ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 3, 1999

ON MODULATED ERGODIC THEOREMS FOR DUNFORD-SCHWARTZ OPERATORS

MICHAEL LIN, JAMES OLSEN AND ARKADY TEMPELMAN

ABSTRACT. We investigate sequences of complex numbers $\mathbf{a} = \{a_k\}$ for which the modulated averages $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ converge in norm for every weakly almost periodic linear operator T in a Banach space. For Dunford-Schwartz operators on probability spaces, we study also the a.e. convergence in L_p. The limit is identified in some special cases, in particular when T is a contraction in a Hilbert space, or when $\mathbf{a} = \{S^k\phi(\xi)\}$ for some positive Dunford-Schwartz operator S on a Lebesgue space and $\phi \in L_2$. We also obtain necessary and sufficient conditions on $\mathbf{a}$ for the norm convergence of the modulated averages for *every* mean ergodic power bounded T, and identify the limit.

1. Introduction and preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, and let T be a Dunford-Schwartz operator on $L_1(\mu)$ (i.e., a contraction of $L_1(\mu)$ which is also a contraction of $L_\infty(\mu)$). T is then also a contraction of each $L_p(\mu)$, $1 \le p \le \infty$, and the Dunford-Schwartz pointwise ergodic theorem yields a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} T^k f$ for every $f \in L_p(\mu)$, $1 \le p < \infty$. Convergence in L_p-norm, for $1 < p < \infty$, follows from the reflexivity of $L_p(\mu)$, and yields L_1-norm convergence for μ finite.

For fixed p, we will be interested in sequences $\mathbf{a} = \{a_k\}$ of complex numbers, which yield *modulated ergodic theorems*—convergence, for every Dunford-Schwartz operator T and every $f \in L_p$, either almost surely or in the mean, of the "modulated" averages of the form $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$. In case the limit

$$L(\mathbf{a}, T)f := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k T^k f$$

exists for every $f \in L_p(\mu)$, we would like to identify it.

If we want convergence of the modulated averages to hold at least for all rotations of the unit circle, we must have that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k \overline{\lambda}^k$ exists for all λ with $|\lambda| = 1$, and denote that limit by $c(\lambda)$. In that case, we say that the sequence $\mathbf{a} = \{a_k\}$ *has Fourier coefficients*, call $c(\cdot) = c(\cdot, \mathbf{a})$ the Fourier function of $\mathbf{a}$, and, following [K-2, p. 72], call $\mathbf{a}$ a *Hartman (almost-periodic) sequence*. The *spectrum* of a Hartman

Received July 15, 1998.
1991 Mathematics Subject Classification. Primary 28D05, 47A35.
The research of the first-named author was partially supported by the Israel Science Foundation.
The research of the second-named author was partially supported by ND EPSCoR through a grant from the National Science Foundation.

sequence $\mathbf{a} = \{a_k\}$ with Fourier function $c(\lambda)$ is the set $\sigma(\mathbf{a}) := \{\lambda\colon c(\lambda) \neq 0\}$. By [K-1], the spectrum is countable. A simple proof for bounded Hartman sequences, due to Boshernitzan, is given in [Ro, Theorem 41]. In fact, the proof applies (precisely) to Hartman sequences in the class W_2 defined below.

For $1 \leq p < \infty$, let W_p be the class of complex sequences $\mathbf{a} = \{a_k\}$ such that the seminorm $\|\mathbf{a}\|_{W_p}$, given by $(\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_k|^p)^{\frac{1}{p}}$, is finite. W_∞ denotes all bounded sequences. Clearly $W_{p_1} \subset W_{p_2}$ for $1 \leq p_2 < p_1 \leq \infty$, and positive Hartman sequences are in W_1. We denote by W_{1^+} the W_1-seminorm closure of $\cup_{p>1} W_p$. An adaptation of the proof given by Marcinkiewicz [M] for functions defined on $\mathbf{R}$ (see also [Le, Theorem 5.10.1]) shows that all the sequence spaces W_p are complete.

We look also at operators more general than Dunford-Schwartz operators, and the problem is, for a power-bounded operator T on a Banach space X and a sequence $\mathbf{a} = \{a_k\}$, to obtain the norm convergence

$$L(\mathbf{a}, T)x := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k T^k x \text{ exists for every } x \in X. \tag{1.1}$$

When $\mathbf{a} \in W_1$, we have $\sup_n \|\frac{1}{n}\sum_{k=1}^{n} a_k T^k\| < \infty$ for every T power bounded in a Banach space X, so the set of $x \in X$ for which $\frac{1}{n}\sum_{k=1}^{n} a_k T^k x$ converges is closed.

The next proposition follows from Lemma 1 of [T-1] (see also [ÇLO] for its second part).

PROPOSITION 1.1. *Let T be a power-bounded operator in X. If $\mathbf{a}^{(N)} \in W_1$ satisfies* (1.1) *for each N, and $\|\mathbf{a}^{(N)} - \mathbf{a}\|_{W_1} \to 0$, then also $\mathbf{a}$ (which is necessarily in W_1) satisfies* (1.1), *and* $\lim_N \|L(\mathbf{a}^{(N)}, T) - L(\mathbf{a}, T)\| = 0$.

By Proposition 1.1, the set of Hartman sequences in W_1 is a *closed* subspace of W_1.

Recall that T is called *weakly almost periodic* (WAP) if for every $x \in X$ the orbit $\{T^k x\}$ is weakly conditionally compact. Power-bounded operators on reflexive spaces are WAP. An important tool in the study of ergodic properties of WAP operators is the Jacobs-Deleeuw-Glicksberg decomposition [Kr, §2.4]:

$$X = [\text{closed lin. span}\{y\colon Ty = \lambda y,\ |\lambda| = 1\}] \oplus \{z\colon T^{n_j} z \to 0 \text{ weakly for some } \{n_j\}\}.$$

THEOREM 1.2. *Let $\mathbf{a} = \{a_k\}$ be a Hartman sequence. If $\mathbf{a} \in W_{1^+}$, then for every weakly almost periodic operator T on a Banach space X,*

$$L(\mathbf{a}, T)x := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k T^k x \text{ exists for every } x \in X.$$

Proof. If $Ty = \lambda y$ with $|\lambda| = 1$, then $L(\mathbf{a}, T)y$ exists, by the existence of Fourier coefficients for $\mathbf{a}$. On the space X_0 of the flight vectors of T (the vectors z with a subsequence of $\{T^n z\}$ converging weakly to 0), the limit in (1.1) exists and is zero for all W_p sequences ($p > 1$), by the proof of Theorem 4.1 of [ÇLO] – for that part the existence of the Fourier coefficients is not needed. Hence, for the flight vectors the limit exists and is zero also when $\mathbf{a} \in W_{1^+}$, by Proposition 1.1.

Thus, convergence holds on the linear manifold Y generated by X_0 and the eigenvectors $\{y\colon Ty = \lambda y,\ |\lambda| = 1\}$. Since Y is dense in X by weak almost periodicity, convergence holds on X (because $\mathbf{a} \in W_1$).

Remark. The W_p sequences which approximate $\mathbf{a} \in W_{1^+}$ need not be Hartman.

Example. A Hartman sequence in W_1 which is not in W_{1^+}.

We define $\mathbf{a} = \{a_k\}$ by $a_k = 0$ if k is not a square, and $a_{j^2} = j$. For $j^2 \le n < (j+1)^2$ we have $\frac{1}{n}\sum_{k=1}^n a_k = \frac{1}{n}\sum_{i=1}^j i \to \frac{1}{2}$, so $\mathbf{a} \in W_1$. Suppose $\mathbf{b} \in W_p$ with $p > 1$, and $\|\mathbf{b} - \mathbf{a}\|_{W_1} < \frac{1}{4}$. Then $\|\mathbf{b}\|_{W_1} > \frac{1}{4}$. We may assume that $b_k = 0$ for k not a square, as this will only improve the approximation. For $n = j^2$, using Hölder's inequality we have

$$\frac{1}{n}\sum_{i=1}^n |b_i| = \frac{1}{j^2}\sum_{k=1}^j |b_{k^2}| \le \frac{1}{j}\left(\frac{1}{j}\sum_{k=1}^j |b_{k^2}|^p\right)^{1/p} = \frac{1}{j^{1-1/p}}\left(\frac{1}{n}\sum_{i=1}^n |b_i|^p\right)^{1/p} \to 0.$$

This contradiction shows that $\mathbf{a} \notin W_{1^+}$. We now show that $\mathbf{a}$ is Hartman. Let λ have an irrational angle. By Weil's equidistribution theorem for the squares [KuN, p. 27], $\frac{1}{n}\sum_{k=1}^n \lambda^{k^2} \to 0$. Using Abel's summation by parts, we can prove that *if* $\frac{1}{n}\sum_{k=1}^n d_k \to 0$, *then* $\frac{1}{n^2}\sum_{k=1}^n k d_k \to 0$. Hence, for λ with irrational angle, our $\mathbf{a}$ satisfies $\frac{1}{n}\sum_{k=1}^n a_k\lambda^k = \frac{j^2}{n}\frac{1}{j^2}\sum_{k=1}^j k\lambda^{k^2} \to 0$, where $j = [\sqrt{n}]$. For λ a root of unity of order t, the convergence is shown by representing each $k = ts + r$ with $0 \le r < t$; we omit the computations.

Recall that T is called *almost periodic* if for every $x \in X$ the orbit $\{T^k x\}$ is conditionally compact (in the norm). In that case, $\|T^n x\| \to 0$ for every $x \in X_0$ (this property characterizes the almost periodic operators among the weakly almost periodic ones), and (1.1) holds for any Hartman sequence $\mathbf{a} \in W_1$.

PROPOSITION 1.3. *Let $\mathbf{a} = \{a_k\}$ be a sequence of complex numbers. If for every almost periodic operator T in a Banach space* $\sup_n \|\frac{1}{n}\sum_{k=1}^n a_k T^k\| < \infty$, *then* $\mathbf{a} \in W_1$.

Proof. Define T in c_0 (the space of sequences converging to 0) by the shift $T(\{x_k\}) = \{x_{k+1}\}$. Clearly T^n converges to 0 strongly, so T is almost periodic. For $\mathbf{a} \ne 0$, define $\mathbf{x}^{(n)} = (0, \operatorname{sign} a_1, \operatorname{sign} a_2, \dots, \operatorname{sign} a_n, 0, 0, \dots) \in c_0$ (where $\operatorname{sign} a = \bar{a}/|a|$ for $a \ne 0$ and $\operatorname{sign} 0 = 0$). Then $\|\mathbf{x}^{(n)}\| = 1$ (for n large enough), and the first coordinate of $\frac{1}{n}\sum_{k=1}^n a_k T^k \mathbf{x}^{(n)}$ is $\frac{1}{n}\sum_{k=1}^n |a_k|$. Hence

$$\sup_n \frac{1}{n}\sum_{k=1}^n |a_k| \le \sup_n \left\|\frac{1}{n}\sum_{k=1}^n a_k T^k \mathbf{x}^{(n)}\right\| \le \sup_n \left\|\frac{1}{n}\sum_{k=1}^n a_k T^k\right\| < \infty.$$

PROPOSITION 1.4. *A sequence* $\mathbf{a} = \{a_k\}$ *satisfies* (1.1) *for every almost periodic operator* T *in a Banach space* X *if and only if* $\mathbf{a}$ *is a Hartman sequence in* W_1.

