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COBOUNDARIES FOR COMMUTING TRANSFORMATIONS

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To Alexandra Bellow on the occasion of her retirement from teaching.

ABSTRACT. Let τ and σ be two commuting ergodic measure preserving transformations of a probablity space, and $\mathrm{Cob}(\tau)$, $\mathrm{Cob}(\sigma)$ be the sets of their coboundaries. We show that the inclusion $\mathrm{Cob}(\sigma) \subseteq \mathrm{Cob}(\tau)$ holds if and only if $\sigma = \tau^n$ for some $n \in \mathbb{Z}$. The transformations τ and σ have exactly the same coboundaries if and only if $\sigma = \tau^{\pm 1}$. Some related results and open questions are discussed.

1. Introduction and statements of the main results

Let τ be an invertible measure preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. A measurable real-valued function f on X is called *a (measurable) coboundary* for τ with *transfer-function* g if

$$f(x) = g(x) - g(\tau x) \text{ a.e.,} \tag{1}$$

and g is measurable. Two measurable functions, f_1 and f_2, are called *cohomologous* if their difference $f_1 - f_2$ is a measurable coboundary.

Let $\mathrm{Cob}(\tau)$ denote the set of all measurable coboundaries for τ, and for any p, $1 \leq p \leq \infty$, let $\mathrm{Cob}_p(\tau)$ be the subset of $\mathrm{Cob}(\tau)$ for which the transfer-functions are in L^p (the subset of L^p*-coboundaries*). If the space X has an additional structure of a compact metric space, the measure μ is Borel, and τ is a homeomorphism preserving μ, we denote by $\mathrm{Cob}_C(\tau)$ the set of all (continuous) coboundaries for τ with continuous transfer-functions. For $n = 1, 2, \ldots$ let $s_n(f; \tau)$ denote the sum $\sum_{k=0}^{n-1} f \circ \tau^k$.

Throughout this note we assume τ to be ergodic. In this case the transfer-function g, for a given coboundary f, is defined uniquely up to a constant. If $f \in L^1(X)$ is a coboundary, then $\int f = 0$, even though the transfer-function g may be non-integrable [A].

It is well known, in the frameworks of measurable, topological and smooth dynamics, that various stochastic and dynamical properties of a dynamical system are intimately related to its cohomology. In the purely measure-theoretical setting, it can be shown that some basic properties of a measure preserving transformation τ admit

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simple characterizations in terms of the sets $\mathrm{Cob}(\tau)$, $\mathrm{Cob}_p(\tau)$. For example, τ is ergodic iff for any p, $1 \leq p < \infty$, the set $\mathrm{Cob}_p(\tau)$ is dense in $L_0^p = \{f \in L^p\colon \int f = 0\}$; τ is weakly mixing iff the only function $f \in \mathrm{Cob}(\tau)$ taking only two values (mod 0) is zero [Ha].

K.Schmidt [Sc] (see also [He]) gave a necessary and sufficient condition for a real-valued function f to be a coboundary. To formulate this condition, we call a sequence $\{\varphi_n\}$ of real-valued functions on X *stochastically bounded* if for every $\varepsilon > 0$ there exists $N > 0$ such that for each n,

$$\mu(\{x \in X\colon \ |\varphi_n(x)| > N\}) < \varepsilon.$$

The Schmidt criterion states that *a measurable function f is a coboundary for τ if and only if the sequence $\{s_n(f;\tau)\}$ is stochastically bounded.*

Unfortunately, for a concrete transformation τ and concrete f the verification of stochastical boundedness of $\{s_n(f;\tau)\}$ is often not an easy task; therefore, the description of the set $\mathrm{Cob}(\tau)$, for a given τ, can be very nontrivial.

In this note we will be concerned with the question: to what extent the sets $\mathrm{Cob}(\tau)$, $\mathrm{Cob}_p(\tau)$, $\mathrm{Cob}_C(\tau)$ determine τ itself. In particular, if two ergodic transformations act on the same space, we want to know when they can have the same coboundaries. We restrict ourselves to the case of two commuting transformations; in this case a complete answer will be provided.

It is worth mentioning that the problem of "recovering" various properties of a dynamical system from a cohomological information was one of the themes of several recent papers devoted to the connections between C^*-algebras and topological dynamics (see [HPS], [GPS], [GW] and references therein). The approach in these papers is different from ours in many aspects: their setting is topological and they study Cantor dynamical systems; the cohomological information is represented by the dimension group of a transformation rather than the set of its coboundaries.

Returning to our question, note that if $\sigma = \tau^{-1}$, then $\mathrm{Cob}(\tau) = \mathrm{Cob}(\sigma)$, and $\mathrm{Cob}_p(\tau) = \mathrm{Cob}_p(\sigma)$. Indeed, every function f that can be written in the form $f = g - g \circ \tau$, can also be written as $f = g_1 - g_1 \circ \sigma$, with $g_1 = -g \circ \tau$.

We will show that the converse is also true if the space X on which τ and σ act is non-atomic. Non-atomicity is certainly needed for this fact, since for X consisting of finitely many points, ergodic transformations are cyclic permutations, and their coboundaries are all functions with zero mean.

THEOREM 1. *Suppose τ and σ are two commuting invertible ergodic measure preserving transformations of a non-atomic probability space X. They have the same coboundaries if and only if $\sigma = \tau^{\pm 1}$.*

If $\sigma = \tau^n$, $n > 0$, and $f \in \mathrm{Cob}(\sigma)$ (or $f \in \mathrm{Cob}_p(\sigma)$) with $f = g - g \circ \sigma$, then it is easy to check that $f = g_1 - g_1 \circ \tau$, with $g_1 = s_n(g;\tau)$. Hence, $f \in \mathrm{Cob}(\tau)$ (or $f \in \mathrm{Cob}_p(\tau)$). It turns out that the converse to this statement is also true under the same assumptions on τ, σ and X.

