(M)$ and $W \cap g \cap \partial N(M) = \{q\}$, where g is the geodesic tangent to $\partial N(M)$ at q. To do that, let β: $(-\epsilon, \epsilon) \to S^2(1)$ be a C^2 parametrization by arc length of $\partial N(M)$ in a neighbourhood of q with $\beta(0) = q$. Let $n(s)$ be the exterior unit normal to $\partial N(M)$ at $\beta(s)$. We know that for all $s_0 \in (-\epsilon, \epsilon)$, $k_g(\beta(s_0)) \geq 0$ and there is a dense subset set of $(-\epsilon, \epsilon)$ where k_g $(\beta) > 0$. We take the Beltrami map whose domain is the hemisphere having q as pole, that is, B is the radial projection from this hemisphere over the tangent plane to $S^2(1)$ at q. It is easy to see that B preserves the sign of the geodesic curvature. Set $\lambda = B(\beta)$. Parametrizing λ by arc length and defining $f(s) = \langle \lambda(s), m(0) \rangle$, where $m(s)$ is the unitary normal vector to λ at $\lambda(s)$ and such that $m(0) = dB_q(n(0))$, we obtain $f(0) = f'(0) = 0$ and

$$f''(s) = k(s) \langle m(s), m(0) \rangle$$

where $k(s)$ is the plane curvature of λ with respect to m. Since $f''(s) \geq 0$, for all s in a neighbourhood of 0 and $f'' > 0$ in a dense subset of this neighbourhood, it follows that $f(s) > 0$ in a neighbourhood of 0, $s \neq 0$. This proves that λ is locally strictly convex at q and this implies that β is strictly locally convex at q. This concludes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let us suppose that $N(M) \neq S^2(1)$. Let C be a connected component of $\partial N(M)$. Since $\partial N(M) \subset N(\partial M)$, it follows, by the injectivity of $N|_{\partial M}$, that $C \subset N(\tilde{C})$ where $\tilde{C}$ is a connected component of ∂M, and simple topological arguments prove that in fact $C = N(\tilde{C})$. From Lemma 2.1, it is enough to prove that C is regular and at each point q of C, the geodesic tangent to C at q is locally contained in $N(M)$. Let $p \in \tilde{C}$ be such $N(p) = q$, and let X: $U \to \overline{M}$ be a local parametrization around p as before. Let $\gamma(t) = (u(t), v(t))$, $t \in (-\epsilon, \epsilon)$ be a regular analytic parametrization of $\tilde{C}$ in a neighbourhood of p, $\gamma(0) = p$ and set $\alpha(t) = N(\gamma(t))$. We have

$$\alpha'(t) = u'N_u + v'N_v = -2v'X_v$$

so that $N|_{\tilde{C}}$ is not an immersion at p if and only if $v'(0) = 0$.

If $v'(0) \neq 0$ then C is regular at q and, according to the proof of Theorem 1.1, the geodesic tangent to C at q is locally contained in $N(M)$.

If $v'(0) = 0$ then, from the analyticity of $v(t)$, $v'(t) \neq 0$ in a neighbourhood of $t = 0$ (otherwise $v' = 0$ in a neighbourhood of $t = 0$, implying that $\alpha(t) = q$, for all t in this neighbourhood, contradicting the injectivity of $N|_{\tilde{C}}$).

If $v'(t)$ doesn't change sign in a neighbourhood of $t = 0$, we can write $v(t) = at^n + O(t^{n+1})$, $a \neq 0$, n odd. Therefore, setting $t(s) = s^{1/n}$, it is easy to see that $\alpha(t(s))$ is a C^1 regular reparametrization of α which is C^∞at $s \neq 0$. Moreover, according to the proof of Theorem 1.1, the geodesic curvature of $\alpha(t(s))$ at $s \neq 0$ is positive (with respect to the exterior of $N(M)$). It follows from an argument similar to that used in the final part of the proof of Theorem 1.1, that the geodesic tangent to C at q is locally contained in $N(M)$ (although now, the curve $\alpha(t(s))$ is C^2 at $s \neq 0$ but just C^1 at $s = 0$).