Proof. Let $\mathbf{a}$ satisfy (1.1) for every almost periodic T. Then $\mathbf{a} \in W_1$ by the previous theorem, and it is Hartman because rotations on the unit circle $\Gamma := \{\lambda \in \mathbf{C}: |\lambda| = 1\}$ yield almost periodic operators on $C(\Gamma)$. The converse was observed above.

A complex-valued function $\mathbf{t}$ on the integers is called a *trigonometric polynomial* if there exist complex numbers $\lambda_1, \dots, \lambda_n$ with all $|\lambda_j| = 1$, and complex numbers $b_1, \dots, b_n$, such that $\mathbf{t}(k) = b_1\lambda_1^k + \cdots + b_n\lambda_n^k$ for all k. If $1 \leq p < \infty$ and $\mathbf{a}$ is in the W_p closure of the trigonometric polynomials, $\mathbf{a}$ is called *p-Besicovitch*. 1-Besicovitch sequences will be called just Besicovitch sequences, and every p-Besicovitch sequence is Besicovitch. Since trigonometric polynomials are bounded Hartman sequences, every Besicovitch sequence is in W_{1^+}, and Proposition 1.1 shows that it is a Hartman sequence, and also that it has countable spectrum. However, there are many bounded Hartman sequences which are not Besicovitch [K-2, p. 73]. The second part of the next proposition yields specific constructions.

PROPOSITION 1.5. *Let* θ *be an ergodic measure preserving transformation of a probability space* $(\Omega, \mathcal{F}, \mu)$. *If* $\phi \in L_p(\mu)$ *for* $1 \leq p \leq \infty$, *then for a.e.* ω, *the sequence* $\mathbf{a} = \{\phi(\theta^k\omega)\}$ *is a Hartman sequence, which is in* W_p. *Furthermore, if* θ *is weakly mixing, and* ϕ *is non-constant, then for a.e.* ω *the sequence* $\mathbf{a}$ *is a non-Besicovitch Hartman sequence.*

Proof. When $p < \infty$, the sequence $\{\phi(\theta^k\omega)\}$ is a.e. in W_p by the pointwise ergodic theorem applied to $|\phi|^p$. The Wiener-Wintner Theorem (e.g., [W]) implies that for almost every ω, the sequence $\mathbf{a} = \{\phi(\theta^k\omega)\}$ is a Hartman sequence. Its proof also shows that for a.e. ω, the Fourier coefficients of $\mathbf{a}$ are $E(\lambda)\phi(\omega)$, where $E(\lambda)$ is the ergodic projection of $L_p(\mu)$ on the eigenspace of λ, and thus the spectrum of $\mathbf{a}$ is $\{\lambda : E(\lambda)\phi \neq 0\}$. This shows that if θ is weakly mixing, the Fourier coefficients of $\mathbf{a}$, except at 1, are all zero (see also [BO, Theorem 5.2]), so the sequence is not Besicovitch.

Remarks. 1. The first part of the proposition is true without ergodicity, when the probability space is a Lebesgue space.

2. For $p = 1$, the sequences $\mathbf{a}$ obtained in the proposition are in fact in W_{1^+} (see details in the proof of Theorem 3.6 below).

3. Proposition 1.5 is true also for $\mathbf{a} = \{P^k\phi(\omega)\}$, when P is a transition probability operator for which μ is an ergodic invariant probability (this follows from the construction of the Markov shift — see [ÇLO]). When the space is a Lebesgue space, we can replace P by any Dunford-Schwartz operator on $L_1(\Omega, \mu)$ [ÇLO].

The following lemma, known for 2-Besicovitch sequences, is proved by standard approximations (using Hölder's inequality).

LEMMA 1.6. *Let $1 < p < \infty$ and $q = p/(p-1)$. If $\mathbf{a} \in W_p$ is a Hartman sequence and $\mathbf{b}$ is q-Besicovitch, then $\lim_n \frac{1}{n}\sum_{k=1}^n a_k b_k$ exists.*

It follows from Lemma 1.6 that $< \mathbf{a}, \mathbf{b} > := \lim_n \frac{1}{n}\sum_{k=1}^n a_k \bar{b}_k$ exists when $\mathbf{a}$ and $\mathbf{b}$ are 2-Besicovitch, and the completeness theorem yields that the equivalence classes of 2-Besicovitch sequences form a Hilbert space with inner product $< \mathbf{a}, \mathbf{b} >$, in which the sequences $\{\lambda^k\}$ ($|\lambda| = 1$) form an (uncountable) orthonormal basis. Thus, for a 2-Besicovitch sequence $\mathbf{a}$ with spectrum $\{\lambda_j\}$ we have $\|\mathbf{a}\|_2^2 = \sum_j |c(\lambda_j)|^2$, and it follows from the Riesz-Fisher theorem that for any sequence $\{\lambda_j\}$ with $|\lambda_j| = 1$ and $\{c_j\}$ with $\sum_j |c_j|^2 < \infty$ there is a 2-Besicovitch sequence $\mathbf{a}$ with spectrum $\{\lambda_j\}$ and $c(\lambda_j) = c_j$ (see [Bes, p. 110] for functions defined on $\mathbf{R}$).

2. Series representation of the limit

In this section we study the problem of identifying the limit in Theorem 1.2. We saw that for the flight vectors of a weakly almost periodic operator T the limit in (1.1) exists and is zero for any $\mathbf{a} \in W_{1^+}$. Thus, on the dense linear manifold Y generated by the space X_0 of flight vectors and the eigenvectors $\{y\colon Ty = \lambda y,\ |\lambda| = 1\}$ the value of the limit operator $L(\mathbf{a}, T)$ is known: the limit is 0 for the flight vectors, and obviously, for any Hartman sequence, $L(\mathbf{a}, T)(\sum_{j=1}^k y_j) = \sum_{j=1}^k c(\bar{\lambda}_j) y_j$ when $Ty_j = \lambda_j y_j$ with $|\lambda_j| = 1$.

For T power-bounded in X with $\bar{\lambda}T$ mean ergodic, $|\lambda| = 1$, the limit $E(\lambda, T)x = \lim_n \frac{1}{n}\sum_{k=1}^n \bar{\lambda}^k T^k x$ is the projection onto the eigenspace $X_\lambda = \{y\colon Ty = \lambda y\}$, along $\overline{(\lambda I - T)X}$. When T is understood, we write $E(\lambda)$ for $E(\lambda, T)$. It is immediate that if λT is mean ergodic for every $|\lambda| = 1$ (we call T *totally mean ergodic*), then for $\lambda_1 \neq \lambda_2$ we have $E(\lambda_1)E(\lambda_2) = 0$. For such a T, for any $x \in X$ the set of λ of unit modulus with $E(\lambda)x \neq 0$ is countable (since in the separable T-invariant subspace generated by $\{T^n x\}$, the restriction of T has at most countably many eigenvalues [Ja]).

THEOREM 2.1. *Let $\mathbf{a} = \{a_k\}$ be a Hartman sequence which is in W_1. Then (1.1) holds for any contraction T in a Hilbert space H, and we have*

$$L(\mathbf{a}, T)x = \sum_{|\lambda|=1} c(\lambda) E(\bar{\lambda}) x \tag{2.1}$$

(with countably many non-zero terms, and strong unconditional convergence of the series).

Proof. For any contraction T in H, the $E(\lambda, T)$ are orthogonal projections, by the mean ergodic theorem. Orthogonality of the eigenspaces yields directly that for fixed $x \in H$ only countably many $E(\lambda)x$ are non-zero. Since $|c(\lambda)| \leq \|\mathbf{a}\|_{W_1}$ for $|\lambda| = 1$, the orthogonal series on the right hand side of (2.1) converges in norm.

We first prove the theorem for U unitary. By the spectral theorem, for every $x \in H$ we have a (vector) measure σ_x on the unit circle such that

$$\frac{1}{n}\sum_{k=1}^{n} a_k U^k x = \int_{\{\lambda:|\lambda|=1\}} \frac{1}{n}\sum_{k=1}^{n} a_k \lambda^k d\sigma_x(\lambda).$$

Since $\sup_n |\frac{1}{n}\sum_{k=1}^{n} a_k\lambda^k| \leq \sup_n \frac{1}{n}\sum_{k=1}^{n} |a_k| < \infty$, the strong convergence follows from Lebesgue's bounded convergence theorem, with $L(\mathbf{a}, U)x = \int_{\{\lambda:|\lambda|=1\}} c(\bar{\lambda})d\sigma_x$. But $c(\lambda) = 0$ except for countably many values λ_j, so

$$L(\mathbf{a}, U)x = \sum_j c(\lambda_j)\sigma_x(\{\bar{\lambda}_j\}) = \sum_j c(\lambda_j)E(\bar{\lambda}_j)x.$$

Now let T be a contraction in H. By the dilation theorem, there exist a Hilbert space H_1 containing H and a unitary operator U in H_1, such that $T^k = PU^k$ for every k, where P is the orthogonal projection from H_1 onto H. Hence $\frac{1}{n}\sum_{k=1}^{n} a_k T^k x = P(\frac{1}{n}\sum_{k=1}^{n} a_k U^k x)$ converges in norm for every $x \in H$, by continuity of P, and (since clearly $E(\lambda, T) = P\,E(\lambda, U)$) we obtain

$$L(\mathbf{a}, T)x = P\,L(\mathbf{a}, U)x = \sum_j c(\lambda_j)P\,E(\bar{\lambda}_j, U)x = \sum_j c(\lambda_j)E(\bar{\lambda}_j, T)x.$$

Remarks. 1. For 2-Besicovitch sequences, the theorem was proved in [T-1] for T unitary (in the context of unitary representations of LCA groups). It was extended to contractions in a Hilbert space in [O] (still for 2-Besicovitch sequences), but without mentioning that $P\,E(\lambda, U) = E(\lambda, T)$.

2. Even for a contraction in a Hilbert space, the method of [ÇLO] yields (1.1) and (2.1) only for a Hartman sequence $\mathbf{a}$ in W_{1+}. In the more general case of $\mathbf{a} \in W_1$, it is not clear how to use that method to prove convergence to 0 on the space H_0 of flight vectors.

3. The special case of the theorem obtained in [T-1] was applied in [T-2] to the consistency of least square estimators in linear regression models with 2-Besicovitch regressors. Applications of Theorem 2.1 (and of Theorem 2.7 below) to more general regressors will appear elsewhere.

For a Hartman sequence $\mathbf{a}$ in W_{1+} and T weakly almost periodic in X, we saw that the limit $L(\mathbf{a}, T)x$ equals the right-hand side of (2.1) for x in the dense linear manifold Y generated by X_0 and $\{y\colon Ty = \lambda y,\ |\lambda| = 1\}$. Thus, by continuity of the limit operator (which is defined on all the space by Theorem 1.2), the problem of the identification of the limit is reduced to proving the convergence of the right hand-side of (2.1) for *every* $x \in X$.

THEOREM 2.2. *Let* $\mathbf{a} = \{a_k\}$ *be a 2-Besicovitch sequence with spectrum* $\{\lambda_j\}$. *Let* T *be a totally mean ergodic power-bounded operator in a Banach space* X. *Then*

$\sum_{j=1}^{\infty} c(\lambda_j)E(\overline{\lambda}_j)$ *converges in operator norm, and for every* $x \in X$,

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n} a_k T^k x - \sum_{j=1}^{\infty} c(\lambda_j)E(\overline{\lambda}_j)x\right\| = 0.$$

Proof. Since $\{a_k\}$ is 2-Besicovitch, $\sum_{j=1}^{\infty}|c(\lambda_j)|^2 < \infty$. Define $\mathbf{a}^{(N)} = \{a_k^{(N)}\}_{k\geq 1}$ by $a_k^{(N)} = \sum_{j=1}^{N} c(\lambda_j)\lambda_j^k$. Then

$$\|\mathbf{a}-\mathbf{a}^{(N)}\|_{W_1}^2 \leq \|\mathbf{a}-\mathbf{a}^{(N)}\|_{W_2}^2 = \sum_{j=N+1}^{\infty} |c(\lambda_j)|^2 \to_{N\to\infty} 0.$$

Since T is totally mean ergodic, for every $x \in X$ we have

$$\begin{aligned} L(\mathbf{a}^{(N)}, T)x &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k^{(N)} T^k x \\ &= \lim_{n\to\infty}\sum_{j=1}^{N} c(\lambda_j)\left(\frac{1}{n}\sum_{k=1}^{n}\lambda_j^k T^k x\right) = \sum_{j=1}^{N} c(\lambda_j)E(\overline{\lambda}_j)x\,. \end{aligned}$$

Proposition 1.1 now yields both assertions of the theorem.

Remarks. 1. Any weakly almost periodic operator is power-bounded and totally mean ergodic. The identification of the limit for T WAP follows from (the ideas of) [LO].

2. By Proposition 1.1 (see also [ÇLO]), the convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k x$ holds even for a 1-Besicovitch sequence $\mathbf{a}$. The difficulty in obtaining the identification of the limit for every x is in proving that $\lim_{n\to\infty}\|\mathbf{a}-\mathbf{a}^{(N)}\|_{W_1} = 0$ for the $\{\mathbf{a}^{(N)}\}$ defined in the previous proof. It is an interesting open problem at the moment whether we have this latter convergence even for p-Besicovitch sequences, $1 < p < 2$.

3. Any contraction T in L_1 with mean ergodic (ME) modulus is totally ME, by [ÇL], since the modulus of λT is that of T.

Example. A totally mean ergodic Markov operator on $C(K)$ *which is not WAP.* The example was provided by I. Kornfeld. Let θ be a uniquely ergodic minimal homeomorphism of a compact metric space K, with invariant probability μ, such that θ is weakly mixing in $L_2(\mu)$ (such homeomorphisms exist—by Jewett's theorem, every weakly mixing probability preserving invertible transformation has a topological model with such a θ). Define P on $C(K)$ by $Pf = f\circ\theta$. Since θ is uniquely ergodic, P is ME. Since θ is weakly mixing, mean ergodicity of λP, for every $\lambda \neq 1$ with $|\lambda| = 1$, follows from [As-2]; see [R] (and also [W]). Thus, P is totally mean ergodic. P is irreducible, by minimality of θ. If P were weakly almost periodic, it would be almost periodic [Kr, p. 182], and by weak mixing we would then have $\|P^n f\|_\infty \to 0$ for any $f \in C(K)$ with $\int f\,d\mu = 0$, contradicting the invertibility of P. Hence P is not WAP.

THEOREM 2.3. *Let $\{c_j\}$ be a sequence of complex numbers with $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, and let T be a totally mean ergodic power-bounded operator in a Banach space X. Then for any sequence (of unimodular eigenvalues) $\{\lambda_j\}$, the series $\sum_{j=1}^{\infty} c_j E(\bar{\lambda}_j)x$ converges strongly for every $x \in X$.*

Proof. Fix the operator T and the sequence $\{\lambda_j\}$. Since $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ by assumption, by the Riesz-Fisher theorem (see §1) there exists a 2-Besicovitch sequence $\mathbf{b}$ with spectrum $\{\lambda_j\}$ and Fourier function $c(\lambda)$, such that $c(\lambda_j) = c_j$ for every j. By the previous theorem we have the required convergence.

Remarks. 1. The stronger condition $\sum_{j=1}^{\infty} |c_j| < \infty$ yields the theorem trivially.
2. For a contraction T in a Hilbert space, the orthogonality of $\{E(\lambda_j)\}$ yields the convergence of $\sum_{j=1}^{\infty} c_j E(\bar{\lambda}_j)x$ for any bounded sequence $\{c_j\}$.

THEOREM 2.4. *Let $\mathbf{a} = \{a_k\} \in W_{1^+}$ be a Hartman sequence. If $\sum_{\lambda_j \in \sigma(\mathbf{a})} |c(\lambda_j)|^2 < \infty$, then for any weakly almost periodic operator T in a Banach space X and $x \in X$, we have*

$$L(\mathbf{a}, T)x = \sum_{\lambda_j \in \sigma(\mathbf{a})} c(\lambda_j) E(\bar{\lambda}_j)x$$

(with countably many non-zero terms, and unconditional strong convergence of the series).

Proof. The series converges to $L(\mathbf{a}, T)x$ for x in a dense subspace. The spectrum of $\mathbf{a}$ is countable, and let $\{\lambda_j\}$ be an enumeration. Since $\sum_{j=1}^{\infty} |c(\lambda_j)|^2 < \infty$ by assumption, the previous theorem yields that the series converges for every x, which is equivalent to the claimed equality.

Remarks. 1. The condition $\sum_{j=1}^{\infty} |c(\lambda_j)|^2 < \infty$ does not imply boundedness of $\{a_k\}$. For example, let $a_k = 1$ if $k \neq 2^n$, $a_{2^n} = n + 1$. Then the spectrum consists only of $\lambda = 1$.
2. Sequences $\mathbf{a}$ obtained from ergodic probability preserving transformations as in Proposition 1.5, with $\phi \in L_2$, satisfy the hypotheses of Theorem 2.4. A more general case is treated in Theorem 3.15 in the next section.

If T is an operator in an L_p space, we can ask also for a.e. convergence on the right-hand side of (2.1). This question seems to be independent of the question of a.e. convergence in (1.1) (treated in the next section; see [ÇLO], where earlier references are given).

THEOREM 2.5. *Let T be a contraction of $L_2(\Omega, \mu)$. If $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, then for every sequence $\{\lambda_j\}$ of unimodular complex numbers and for every $f \in L_2$, the series $\sum_{j=1}^{\infty} c_j E(\bar{\lambda}_j) f$ is absolutely convergent a.e., and also converges unconditionally in L_2-norm.*

Proof. By the definitions, $\lambda_j T(E(\bar{\lambda}_j)f) = E(\bar{\lambda}_j)f$. Hence the sequence $\{E(\bar{\lambda}_j)f\}$ is orthogonal in L_2. Each $E(\bar{\lambda}_j)$ is an orthogonal projection, and $\{E(\bar{\lambda}_j)L_2\}$ are orthogonal subspaces. Hence for every n,

$$\|f\|^2 \geq \left\|\sum_{j=1}^{n} E(\bar{\lambda}_j)f\right\|^2 = \sum_{j=1}^{n} \|E(\bar{\lambda}_j)f\|^2,$$

so $\int \sum_{j=1}^{n} |E(\bar{\lambda}_j)f|^2\,d\mu \leq \|f\|^2$. Hence by Lebesgue's theorem $\sum_{j=1}^{\infty} |E(\bar{\lambda}_j)f|^2(w)\} < \infty$ a.e. By the Cauchy Schwarz inequality,

$$\left[\sum_{j=1}^{n} |c_j E(\bar{\lambda}_j)f(w)|\right]^2 \leq \left(\sum_{j=1}^{n} |c_j|^2\right)\left(\sum_{j=1}^{n} |E(\bar{\lambda}_j)f(w)|^2\right).$$

Hence the series is a.e. absolutely convergent, since $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ by assumption.

Since $\lim_j c_j = 0$, $\sum_{j=1}^{\infty} \|c_j E(\bar{\lambda}_j)f\|^2 \leq \|f\|^2 \max_j |c_j|^2$, and the orthogonality yields the unconditional norm convergence of $\sum_{j=1}^{\infty} c_j E(\bar{\lambda}_j)f$.

A function $f \in L_p$ is called an L_p-flight vector ($1 \leq p < \infty$) for a Dunford-Schwartz operator T in a probability space, if there is a subsequence such that $T^{k_j} f$ converges to zero weakly in L_p. The last part of the following proposition is probably known (its first part is standard), but we have no reference for it.

PROPOSITION 2.6. *Let T be a Dunford-Schwartz operator in a probability space. Then T is weakly almost periodic in L_1, every L_p-flight vector f is a flight vector in L_1, and the set of L_2-flight vectors is dense in the set of L_1-flight vectors.*

Proof. For any contraction in a Banach space, standard approximation arguments show that the set of vectors with weakly sequentially compact orbits is closed. For $f \in L_2$ the sequence $\{T^n f\}$ is weakly sequentially compact in L_2, hence, since $L_\infty \subset L_2 \subset L_1$, it is also weakly sequentially compact in L_1. Hence T is weakly almost periodic.

The Jacobs-Deleeuw-Glicksberg decomposition of T in L_1 yields a bounded projection E_0 on the space of L_1-flight vectors. The same decomposition in L_p (and its uniqueness) yields that if $f \in L_p$, then $E_0 f \in L_p$. Now let $f \in L_1$ be a flight vector, and let $f_i \in L_2$ converge to f in L_1. Then $f = E_0 f = \lim_i E_0 f_i$.

THEOREM 2.7. *Let $\mathbf{a} = \{a_k\}$ be a Hartman sequence which is in W_1. Then for every Dunford-Schwartz operator T in a probability space and every $f \in L_p$, $1 \leq p < \infty$, $L(\mathbf{a}, T)f := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k T^k f$ exists in L_p-norm for every $f \in L_p$, and $L(\mathbf{a}, T)f = 0$ for every L_1-flight vector f. For $f \in L_2$ we have (with L_2 unconditional convergence of the series)*

$$L(\mathbf{a}, T)f = \sum_{\lambda_j \in \sigma(\mathbf{a})} c(\lambda_j) E(\bar{\lambda}_j) f \tag{2.2}$$

Furthermore, if the Fourier function of $\mathbf{a}$ *satisfies* $\sum_{\lambda_j \in \sigma(\mathbf{a})} |c(\lambda_j)|^2 < \infty$, *then* (2.2) *holds for every* $f \in L_p$, *with* L_p*-norm unconditional convergence of the series, and for* $f \in L_2$, *the series converges also a.e.*

Proof. T is a contraction of L_2, so by Theorem 2.1 $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k T^k f$ exists in L_2 norm for every $f \in L_2$. This implies convergence in L_p-norm, for $1 \le p < 2$ and $f \in L_2$. Since the sequence of operators $\{\frac{1}{n} \sum_{k=1}^{n} a_k T^k\}$ is bounded in norm, we have the L_p convergence for every $f \in L_p$ when $1 \le p < 2$.

Now let $p = 2 + \alpha$. Fix $f \in L_\infty$, and put $g = L(\mathbf{a}, T) f$. Then $g \in L_\infty$, and

$$\begin{aligned}
\left\| g - \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right\|_p^p &= \int \left| g - \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right|^p d\mu \\
&= \int \left| g - \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right|^2 \left| g - \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right|^\alpha d\mu \\
&\le \left\| g - \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right\|_2^2 \left[\|g\|_\infty + \sup_n \frac{1}{n} \sum_{k=1}^{n} |a_k| \|f\|_\infty \right]^\alpha \\
&\to_{n\to\infty} 0.
\end{aligned}$$

Hence we have convergence in L_p norm for bounded functions, and therefore, as before, for every $f \in L_p$.

The limit operator $L := L(\mathbf{a}, T)$ on L_1 is bounded, and $Lf = 0$ for any L_2-flight vector f. Since the L_2-flight vectors are dense in the L_1-flight vectors, $Lf = 0$ for every flight vector $f \in L_1$. Since T is a contraction in L_2, (2.2) holds for $f \in L_2$ by Theorem 2.1.