Now we prove that the case where $v'(t) < 0$ for $t \in (-\epsilon, 0)$ and $v'(t) > 0$ for $t \in (0, \epsilon)$ can not occur. Let V be an open set containing q such that $V \backslash N(\gamma)$ has two connected components, V_1 and V_2. We also require, by choosing V small enough, that one of the connected component is contained in $N(M)$ and the other is disjoint to $N(M)$. Let H be the open hemisphere of $S^2(1)$ whose boundary is the geodesic tangent to $X_u(p)$ at q and whose exterior normal at q is $X_v(p)$. We observe that $N(\gamma(t)) \in H$ for $t \neq 0$, $t \approx 0$. In fact, setting $f(t) = \langle N(\gamma(t)) - q, X_v(p)\rangle$, we obtain $f'(t) = -2v'(t)\langle X_v(\gamma(t)), X_v(p)\rangle$ so that $f'(t) > 0$ for $t \in (-\epsilon, 0)$ and $f'(t) < 0$ for $t \in (0, \epsilon)$, that is, $f(t) < 0$, $t \in (-\epsilon, \epsilon) - \{0\}$. This proves our claim.

Let V_1 be the connected component of $V \backslash N(\gamma)$ which is contained in H, and let us consider the one-parameter family of functions

$$g_t(s) = \langle N(u(t), v(t) + s) - q, X_v(p)\rangle$$

We then have $g_t(0) = f(t) < 0$ if $t \neq 0$ and $= 0$ if $t = 0$. Also,

$$g_t'(0) = -(1 + e^{-2w(u(t),v(t))})\langle X_v(u(t), v(t)), X_v(p)\rangle = -2\langle X_v(u(t), v(t)), X_v(p)\rangle$$

so that we can take $\delta > 0$ such that $g_t'(0) < -1/4$ for all $|t| \leq \delta$. Since

$$\lim_{s \to 0} \frac{g_t(s) - g_t(0)}{s} = g_t'(0) < -\frac{1}{4},$$

there is $\lambda > 0$ such that if $0 \leq s \leq \lambda$ and $|t| \leq \delta$ then $g_t(s) < -(1/4)s + g_t(0) < 0$ if $(s, t) \neq (0, 0)$. Furthermore, we can assume that $g_t(s) \in V$ for $0 \leq s \leq \lambda$ and $|t| \leq \delta$.

Let $s\colon [-\delta, \delta] \to [0, \lambda]$ be a continuous function satisfying $s(-\delta) = s(\delta) = 0$ and $s(t) > 0$ for $t \in (-\delta, \delta)$. The curve $\beta(t) := N(u(t), v(t) + s(t))$ belongs to $\overline{V_1} \backslash \{q\}$ for $t \in (-\delta, \delta)$, since $\beta(-\delta) = N(\gamma(-\delta))$, $\beta(\delta) = N(\gamma(\delta))$, and $\beta([-\delta, \delta]) \subset H$ (note that $q \notin H$). Furthermore, at $t = 0$, one has

$$\lim_{s \to 0} \frac{g_0(s)}{s} = -\frac{1}{2}$$

so that $g_0(s) > 0$ for $s < 0$, $s \approx 0$, showing that the curve $N(u(0), v(0) + s)$ is contained in V_2 for these values of s. We therefore conclude that there exist points of $N(M)$ belonging to both V_1 and V_2, which is impossible since $N(\gamma)$ is at the boundary of $N(M)$. This concludes the proof of Theorem 1.2.

Proof of Corollary 1.3. If $\partial N(M)$ has just one connected component C, then it follows that C is contained in an hemisphere of $S^2(1)$ so that $N(M)$ contains an equator of the sphere. Suppose that $\partial N(M)$ contains two connected components C_1 and C_2.

We know that C_1 is entirely contained in an open hemisphere H of $S^2(1)$. Let p be the center of H and let p_0 be a point belonging to the connected component of $S^2(1)\backslash N(M))$ having C_1 as boundary. Let B_p: $H \to \pi$ be the Beltrami map at p; that is, π is the tangent plane to $S^2(1)$ at p and $B_p(x)$ is the radial projection of x over π. Given $t \in [1, \infty)$, set

$$C_t = B_p^{-1}(t(B_p(C_1) - B_p(p_0)) + B_p(p_0)).$$

If $C_t \cap C_2 = \emptyset$ for all $t \geq 1$ then $C_t \subset N(M)$ for all $t \geq 1$ so that the geodesic

$$g := \lim_{t\to\infty} C_t$$

is contained in $N(M)$. If $C_t \cap C_2 \neq \emptyset$ for some $t > 1$, set

$$t_0 := \inf\{t | C_t \cap C_2 \neq \emptyset\}$$

and choose $q \in C_{t_0} \cap C_2$. Let g be the geodesic of $S^2(1)$ passing through q and tangent to both C_{t_0} and C_2. From Lemma 2.1, g intersects C_2 only at q, and is locally contained in $N(M)$. Since the curves C_t are obtained by homotheties and the Beltrami map is a geodesic preserving map, the geodesic g intersects C_{t_0} only at q. It follows that g is entirely contained in $N(M)$, finishing the proof of the corollary.

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