Assume now $\sum_{j=1}^{\infty} |c(\lambda_j)|^2 < \infty$. For $1 \le p < \infty$, the right hand side of (2.2) converges in L_p by Theorem 2.3, and equals $L(\mathbf{a}, T)$ on the dense subspace L_∞, so (2.2) holds.

Finally, the a.e. convergence of the series, for $f \in L_2$, follows from Theorem 2.5.

COROLLARY 2.8. *Let* T *be a Dunford-Schwartz operator in a probability space, and let* $\mathbf{a} = \{a_k\}$ *be* 2*-Besicovitch with spectrum* $\{\lambda_j\}$. *Then for every* $f \in L_2$, *the series* $\sum_{j=1}^{\infty} c(\lambda_j) E(\overline{\lambda}_j) f$ *converges a.e.* (*and in* L_2) *to* $L(\mathbf{a}, T) f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k T^k f$.

Problems. 1. Under the assumptions of the corollary, is the a.e. convergence of the series in (2.2) valid for $f \in L_1$, or at least for $f \in L_p$, $p > 1$? (Theorem 2.7 yields L_p-norm convergence).

2. If T is a Dunford-Schwartz contraction in a probability space, for which functions can we obtain a.e. or norm convergence of the series $\sum_{j=1}^{\infty} c_j E(\overline{\lambda}_j) f$, when $\sum_{j=1}^{\infty} |c_j|^p < \infty$ for some $p > 1$? Of course, the norm convergence problem is only for $p > 2$.

For a positive Dunford-Schwartz operator τ in a probability space (i.e., a Markov operator with invariant probability μ), $E(1,\tau)f$ is $E_\mu(f \mid \mathcal{I})$—the conditional expectation of f with respect to the σ-algebra $\mathcal{I}$ of τ-invariant sets [Kr, p. 129]. M. Akcoglu has noted that for a Dunford-Schwartz operator T in a probability space, the structure of L_1-contractions given in [ABr] (see also [Kr, p. 163]), and the fact that T and λT have the same linear modulus, yield the following representation of the projections $E(\lambda, T)$.

THEOREM 2.9. *Let T be a Dunford-Schwartz operator in a probability space $(\Omega, \mathcal{F}, \mu)$ with linear modulus τ, such that $\tau 1 = 1$, and let $\mathcal{I}$ be the σ-algebra of τ-invariant sets. Then for every λ with $|\lambda| = 1$ there exist a set $A_\lambda \in \mathcal{I}$ and a complex-valued $\mathcal{I}$-measurable function h_λ, with $|h_\lambda| = 1_{A_\lambda}$, such that*

$$E(\lambda, T)f = \overline{h}_\lambda E(1,\tau)(h_\lambda f) = \overline{h}_\lambda E_\mu(h_\lambda f \mid \mathcal{I}) \quad \forall f \in L_1.$$

Remark. If, in the above theorem, $\tau 1 \le 1$, then both the conservative and dissipative parts of τ are absorbing. Thus, Theorem 2.9 applies to the restriction of T to the conservative part of τ. On its dissipative part D, τ has no fixed points, so $1_D E(\lambda, T)f = 0$ a.e. for every $f \in L_1$.

THEOREM 2.10. *Let T be a totally mean ergodic power-bounded operator on $L_1(\Omega, \mu)$. If $\sum_{j=1}^\infty |c_j| < \infty$, then for every sequence $\{\lambda_j\}$ of unimodular complex numbers and for every $f \in L_1$, the series $\sum_{j=1}^\infty c_j E(\overline{\lambda}_j) f$ is absolutely convergent a.e., and in L_1.*

Proof. Define $M := \sup_k \|T^k\|$. Clearly $\|E(\lambda)\| \le M$ for $|\lambda| = 1$. For every n we have

$$\int \sum_{j=1}^n |c_j E(\lambda_j) f| \, d\mu = \sum_{j=1}^n |c_j| \, \|E(\lambda_j) f\|_1 \le M \|f\|_1 \sum_{j=1}^\infty |c_j|.$$

Hence we have the L_1 absolute convergence, and by Lebesgue's theorem also the a.e. absolute convergence.

Remark. If T is a contraction of $L_1(\mu)$ with mean ergodic linear modulus, then it is totally mean ergodic [ÇL].

3. Pointwise modulated ergodic theorems for Dunford-Schwartz contractions

Pointwise modulated ergodic theorems in L_p $(p > 1)$, for LCA group actions, were first obtained by Tempelman [T-1], with modulation by (not necessarily bounded) q-Besicovitch sequences (where $q = p/(p-1)$ is the dual index). For T induced by a measure-preserving transformation, Ryll-Nardzewski [RN] obtained (independently of [T-1]) pointwise convergence for L_1 functions, with modulation by *bounded Besicovitch* sequences (denoted ∞-Besicovitch; these are in fact p-Besicovitch for every $1 \le p < \infty$ [BeLo],[JO],[LO]). Baxter and Olsen [BO] showed that a bounded sequence which modulates pointwise all L_1 functions for any measure preserving transformation also modulates for all Dunford-Schwartz operators. Most of the subsequent research was for L_1 functions with modulation by bounded sequences (see [ÇLO] for aditional references). The celebrated "return times theorem" [BoFKaOr] shows that the sequences generated by measure preserving transformations (as in Proposition 1.5) are pointwise modulating sequences.

B. Weiss (oral communication) has noted that there are bounded Hartman sequences $\{a_k\}$ for which $\{\frac{1}{n}\sum_{k=1}^n a_k f \circ \theta^k\}$ may fail to converge a.e. for some probability preserving θ and $f \in L_\infty$: Thouvenot and Weiss (unpublished) have constructed a weakly mixing shift invariant probability on $\mathcal{X} := \{1, 2, 3, 4\}^{\mathbf{N}}$ and a corresponding generic point $\mathbf{a}$, which is a Hartman sequence by [OrWe, pp. 120–121], with that property.

PROPOSITION 3.1. *Let T be a Dunford-Schwartz operator in a probability space (Ω, μ). Fix p, $1 \le p < \infty$, and let $q = \frac{p}{p-1}$. If $\mathbf{a} = \{a_k\}$ is a sequence in W_q (with $W_\infty = \ell_\infty$), such that $\frac{1}{n}\sum_{k=1}^n a_k T^k f$ converges a.e. for every $f \in L_\infty(\mu)$, then $\frac{1}{n}\sum_{k=1}^n a_k T^k f$ converges a.e. for every $f \in L_p(\mu)$.*

Proof. We first assume $1 < p < \infty$, and fix $\{a_k\}$ as in the statement of the theorem. For $f \in L_p(\Omega, \mu)$, $\frac{1}{n}\sum_{k=1}^n \tau^k(|f|^p)$ converges a.e. by the Dunford-Schwartz theorem, where τ is the linear modulus of T. By Hölder's inequality, for a.e. w we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^n a_k T^k f(w)\right| &\le \left(\frac{1}{n}\sum_1^n |a_k|^q\right)^{\frac{1}{q}} \left(\frac{1}{n}\sum_1^n |T^k f(w)|^p\right)^{\frac{1}{p}} \\ &\le \left(\frac{1}{n}\sum_{k=1}^n |a_k|^q\right)^{\frac{1}{q}} \left(\frac{1}{n}\sum_{k=1}^n [\tau^k|f|(w)]^p\right)^{\frac{1}{p}} \\ &\le \left(\frac{1}{n}\sum_{k=1}^n |a_k|^q\right)^{\frac{1}{q}} \left(\frac{1}{n}\sum_{k=1}^n \tau^k(|f|^p)(w)\right)^{\frac{1}{p}}, \qquad (3.1)\end{aligned}$$

with the last inequality by $|\tau f|^p \le \tau(|f|^p)$ a.e. [Kr, p. 65, Lemma 7.4]. Hence, for every $f \in L_p$ we have

$$\sup_n \left|\frac{1}{n}\sum_{k=1}^n a_k T^k f(w)\right| < \infty \text{ a.e.}$$

By assumption, $\frac{1}{n}\sum_{k=1}^n a_k T^k f(w)$ converges a.e. for every $f \in L_\infty(\Omega, \mu)$, so the Banach principle now yields a.e. convergence for all $f \in L_p$.

Now let $p = 1$, so $q = \infty$. Then for a.e. w we have

$$\sup_n \left| \frac{1}{n} \sum_{k=1}^{n} a_k T^k f(w) \right| \le \|\mathbf{a}\|_\infty \sup_n \frac{1}{n} \sum_{k=1}^{n} \tau^k |f|(w) < \infty,$$

and convergence again follows from the Banach principle.

PROPOSITION 3.2. *Let T be a contraction of $L_\infty(\mu)$, let $\mathbf{b}^{(N)} = \{b_k^{(N)}\}$ be sequences in W_1 such that for every N, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} b_k^{(N)} T^k f$ exists a.e. for every $f \in L_\infty$, and let $\mathbf{a} = \{a_k\}$ with $\lim_{N\to\infty} \|\mathbf{b}^{(N)} - \mathbf{a}\|_{W_1} = 0$. Then for every $f \in L_\infty$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k T^k f$ exists a.e., and the limit operator $L(\mathbf{a}, T)f$ is bounded on L_∞.*

Proof. Fix $f \in L_\infty$. We prove the convergence of $\frac{1}{n} \sum_{k=1}^{n} a_k T^k f$ by the method of [JO] (which applies a.e. the inequalities obtained in [T-1] for norms): for a.e. w,

$$\begin{aligned}
&\left| \frac{1}{n} \sum_{k=1}^{n} a_k T^k f(w) - \frac{1}{m} \sum_{k=1}^{m} a_k T^k f(w) \right| \\
&\qquad \le \left| \frac{1}{n} \sum_{k=1}^{n} (a_k - b_k^{(N)}) T^k f(w) \right| \\
&\qquad\quad + \left| \frac{1}{n} \sum_{k=1}^{n} b_k^{(N)} T^k f(w) - \frac{1}{m} \sum_{k=1}^{m} b_k^{(N)} T^k f(w) \right| \\
&\qquad\quad + \left| \frac{1}{m} \sum_{k=1}^{m} (a_k - b_k^{(N)}) T^k f(w) \right| \\
&\qquad \le \frac{1}{n} \sum_{k=1}^{n} |a_k - b_k^{(N)}| \|f\|_\infty + \frac{1}{m} \sum_{k=1}^{m} |a_k - b_k^{(N)}| \|f\|_\infty \\
&\qquad\quad + \left| \frac{1}{n} \sum_{k=1}^{m} b_k^{(N)} T^k f(w) - \frac{1}{m} \sum_{k=1}^{m} b_k^{(N)} T^k f(w) \right| .
\end{aligned}$$

Hence $\left\{ \frac{1}{n} \sum_{k=1}^{n} a_k T^k f(w) \right\}$ is a Cauchy sequence, and hence converges, for a.e. w. Denote this limit by $Lf(w)$. Clearly, $\|L\|_\infty \le \|\mathbf{a}\|_{W_1}$, since we have a.e.

$$|Lf(w)| \le \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |a_k| \|f\|_\infty .$$

THEOREM 3.3. *Let T be a Dunford-Schwartz contraction in a probability space, and let $\mathbf{a} = \{a_k\}$ be a 1-Besicovitch sequence. Then for every $f \in L_\infty$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k T^k f$ exists a.e., and the limit operator Lf is bounded on L_∞.*

Proof. Let $\mathbf{b}^{(N)} = \{b_k^{(N)}\}$ be trigonometric polynomials with

$$\lim_{N\to\infty} \|\mathbf{b}^{(N)} - \mathbf{a}\|_{W_1} = 0.$$

By the Dunford-Schwartz theorem (for each λT), $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n b_k^{(N)} T^k f$ exists a.e. (and equals $L_N f = \sum_{j=1}^{J_N} c(\lambda_j^{(N)}) E(\overline{\lambda}_j^{(N)}) f$, where $\{\lambda_j^{(N)} : 1 \le j \le J_N\}$ is the spectrum of $\mathbf{b}^{(N)}$). The Theorem now follows from the previous proposition.

Remark. L_1-norm convergence holds for every $f \in L_1$ [ÇLO].

COROLLARY 3.4. *If in the above theorem, $\{a_k\}$ is 2-Besicovitch with spectrum $\{\lambda_j\}$, then*

$$\left\| Lf - \sum_{j=1}^{n} c(\lambda_j) E(\overline{\lambda}_j) f \right\|_\infty \to_{n\to\infty} 0$$

for every $f \in L_\infty$.

Proof. For the approximating sequence $\mathbf{b}^{(N)}$ in the proof, take $\mathbf{a}^{(N)}$ of Theorem 2.2.

THEOREM 3.5. *Fix p, $1 \le p \le \infty$, and let $q = \frac{p}{p-1}$. If $\{a_k\}$ is q-Besicovitch, then for every Dunford-Schwartz operator T in a probability space (Ω, μ) and every $f \in L_p(\mu)$, $\frac{1}{n}\sum_{k=1}^n a_k T^k f$ converges a.e.*

Proof. For $p = \infty$, $q = 1$ and this is Theorem 3.3. For $p < \infty$, combine Proposition 3.1 and Theorem 3.3.

Remarks. 1. In fact, we have a weak maximal inequality in L_p, $(1 \le p < \infty)$:

$$\mu\left\{w\colon \sup_n \left|\frac{1}{n}\sum_{k=1}^n a_k T^k f(w)\right| \ge \alpha\right\} \le \mu\left\{w\colon \sup_n \left[\frac{1}{n}\sum_{k=1}^n \tau^k(|f|^p)(w)\right]^{\frac{1}{p}} \ge \frac{\alpha}{C(\mathbf{a})}\right\}$$
$$\le \frac{K(\mathbf{a})}{\alpha^p}\int |f|^p\, d\mu$$

by the maximal inequalities for the Cesaro averages in L_1 (the constant $K(\mathbf{a})$ depends only on $\mathbf{a} = \{a_k\}$, not on T or f).

2. The theorem is true without requiring μ to be finite. The modulus τ splits the space into two absorbing sets: on one of which τ has an equivalent finite invariant measure, and Theorem 3.3 applies; on the other τ has no absolutely continuous invariant measure, so on that set $\frac{1}{n}\sum_{k=1}^n \tau^k f$ converges to zero for all $f \in L_p$ $(1 \le p < \infty)$, and our inequalities (3.1) imply the result.

3. The theorem was first proved in [T-1], for $1 < p < \infty$ and T induced by a measure preserving transformation (in the context of group actions of LCA groups), without using the Banach principle. The proof of [T-1] can be adapted to our situation (using the inequality $[\tau|f|]^p \le \tau(|f|^p)$), but it does not give the theorem for $p = 1$. For $p = 1$ and T induced by a measure preserving transformation, the theorem is due to Ryll-Nardzewski [RN] (with a different proof).

THEOREM 3.6. *Let $(\mathcal{X}, m)$ be a Lebesgue space, and let S be a Dunford-Schwartz operator on $L_1(m)$. Let $1 \le p \le \infty$ with dual index q. Then for $\phi \in L_q(m)$ there exists a null set $\mathcal{Z}$, such that for $\xi \notin \mathcal{Z}$ the sequence $a_k = S^k\phi(\xi)$ is in W_q, and has the property that for every Dunford-Schwartz operator T on a probability space (Ω, μ) and every $f \in L_p(\mu)$ we have a.e. convergence of $\frac{1}{n}\sum_{k=1}^n a_k T^k f$.*

Proof. For $p = 1$ (W_∞ is ℓ_∞), this is Theorem 3.2 of [ÇLO], which depends on the return times theorem [BoFKaOr] (and on [BO]). Fix $1 < p \le \infty$, so $1 \le q < \infty$, and fix $\phi \in L_q(m)$. Let $\{\phi_N\}$ be a sequence of bounded measurable functions on $\mathcal{X}$ with $|\phi_N| \le |\phi|$ a.e., which converges to ϕ pointwise and in $L_q(m)$-norm. Let $\tilde{S}$ be the linear modulus of S. For a.e. ξ we have (using [Kr, Lemma 1.7.4] again)

$$\limsup_n \frac{1}{n}\sum_{k=1}^n |S^k\phi(\xi)|^q \le \limsup_n \frac{1}{n}\sum_{k=1}^n [\tilde{S}^k|\phi|(\xi)]^q \le \lim_n \frac{1}{n}\sum_{k=1}^n \tilde{S}^k(|\phi|^q)(\xi),$$

which is finite a.e. by the pointwise ergodic theorem, so for a.e. ξ the sequence $\{S^k\phi(\xi)\}$ is in W_q.

Let C and D be the conservative and dissipative parts of $\tilde{S}$, which are both absorbing [Kr, p. 131]. Since the limit in the ergodic theorem is zero on D and on the conservative absorbing set C_0 on which $\tilde{S}$ has no finite invariant measure, for a.e. $\xi \in D \cup C_0$ the sequence $\{S^k\phi(\xi)\}$ has W_q seminorm 0, and (3.1) yields the desired convergence.

Let C_1 be the maximal support of a finite invariant measure for $\tilde{S}$, and let $0 \le \psi \in L_1(m)$ with $\{\psi > 0\} = C_1$ and $\tilde{S}\psi = \psi$. For the behavior for $\xi \in C_1$ we may assume $\mathcal{X} = C_1$, since C_1 is also absorbing. We already know that for a.e. ξ the sequence $\{S^k\phi(\xi)\}$ is in W_q. Let E be the conditional expectation with respect to the invariant σ-field of $\tilde{S}$, in the space $L_1(\psi\, dm)$. Using the identification of the limit in the ergodic theorem, for a.e. ξ we obtain

$$\begin{aligned}\limsup_n \frac{1}{n}\sum_{k=1}^n |S^k\phi(\xi) - S^k\phi_N(\xi)|^q &\le \limsup_n \frac{1}{n}\sum_{k=1}^n [\tilde{S}^k(|\phi - \phi_N|)(\xi)]^q \\ &\le \lim_n \frac{1}{n}\sum_{k=1}^n \tilde{S}^k(|\phi - \phi_N|^q)(\xi) \\ &= \psi(\xi)E(\psi^{-1}|\phi - \phi_N|^q)(\xi).\end{aligned}$$

Since $\psi^{-1}f \in L_1(\psi dm)$ for $f \in L_1(m)$, the bounded convergence theorem for conditional expectations yields $\lim_N E(\psi^{-1}|\phi - \phi_N|^q)(\xi) = 0$ a.e. By Theorem 3.2 of

[ÇLO], for a.e. ξ the sequence $\{S^k\phi_N(\xi)\}_{k\geq 1}$ can be taken as $\mathbf{b}^{(N)}$ in Proposition 3.2, good for all Dunford-Schwartz operators. Since we have only countably many relations, it follows from Proposition 3.2 that for a.e. ξ, the sequence $a_k = S^k\phi(\xi)$ satisfies the hypothesis of Proposition 3.1 for all Dunford-Schwartz operators. For $p = \infty$ this is the statement of our theorem; for $1 < p < \infty$ we now use Proposition 3.1 to conclude our theorem.

Remarks. 1. The first part of Theorem 3.6 is a generalization of Proposition 1.5.

2. Applying the theorem to rotations, we see that $\{S^k\phi(\xi)\}$ is Hartman. When $q = 1$, the proof shows that for $\phi \in L_1$, the sequence $\{S^k\phi(\xi)\}$ is in W_{1^+} (all for a.e ξ).

3. The main interest in [ÇLO] was in obtaining bounded a.e. modulating sequences. The previous theorem suggests that positive L_q contractions may yield a.e. modulating sequences for positive L_p contractions (when $\frac{1}{p}+\frac{1}{q} = 1$). We now study this problem.

Definition. A contraction T of $L_p(\Omega, \mathcal{F}, \mu)$ of a σ-finite measure space, $1 < p < \infty$, is called *positively dominated* if there exists a positive contraction $\tilde{T}$ of L_p, such that $|Tf| \leq \tilde{T}(|f|)$ a.e. for every $f \in L_p$.

The proofs of the following well-known lemmas are sketched for completeness.

LEMMA 3.7. *Let $1 < p < \infty$ with dual index $q = p/(p-1)$, and let $\tilde{T}$ be a positive contraction of $L_p(\mu)$. If $0 \leq \phi \in L_p$, then $\phi^{p-1} \in L_q$, and $\tilde{T}\phi = \phi$ if and only if $\tilde{T}^*(\phi^{p-1}) = \phi^{p-1}$.*

Proof. The first part is easy. When $\tilde{T}\phi = \phi$, use the equality in Hölder's inequality in

$$\int \phi^p \, d\mu = \int \phi^{p-1}\tilde{T}\phi \, d\mu = \int \phi\tilde{T}^*(\phi^{p-1}) \, d\mu \leq \|\phi\|_p \|\tilde{T}^*(\phi^{p-1})\|_q$$
$$\leq \|\phi\|_p \|\phi^{p-1}\|_q = \int \phi^p \, d\mu$$

LEMMA 3.8. *Let $1 < p < \infty$, and let T be a positively dominated contraction of $L_p(\Omega, \mu)$. Then Ω is decomposed as $\Omega = \Omega_0 \cup \Omega_1$, such that each $L_p(\Omega_i)$ is $\tilde{T}$ and T invariant, there is $0 \leq \psi \in L_q$ with $\tilde{T}^*\psi = \psi$ and $\{\psi > 0\} = \Omega_1$, and $\|\frac{1}{n}\sum_{k=1}^n \tilde{T}^k|f|\,\|_p \to 0$ for every $f \in L_p(\Omega_0)$.*

Proof. Ω_1 is the maximal support of $\tilde{T}$-invariant functions, which by the previous lemma is the maximal support of $\tilde{T}^*$-invariant functions. Ω_0 is the complement of Ω_1. The assertions are now easily checked (using the mean ergodic theorem for the last one).

LEMMA 3.9. *Let $1 < p < \infty$ with dual index q, and let T be a positively dominated contraction of $L_p(\Omega, \mu)$. If there is $\psi \in L_q$ with $\psi > 0$ a.e. and $\tilde{T}^*\psi = \psi$, then the operator $\hat{T}$ defined on L_∞ by $\hat{T}(f) = \psi^{1-q}T(f\psi^{q-1})$ can be extended to a Dunford-Schwartz operator on $L_1(\Omega, \nu)$, where $d\nu/d\mu = \psi^q$.*

Proof. By Lemma 3.7, $\tilde{T}(\psi^{q-1}) = \psi^{q-1} \in L_p$, so for $f \in L_\infty$ we have

$$|\hat{T}f| = \psi^{1-q}|T(f\psi^{q-1})| \leq \psi^{1-q}\tilde{T}(|f|\psi^{q-1}) \leq \|f\|_\infty.$$

$\hat{T}$ can be extended to a contraction of $L_1(\nu)$ since

$$\begin{aligned}\int |\hat{T}f|d\nu &= \int |\hat{T}f|\psi^q\, d\mu \leq \int \psi^{1-q}\tilde{T}(|f|\psi^{q-1})\psi^q\, d\mu = \int |f|\psi^{q-1}\tilde{T}^*\psi\, d\mu \\ &= \int |f|\psi^q\, d\mu.\end{aligned}$$

THEOREM 3.10. *Let $1 < p < \infty$ with dual index q, and let $\mathbf{a} = \{a_k\} \in W_s$ for some $q < s < \infty$. Assume that for every Dunford-Schwartz operator T in a probability space we have*

$$\frac{1}{n}\sum_{k=1}^{n} a_k T^k f \text{ converges a.e. } \forall f \in L_p. \tag{3.2}$$

Then (3.2) is satisfied by any positive contraction T of L_p of an atomless measure space. If $a_k \geq 0\ \forall k$, then (3.2) holds for every positively dominated contraction T of L_p of an atomless measure space.

Proof. Let T be a positively dominated contraction of $L_p(\Omega, \mu)$. By Lemma 3.8, the problem is reduced to two cases: either (i) $\tilde{T}^*$ has an invariant function $\psi \in L_q$ with $\psi > 0$ a.e., or (ii) $\tilde{T}^*$ has no invariant functions at all.

Case (i). $\hat{T}$ defined in Lemma 3.9 is a Dunford-Schwartz operator in $L_1(\Omega, \nu)$, with $d\nu/d\mu = \psi^q$. For $f \in L_p(\mu)$ we have $\int |f\psi^{1-q}|^p\psi^q\, d\mu = \int |f|^p\psi^{(1-q)p+q}\, d\mu = \int |f|^p\, d\mu < \infty$. Hence $f\psi^{1-q} \in L_p(\nu)$, and since $\psi^{1-q}T^k(f) = \hat{T}^k(f\psi^{1-q})$ for every k, application of (3.2) to $\hat{T}$ yields (3.2) for T.

Case (ii). We first prove (3.2) for T a *positive isometry* of $L_p(\mu)$ (with no invariant functions). By Lamperti's Theorem [La], there is a non-singular measurable transformation θ on (Ω, μ) such that

$$Tf(\omega) = f(\theta\omega)\left[\frac{d\mu}{d(\mu\theta^{-1})}(\theta\omega)\right]^{1/p} \tag{3.3}$$

Let $t = s/(s-1)$ (where $q < s < \infty$ with $\mathbf{a} \in W_s$), so $1 < t < p$. Define $Qg(\omega) = g(\theta\omega)[\frac{d\mu}{d(\mu\theta^{-1})}(\theta\omega)]^{t/p}$. Then Q is a positive isometry of $L_{p/t}(\mu)$. Clearly

$f \geq 0$ is in L_p if and only if $f^t \in L_{p/t}$, and then $Q(f^t) = [Tf]^t$. Hence Q has no invariant functions, and the pointwise ergodic theorem for Q [IT] (see also [Kr, p. 186]) yields

$$\frac{1}{n}\sum_{k=1}^{n} |T^k f|^t \leq \frac{1}{n}\sum_{k=1}^{n} [T^k|f|]^t \leq \frac{1}{n}\sum_{k=1}^{n} Q^k(|f|^t) \to 0 \text{ a.e. } \forall f \in L_p.$$

This implies (3.2), since $\mathbf{a} \in W_s$, and, by Hölder's inequality, for $f \in L_p(\mu)$ we have,

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k T^k f\right| \leq \left[\frac{1}{n}\sum_{k=1}^{n} |T^k f|^t\right]^{1/t} \times \left[\frac{1}{n}\sum_{k=1}^{n} |a_k|^s\right]^{1/s} \to 0.$$

We now turn to the general case of $\tilde{T}$ having no invariant functions. By the dilation theorem for positive contractions of L_p [AS], there exists a larger space $L_p(\Omega', \mu')$, an isometry R of $L_p(\mu')$, and a positive isometric embedding D of $L_p(\mu)$ into $L_p(\mu')$, such that $D\tilde{T}^k = ER^k D$ for every $k \geq 0$, with E a conditional expectation operator. By what we have proved above and by case (i), we already have (3.2) for every positive isometry of L_p, so we apply it to R to conclude that $\frac{1}{n}\sum_{k=1}^{n} a_k R^k Df$ converges a.e. for every $f \in L_p(\mu)$. Again, let $t = s/(s-1)$, so $1 < t < p$, and let Q be the isometry of $L_{p/t}(\mu')$ as defined before (now for R instead of T). Since $\{a_k\} \in W_s$, and $|Df|^t \in L_{p/t}$ implies $\sup_n \frac{1}{n}\sum_{k=1}^{n} Q^k(|Df|^t) \in L_{p/t}$ by [IT] (see [Kr, p. 186]), we have

$$\begin{aligned} \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} a_k R^k Df\right| &\leq \sup_n \left[\frac{1}{n}\sum_{k=1}^{n} |a_k|^s\right]^{1/s} \cdot \sup_n \left[\frac{1}{n}\sum_{k=1}^{n} |R^k Df|^t\right]^{1/t} \\ &\leq \sup_n \left[\frac{1}{n}\sum_{k=1}^{n} |a_k|^s\right]^{1/s} \cdot \sup_n \left[\frac{1}{n}\sum_{k=1}^{n} Q^k(|Df|^t)\right]^{1/t} \in L_p(\mu'). \end{aligned}$$

By the dominated convergence theorem for conditional expectations, $\tilde{T}$ satisfies (3.2). Since $\tilde{T}$ has no invariant functions, $\tilde{T}f = \lambda f$ for $f \in L_p$ and $|\lambda| = 1$ implies $f = 0$ a.e. Hence all functions of $L_p(\mu)$ are flight vectors for $\tilde{T}$. Applying our assumption to rotations of the circle, we see that $\mathbf{a}$ is a Hartman sequence. Since $\mathbf{a} \in W_s$ with $s > 1$, Theorem 4.1 of [ÇLO] (see Theorem 1.2) shows that $\|\frac{1}{n}\sum_{k=1}^{n} a_k \tilde{T}^k f\|_p \to 0$ for every $f \in L_p$. Hence also the a.e. convergence is to 0. This completes the proof of (3.2) for positive contractions of L_p.

Now assume that $a_k \geq 0$ for every k. For T positively dominated, we have to prove (3.2) only in case (ii) (i.e., $\tilde{T}^*$ has no invariant functions). But in that case, by the above, we have

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k T^k f\right| \leq \frac{1}{n}\sum_{k=1}^{n} a_k \tilde{T}^k |f| \to 0 \text{ a.e.}$$

Remarks. 1. The proof of case (ii) shows that if $\tilde{T}^*$ has no invariant functions, then the limit in (3.2), and also in (3.4) below, is 0 a.e.

2. In general, the sequences produced in Theorem 3.6 are only in W_q. We do not know if Theorem 3.10 is true if we assume only that $\mathbf{a} \in W_q$.

COROLLARY 3.11. *Let $1 < p < \infty$ with dual index q, and let $\mathbf{a} = \{a_k\} \in W_s$ for every $1 \leq s < q$ (e.g., $\mathbf{a} \in W_q$). Assume that for every Dunford-Schwartz operator T in a probability space and every $r > p$ we have*

$$\frac{1}{n}\sum_{k=1}^{n} a_k T^k f \text{ converges a.e. } \forall f \in L_r. \tag{3.4}$$

Then for every $r > p$, (3.4) *is satisfied by any positive contraction T of L_r. If $a_k \geq 0\ \forall k$, then* (3.4) *holds for every positively dominated contraction T of L_r.*

Proof. Fix $r > p$. Then its dual index r' is less than q, so there is s with $r' < s < q$. We now apply the theorem, with p and q replaced by r and r'.

PROPOSITION 3.12. *Let $1 < p < \infty$ with dual index q, and let S be a positively dominated contraction on $L_q(\mathcal{X}, m)$ of a Lebesgue space. Then for $\phi \in L_q(m)$ there exists a null set $\mathcal{Z}$, such that for $\xi \notin \mathcal{Z}$ the sequence $a_k = S^k\phi(\xi)$ is in W_s for every $s < q$, and has the property that for every Dunford-Schwartz operator T on a probability space (Ω, μ) and every $f \in L_r(\mu)$ with $r > p$ we have a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$.*

Proof. Let $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ be the decomposition obtained by applying Lemma 3.8 to S (interchanging p and q). Then $\mathcal{X}_i$ are also Lebesgue spaces, and we can deal separately with two cases .

(i) $\mathcal{X} = \mathcal{X}_1$. Let $\tilde{S}^*\psi = \psi \in L_p$ with $\psi > 0$ a.e. For $\phi \in L_q$, we have $\int |\phi\psi^{1-p}|^q \psi^p dm = \int |\phi|^q \psi^{(1-p)q+p} dm = \int |\phi|^q dm < \infty$. By Lemma 3.9, $\hat{S}$ is a Dunford-Schwartz operator in $L_1(\psi^p dm)$, and our result follows from applying Theorem 3.6 to $\hat{S}$, since $\psi^{1-p}S^k(\phi) = \hat{S}^k(\phi\psi^{1-p})$ for every k.

(ii) $\mathcal{X} = \mathcal{X}_0$. Let $\phi \in L_q$. Since $\tilde{S}$ has no invariant functions, Akcoglu's theorem yields $\frac{1}{n}\sum_{k=1}^{n} \tilde{S}^k|\phi|(\xi) \to 0$ a.e. Applying Lemma 2.9 of [BO] to $\tilde{S}$, for any rational $s \in [0, q)$ we obtain $\sup_n \frac{1}{n}\sum_{k=1}^{n}[\tilde{S}^k|\phi|(\xi)]^s < \infty$ a.e. We let $\mathcal{Z}$ be the null set where any of the above relations fails. For fixed $\xi \notin \mathcal{Z}$, let $a_k = S^k\phi(\xi)$. Clearly $\{a_k\} \in W_s$ for every $1 \leq s < q$.

Now let T be a Dunford-Schwartz operator in $L_1(\mu)$ of a probability space. For $f \in L_\infty(\mu)$ we have

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k T^k f(\omega)\right| \leq \|f\|_\infty \frac{1}{n}\sum_{k=1}^{n} |a_k| \to 0.$$

Fix $r > p$, and let $f \in L_r(\mu)$. Take t rational, $p < t < r$. Then $s = t/(t-1)$ is also rational, and $s < q$. Applying Lemma 2.9 of [BO] to T and using the choice of $\xi \notin \mathcal{Z}$, from Hölder's inequality we obtain

$$\sup_n \left| \frac{1}{n} \sum_{k=1}^{n} a_k T^k f(\omega) \right| \le \sup_n \left[\frac{1}{n} \sum_{k=1}^{n} |a_k|^s \right]^{1/s} \times \sup_n \left[\frac{1}{n} \sum_{k=1}^{n} |T^k f(\omega)|^t \right]^{1/t} < \infty \text{ a.e..}$$

The Banach principle yields the a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ for every $f \in L_r(\mu)$.

THEOREM 3.13. *Let $1 < p < \infty$ with dual index q, and let S be a positively dominated contraction on $L_q(\mathcal{X}, m)$ of a Lebesgue space. Then for $\phi \in L_q(m)$ there exists a null set $\mathcal{Z}$, such that for $\xi \notin \mathcal{Z}$ the sequence $a_k = S^k\phi(\xi)$ is in W_1, and has the property that for every positively dominated contraction T of $L_r(\Omega, \mu)$ with $r > p$ and every $f \in L_r(\mu)$ we have a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$.*

Proof. For S positive, the theorem follows by combining Proposition 3.12 with Corollary 3.11.

We now look at the general case, with S dominated by the positive contraction $\tilde{S}$ in $L_q(\mathcal{X}, m)$. Fix $\phi \in L_q(\mathcal{X}, m)$, and let $\mathcal{Z}_1$ be the null set obtained for $\tilde{S}$, such that for $\xi \notin \mathcal{Z}_1$, the sequence $b_k = \tilde{S}^k|\phi(\xi)|$ yields the a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} b_k T^k f$ for every positively dominated contraction T of $L_r(\mu)$, $r > p$. Let $\mathcal{Z}_2$ be the null set obtained for S by Proposition 3.12, such that for $\xi \notin \mathcal{Z}_2$, the sequence $a_k = S^k\phi(\xi)$ satisfies the hypothesis of Corollary 3.11. By the domination, $|S^k\phi| \le \tilde{S}^k|\phi|$ a.e. for every k, and let $\mathcal{Z}_0$ be the null set where for some k the inequality does not hold. Define the null set $\mathcal{Z} = \mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2$. For $\xi \notin \mathcal{Z}$ we let $b_k = \tilde{S}^k|\phi(\xi)|$ and $a_k = S^k\phi(\xi)$.

Now fix T positively dominated in $L_r(\mu)$, with $r > p$. The proof of case (i) in Theorem 3.10 yields the a.e. convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ when $\tilde{T}^*$ has an invariant function which is > 0 a.e. By Lemma 3.8, it remains to prove the desired convergence only when $\tilde{T}^*$ has no invariant functions. But in that case we have

$$\left| \frac{1}{n} \sum_{k=1}^{n} a_k T^k f \right| \le \frac{1}{n} \sum_{k=1}^{n} |a_k T^k f| \le \frac{1}{n} \sum_{k=1}^{n} |a_k| \tilde{T}^k |f| \le \frac{1}{n} \sum_{k=1}^{n} b_k \tilde{T}^k |f| \to 0 \text{ a.e.}$$

since the limit in case (ii) of Theorem 3.10 (applied to $\{b_k\}$) is 0 a.e.

Remarks. 1. In fact, by [BO, Lemma 2.9], the sequence $\{a_k\}$ defined in the theorem is in W_s for every $1 \le s < q$. We do not know if it is in W_q.

2. For $\{a_k\}$ defined as in the theorem, with S positive, Assani [As-1] proved the convergence of $\frac{1}{n}\sum_{k=1}^{n} a_k T^k f$ only for T induced by a measure preserving transformation and f bounded. His proof is different.

Problem. Is Theorem 3.13 true also for $r = p$?

COROLLARY 3.14. *Let $(\mathcal{X}, m)$ be a Lebesgue space, and let S be a Dunford-Schwartz operator on $L_1(m)$. Let $1 < p < \infty$ with dual index q. Then for $\phi \in L_q(m)$ there exists a null set $\mathcal{X}_0$, such that for $\xi \notin \mathcal{X}_0$ the sequence $a_k = S^k\phi(\xi)$ is in W_q, and has the property that for every positively dominated contraction T of $L_r(\Omega, \mu)$ with $r > p$ and every $f \in L_r(\mu)$ we have a.e. convergence of $\frac{1}{n}\sum_{k=1}^n a_k T^k f$.*

Proof. Let $\{\phi_N\}$ be as in the proof of Theorem 3.6, and $\mathbf{b}^{(N)} = \{S^k\phi_N(\xi)\}_{k\geq 1}$. We saw in Theorem 3.6 that $\mathbf{a} = \{a_k\}$ is in W_q, and that $\|\mathbf{b}^{(N)} - \mathbf{a}\|_{W_q} \to 0$. The convergence statement follows from Theorem 3.13.

Alternative Proof (without using Proposition 3.12). Fix $r > p$, and let T be a positively dominated contraction of $L_r(\Omega, \mu)$. For $f \in L_r(\mu)$, it follows from Lemma 2.9 of [BO] (with the roles of p and r interchanged) that $M_f(w) := \sup \frac{1}{n}\sum_{k=1}^n |T^k f|^p$ is finite a.e. By Hölder's inequality,

$$\left|\frac{1}{n}\sum_{k=1}^n (a_k - b_k^{(N)})T^k f(w)\right| \leq \left(\frac{1}{n}\sum_1^n |a_k - b_k^{(N)}|^q\right)^{\frac{1}{q}} \left(\frac{1}{n}\sum_1^n |T^k f(w)|^p\right)^{\frac{1}{p}}.$$

By [ÇLO, Theorems 2.4 and 3.2], $\frac{1}{n}\sum_{k=1}^n b_k^{(N)} T^k f(w)$ converges a.e. Hence

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{k=1}^n a_k T^k f(w) - \frac{1}{m}\sum_{k=1}^m a_k T^k f(w)\right| \\
&\quad\leq \left|\frac{1}{n}\sum_{k=1}^n (a_k - b_k^{(N)})T^k f(w)\right| \\
&\qquad + \left|\frac{1}{n}\sum_{k=1}^n b_k^{(N)} T^k f(w) - \frac{1}{m}\sum_{k=1}^m b_k^{(N)} T^k f(w)\right| \\
&\qquad + \left|\frac{1}{m}\sum_{k=1}^m (a_k - b_k^{(N)})T^k f(w)\right| \\
&\quad\leq \left(\frac{1}{n}\sum_1^n |a_k - b_k^{(N)}|^q\right)^{\frac{1}{q}} M_f(w) + \left(\frac{1}{m}\sum_1^m |a_k - b_k^{(N)}|^q\right)^{\frac{1}{q}} M_f(w) \\
&\qquad + \left|\frac{1}{n}\sum_{k=1}^m b_k^{(N)} T^k f(w) - \frac{1}{m}\sum_{k=1}^m b_k^{(N)} T^k f(w)\right|,
\end{aligned}$$

so $\left\{\frac{1}{n}\sum_{k=1}^n a_k T^k f(w)\right\}$ is a Cauchy sequence, and hence converges, for a.e. w.

Remark. For T Dunford-Schwartz, the convergence follows immediately from Theorem 3.6, since $L_r \subset L_p$ for $r > p$. We do not know if in general the result is valid also for $r = p$.

In order to use the results of the previous section for the identification of the limit in Theorem 3.6, we have to find the Fourier function of the sequences $\{a_k\}$ constructed there. This is done by using the extension of the Wiener-Wintner Theorem to Dunford-Schwartz operators (which follows from Theorem 3.2 of [ÇLO], or from Theorem 3.6). The passage from point transformations to Dunford-Schwartz operators in [ÇLO] not only shows that these sequences are Hartman but allows a computation of their Fourier function. We will carry out the details only for S an ergodic positive Dunford-Schwartz operator with $S1 = 1$. Since we assume that $(\mathcal{X}, m)$ is a Lebesgue space, we may remove a null set and obtain that S is induced by a transition probability P, for which m is invariant. Thus, we deal with the case where $S\phi(\xi) = \int \phi(\eta)P(\xi, d\eta)$ for every bounded measurable function ϕ, and the same formula defines $S\phi$ for every positive m-integrable function ϕ, with finite values a.e. by the P-invariance of m. Since the identification of the limit can be obtained also from the norm convergence, we use the following general result.

THEOREM 3.15. *Let S be a positive Dunford-Schwartz operator on L_1 of a Lebesgue space $(\mathcal{X}, m)$, with $S1 = 1$. For $\phi \in L_2(m)$ let $\sigma_\phi := \{\lambda\colon |\lambda| = 1,\ E(\lambda, S)\phi \neq 0\}$. Then there exists a null set $\mathcal{Z}$ such that for $\xi \notin \mathcal{Z}$, the sequence $\mathbf{a}(\xi) = \{S^k\phi(\xi)\}$ is a Hartman sequence in W_2, its spectrum is $\sigma(\mathbf{a}(\xi)) = \sigma_\phi$, $c(\lambda, \mathbf{a}(\xi)) = [E(\lambda, S)\phi](\xi)$ for $\lambda \in \sigma_\phi$, and for every weakly almost periodic operator T on a Banach space X we have*

$$L(\mathbf{a}(\xi), T)x = \sum_{|\lambda|=1} E(\lambda, S)\phi(\xi)E(\bar{\lambda}, T)x = \sum_{\lambda\in\sigma_\phi} E(\lambda, S)\phi(\xi)E(\bar{\lambda}, T)x \quad (3.5)$$

with the series unconditionally convergent in X.

Proof. It follows from Theorem 3.6 that for a.e. ξ, the sequence $\mathbf{a}(\xi)$ is a Hartman sequence in W_2. It follows from the above mentioned Wiener-Wintner Theorem for Dunford-Schwartz operators that for a.e. ξ, the Fourier function of $\mathbf{a}(\xi) = \{S^k\phi(\xi)\}$ is $c(\lambda, \mathbf{a}(\xi)) = [E(\lambda, S)\phi](\xi)$ for $\lambda \in \sigma_\phi$, and $c(\lambda, \mathbf{a}(\xi)) = 0$ for the other unimodular λ (only $\phi \in L_1$ is needed). Hence $\sigma(\mathbf{a}(\xi)) = \sigma_\phi$. Since $\phi \in L_2$, the orthogonality in L_2 of the functions $\{E(\lambda, S)\phi\}$ yields $\sum_{\lambda\in\sigma(\mathbf{a}(\xi))} |c(\lambda, \mathbf{a}(\xi))|^2 < \infty$ (see the proof of Theorem 2.5). Now we can apply Theorem 2.4 to obtain the identification of the limit (3.5), with unconditional strong convergence of the series.

4. Modulated ergodic theorems for mean ergodic contractions

In this section we obtain necessary and sufficient conditions on a sequence $\mathbf{a} = \{a_k\}$ for (1.1) to hold for every power-bounded mean ergodic operator T, and identify the limit.

We shall use the following general weighted ergodic theorem for mean ergodic power-bounded operators. As before, let $E(1, T)x = \lim_n \frac{1}{n}\sum_{k=1}^n T^k x$.

THEOREM 4.1. *Let* $(\alpha_{n,k})_{n>0,\,k\geq 0}$ *be a matrix such that for every* n *the series* $\sum_{k=0}^{\infty}|\alpha_{n,k}|$ *converges. For* T *power-bounded in a Banach space* X *define* $A_n(T)x = \sum_{k=0}^{\infty}\alpha_{n,k}T^k x$. *Then* $A_n(T)x \to E(1,T)x$ *for every power-bounded mean ergodic operator* T *if and only if* $(\alpha_{n,k})_{n>0,\,k\geq 0}$ *satisfies the following three conditions*:

$$\sup_n \sum_{k=0}^{\infty}|\alpha_{n,k}| = K < \infty. \tag{4.1}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\alpha_{n,k} = 1. \tag{4.2}$$

$$\lim_{n\to\infty}\left[|\alpha_{n,0}| + \sum_{k=0}^{\infty}|\alpha_{n,k+1} - \alpha_{n,k}|\right] = 0. \tag{4.3}$$

Proof. The sufficiency of (4.1)–(4.3) is well known (e.g., [Kr, p.251]). (4.2) ensures that $A_n(T)y \to y$ for any fixed point y. (4.3) (which is equivalent to the pair of conditions (W2)+(W3) in [Kr]) yields $\|A_n(T)(I-T)\| \to 0$, and (4.1) shows that $\sup_n \|A_n(T)\| < \infty$, yielding the convergence on all of the space.

Assume now that $A_n(T)x \to E(1,T)x$ for every power-bounded mean ergodic operator T. (4.2) follows by taking T the identity. To prove (4.1), define T on c_0 by $T(\{x_k\}) = \{x_{k+1}\}$, and $\mathbf{x}^{(n,j)} = (\operatorname{sign}\alpha_{n,0}, \operatorname{sign}\alpha_{n,1}, \ldots, \operatorname{sign}\alpha_{n,j}, 0, 0, \ldots)$. Then the first coordinate of $\sum_{k=0}^{\infty}\alpha_{n,k}T^k\mathbf{x}^{(n,j)}$ is $\sum_{k=0}^{j}|\alpha_{n,k}|$. Since $\|\mathbf{x}^{(n,j)}\| \leq 1$, and by the assumption, $\sup_n \|A_n(T)\| = K < \infty$, (4.1) follows from

$$\sum_{k=0}^{j}|\alpha_{n,k}| \leq \left\|\sum_{k=0}^{\infty}\alpha_{n,k}T^k\mathbf{x}^{(n,j)}\right\| \leq K.$$

For (4.3), define S on ℓ_1 by $S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$, put $X = \overline{(I-S)\ell_1}$, and $T = S_{|X}$. Then T is mean ergodic with $E(1,T) = 0$. Denote the unit vectors of ℓ_1 by $\{e_j\}$, and let $\mathbf{x} = (I-S)e_1 = e_1 - e_2$. Since $S^k e_j = e_{j+k}$, we have

$$|\alpha_{n,0}| + \sum_{k=0}^{\infty}|\alpha_{n,k+1} - \alpha_{n,k}| = \left\|\alpha_{n,0}e_1 + \sum_{k=0}^{\infty}(\alpha_{n,k+1} - \alpha_{n,k})e_{k+2}\right\| = \|A_n(T)\mathbf{x}\| \to 0.$$

Remarks. 1. If in (4.2) the value of the limit is α, not necessarily 1, the theorem holds with $A_n(T) \to \alpha E(1,T)$.

2. The necessity of (4.1)–(4.3) seems to have been unnoticed.

PROPOSITION 4.2. *Let* $(\alpha_{n,k})_{n>0,\,k\geq 0}$ *be a matrix as above, and assume* $\lim_{n\to\infty}\alpha_{n,k} = 0$ *for every* k. *If* $A_n(T)x$ *converges for every* T *mean ergodic, then* (4.1) *and* (4.3) *hold,* $\alpha = \lim_n \sum_{k=0}^{\infty}\alpha_{n,k}$ *exists, and for every* T *mean ergodic* $\lim_n A_n(T)x = \alpha E(1,T)x$.

Proof. (4.1) follows from the previous proof. α is obtained by taking T to be the identity. Let S on ℓ_1 and T on $X = \overline{(I - S)\ell_1}$ be defined as before. By the assumption that $\lim_n \alpha_{n,k} = 0$ for every k, the coordinates of $A_n(T)(e_1 - e_2)$ converge to 0. Since $A_n(T)(e_1 - e_2)$ converges in norm, the limit is 0, so (4.3) holds.

THEOREM 4.3. *A sequence* $\mathbf{a} = \{a_k\}$ *satisfies* (1.1) *for every mean ergodic operator T in a Banach space X if and only if* $\mathbf{a}$ *is in* W_1, $\alpha = \lim_n \frac{1}{n}\sum_{k=1}^n a_k$ *exists, and* $\{a_k\}$ *satisfies*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_{k+1} - a_k| = 0. \tag{4.4}$$

If the conditions hold, then $L(\mathbf{a}, T)x = \alpha E(1, T)x$ *for every mean ergodic* T.

Proof. Define $\alpha_{n,k} = \frac{a_{k+1}}{n}$ for $n > 0$ and $0 \le k < n$, and $\alpha_{n,k} = 0$ for $k \ge n$. We can now apply the previous proposition.

Remark. The conditions on $\mathbf{a}$ in the previous theorem are very strong. A sequence $a_k = \phi(\theta^k\xi)$ with θ a probability preserving ergodic transformation typically does not satisfy (4.4), as the limit is $\int |\phi \circ \theta - \phi|$, which is positive for non-constant ϕ.

The convergence statement (1.1) in Theorem 2.1 can be similarly obtained from the following more general result, which is also proved by using the dilation theorem and the spectral theorem.

THEOREM 4.4. *Let* $(\alpha_{n,k})_{n>0,\,k\ge 0}$ *be a matrix as in Theorem* 4.1, *which satisfies* (4.1). *Then* $A_n(T)x$ *converges strongly for every contraction* T *in a Hilbert space* H *and every* $x \in H$ *if and only if* $(\alpha_{n,k})_{n>0,\,k\ge 0}$ *satisfies*

$$c(\lambda) := \lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{n,k}\overline{\lambda}^k \text{ exists } \forall |\lambda| = 1. \tag{4.5}$$

When (4.5) *is satisfied,* $\lim A_n(T)x = E(1, T)x$ *for every* T *if and only if* $c(1) = 1$ *and* $c(\lambda) = 0$ *for every other* $|\lambda| = 1$.

If $c(\lambda) \ne 0$ *only for countably many* λ, *then* $\lim_n A_n(T)x = \sum_{\{\lambda:|\lambda|=1\}} c(\overline{\lambda})E(\lambda, T)x$, *with countably many non-zero terms, and strong convergence of the series.*

Remarks. 1. Most of Theorem 4.4, for *unitary* operators and for $\alpha_{n,k} \ge 0$ with $\sum_k \alpha_{n,k} = 1$ for every n, is proved in [Ro]. The identification of the limit, when $c(\lambda)$ is zero except for a countable number of points, is given there only in a restricted particular case. The special case of convergence to $E(1, T)$, under the assumptions of [Ro], was proved in [BlE] (in the context of unitary representations of LCA groups).

2. Proposition 33 of [Ro] shows that (for two-sided infinite row matrices), (4.5) may be satisfied with $\{\lambda\colon |\lambda| = 1 \text{ and } c(\lambda) \ne 0\}$ uncountable.

3. An example in [BlE] shows that (4.1), and (4.5) with $c(\lambda) = 0$ for $\lambda \ne 1$, do not imply the convergence of $A_n(T)x$ for every mean ergodic contraction.

Acknowledgements 1. Each author is grateful to the other authors' institutions for their hospitality, which made this joint research possible.

2. The second author is grateful to the University of Toronto for its hospitality, and to M. Akcoglu for some helpful discussions.

References

[ABr] M. Akcoglu and A. Brunel, *Contractions on L_1-spaces*, Trans. Amer. Math. Soc. **155** (1971), 315–325.

[AS] M. Akcoglu and L. Sucheston, *Dilations of positive contractions in L_p spaces*, Canadian Math. Bull. **20** (1977), 285–292.

[As-1] I. Assani, "Universal weights from dynamical systems to mean-bounded positive operators on L_p" in *Almost everywhere convergence* II (eds. A. Bellow and R. Jones), Academic Press, Boston, 1991, pp. 9–16.

[As-2] ———, *Uniform Wiener-Wintner theorem for weakly mixing dynamical systems*, unpublished manuscript, 1992.

[BO] J. R. Baxter and J. H. Olsen, *Weighted and subsequential ergodic theorems*, Canadian J. Math. **35** (1983), 145–166.

[BeLo] A. Bellow and V. Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, Trans. Amer. Math. Soc. **288** (1985), 307–345.

[Bes] A. Besicovitch, *Almost periodic functions*, Cambridge University Press, London, 1932 (reprinted Dover, NY, 1954).

[BlE] J. Blum and B. Eisenberg, *Generalized summing sequences and the mean ergodic theorem*, Proc. Amer. Math. Soc. **42** (1974), 423–429.

[BoFKaOr] J. Bourgain, H. Furstenberg, Y. Katznelson, and D. Ornstein, *Return times of dynamical systems (appendix to Bourgain's pointwise ergodic theorems for arithmetic sets)*, IHES Publ. Math. **69** (1989), 47–50.

[ÇL] D. Çömez and M. Lin, "Mean ergodicity of L_1 contractions and pointwise ergodic theorems" in *Almost everywhere convergence II* (eds. A. Bellow and R. Jones), Academic Press, Boston, 1991, pp. 113–126.

[ÇLO] D. Çömez, M. Lin and J. Olsen, *Weighted ergodic theorems for mean ergodic L_1 contractions*, Trans. Amer. Math. Soc. **350** (1998), 101–117.

[IT] A. Ionescu-Tulcea, *Ergodic properties of isometries in L^p spaces*, $1 < p < \infty$, Bull. Amer. Math. Soc. **70** (1964), 366–371.

[Ja] B. Jamison, *Eigenvalues of modulus* 1, Proc. Amer. Math. Soc. **18** (1965), 375–377.

[JO] R. Jones and J. Olsen, *Mutiparameter weighted ergodic theorems*, Canadian J. Math. **46** (1994), 343–356.

[K-1] J. P. Kahane, *Sur les coefficients de Fourier-Bohr*, Studia Math. **21** (1961), 103–106.

[K-2] ———, *Some random series of functions*, second edition, Cambridge University Press, Cambridge, 1985.

[Kr] U. Krengel, *Ergodic theorems*, de Gruyter, Berlin, 1985.

[KuN] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley , New York, 1974.

[La] J. Lamperti, On the isometries of certain function spaces, Pacific J. Math. **8** (1958), 459–466.

[Le] B. M. Levitan, *Almost periodic functions* (in Russian), Moscow, 1953.

[LO] M. Lin and J. Olsen, "Besicovitch functions and weighted ergodic theorems for LCA group actions" in *Convergence in ergodic theory and probability* (eds. V. Bergelson, P. March and J. Rosenblatt), de Gruyter, Berlin, 1996, pp. 277–289.

[M] J. Marcinkiewicz, *Une remarque sur les espaces de M. Besicovitch*, C. R. Acad. Sci. Paris **208** (1939), 157–159.

[O] J. Olsen, "Calculation of the limit in the return times theorem for Dunford-Schwartz operators" in *Ergodic theory and its connections with harmonic analysis* (eds. K. Petersen and I. Salama), Cambridge University Press, London, 1995, pp. 359–368.

[OrWe] D. Ornstein and B. Weiss, *Subsequence ergodic theorems for amenable groups*, Israel J. Math. **79** (1992), 113–127.

[R] E. A. Robinson, *On uniform convergence in the Wiener-Wintner theorem*, J. London Math. Soc. **49** (1994), 493–401.

[Ro] J. Rosenblatt, *Norm convergence in ergodic theory and the behavior of Fourier transforms*, Canadian J. Math. **46** (1994), 184–199.

[RN] C. Ryll-Nardzewski, *Topics in ergodic theory*, Springer Lecture Notes in Math., vol. 472, 1975, pp. 131–156.

[T-1] A. Tempelman, *Ergodic theorems for amplitude modulated homogeneous random fields*, Lithuanian Math. J. **14** (1974), 221–229 (in Russian; English translation in Lith. Math. Transl. **14** (1975), 698–704).

[T-2] ———, *Convergence and consistency of regression estimators*, Doctor of Sciences Dissertation, Vilnius, 1975 (in Russian).

[W] P. Walters, *Topological Wiener-Wintner theorems and a random L^2 ergodic theorem*, Ergodic Theory Dynamical Systems **16** (1996), 179–206.

Michael Lin, Ben-Gurion University of the Negev, Beer-Sheva, Israel
lin@math.bgu.ac.il

James Olsen, North Dakota State University, Fargo, ND 58105
jolsen@plains.nodak.edu

Arkady Tempelman, Penn State University, University Park, PA 16802
arkady@stat.psu.edu