THEOREM 2. *If* $\mathrm{Cob}(\sigma) \subseteq \mathrm{Cob}(\tau)$, *then* $\sigma = \tau^n$ *for some* $n \in \mathbb{Z}$. *If* $\mathrm{Cob}_p(\sigma) \subseteq \mathrm{Cob}_p(\tau)$ *for some* p, $1 \le p \le \infty$, *then* $\sigma = \tau^n$ *for some* $n \in \mathbb{Z}$.

The next theorem is a continuous analog of Theorem 2.

THEOREM 3. *Suppose* (X, μ) *has a structure of compact metric space, the measure* μ *is Borel and non-atomic, and* τ, σ *are commuting minimal (i.e., all orbits are dense) homeomorphisms of* X *for which* μ *is invariant and ergodic. If* $\mathrm{Cob}_C(\sigma) \subseteq \mathrm{Cob}_C(\tau)$, *then* $\sigma = \tau^n$ *for some* $n \in \mathbb{Z}$.

Theorem 1 is an immediate consequence of Theorem 2, and the proof of Theorem 3 differs from the proof of Theorem 2 unessentially. Therefore, we prove Theorem 2 only, and later indicate the modifications necessary for Theorem 3.

2. Proofs

The proof of Theorem 2 is based on a slight modification of the $\mathbb{Z}^2$-Rokhlin lemma. It can be called the "non-free" Rokhlin lemma, since the usual $\mathbb{Z}^2$-Rokhlin lemma [C], [KaW] is formulated for free actions (the assumption of freeness is too restrictive for Theorem 2).

Recall that a $\mathbb{Z}^2$-action generated by τ and σ is *free* if $\mu(\{x \in X\colon \tau^m\sigma^n x = x\}) = 0$ whenever $(m, n) \neq (0, 0)$.

One can easily verify (using the ergodicity of at least one of the transformations τ, σ) that if the $\mathbb{Z}^2$-action generated by τ and σ is not free, then there exists a pair of numbers (p, q), $p \in \mathbb{Z} - \{0\}$, $q \in \mathbb{Z}_+$, with the following properties:

(a) $\tau^p\sigma^q = \mathrm{Id}$ a.e.;
(b) if $\mu(A_{m,n}) > 0$, where $A_{m,n} = \{x \in X\colon \tau^m\sigma^n x = x\}$, then (m, n) is an integer multiple of (p, q).

Since it is clear that the numbers p, q are defined uniquely by the properties (a), (b), we will say in this case that the pair (τ, σ) *is of type* (p, q).

The following statement is a weak form of the non-free $\mathbb{Z}^2$-Rokhlin Lemma.

LEMMA 1. *Suppose that the pair* (τ, σ) *is of type* (p, q). *Let* $M \subseteq X$ *be of positive measure. Then for every pair* N_1, N_2 *of positive integers there is a set* $A \subseteq M$ *such that*:

(i) $\mu(\bigcup_{k=0}^{N_1}\bigcup_{l=0}^{N_2}\tau^k\sigma^l A) \ge \frac{1}{4}\mu(M)$.
(ii) *If* $0 \le k, k' \le N_1$, *and* $0 \le l, l' \le N_2$, *then* $\tau^k\sigma^l A \cap \tau^{k'}\sigma^{l'} A = \emptyset$ *unless* $(k', l') \equiv (k, l) \bmod (p, q)$; *if* $(k', l') \equiv (k, l) \bmod (p, q)$, *then (obviously)* $\tau^k\sigma^l A = \tau^{k'}\sigma^{l'} A$.

Remark. Arguing as in [C], Theorem 3.1, one can obtain from Lemma 1 a complete non-free analog of the $\mathbb{Z}^2$-Rokhlin lemma, with $M = X$ and a condition

(i′) $\mu\left(\bigcup_{k=0}^{N_1}\bigcup_{l=0}^{N_2}\tau^k\sigma^l A\right) \geq 1-\varepsilon$, $\varepsilon > 0$ arbitrary,

replacing the condition (i) of Lemma 1. We do not give the details since this stronger form will not be used in the paper.

Proof of Lemma 1. Both this lemma and its proof are "non-free modifications" of Lemma 3.1 in Conze [C]. For completeness we provide a sketch of the argument.

First note the following general fact. Suppose a group G acts on X by measure preserving transformations $\{\tau_g\}$, $g \in G$, and $\mathcal{F} = \{g_1, g_2, \dots g_n\}$ is a finite subset of G. Then every set $M \subseteq X$ of positive measure contains a subset $A \subseteq M$, also of positive measure, with the property that whenever a pair (i, j) satisfies $1 \leq i, j \leq n$ and $g_i^{-1}g_j$ does not have fixed points mod 0, we have $\tau_{g_i}A \cap \tau_{g_j}A = \emptyset$.

We apply this fact to the set $\mathcal{F} = \{(m, n) \in \mathbb{Z}^2\colon 0 \leq m \leq N_1, 0 \leq n \leq N_2\}$, and take the corresponding set A to be maximal with respect to inclusions mod 0. Then we must have

$$\bigcup_{k=-N_1}^{N_1}\bigcup_{l=-N_2}^{N_2}\tau^k\sigma^l A \supseteq M \,(\text{mod } 0) \tag{2}$$

since otherwise one could apply the above mentioned general fact once again, this time to the set $N = M - \bigcup_{k=-N_1}^{N_1}\bigcup_{l=-N_2}^{N_2}\tau^k\sigma^l A$, and get a set $C \subset N$ of positive measure having the same properties of disjointness of its images as A. Adding this set to A, one would get a bigger subset $B = A \cup C$ of M, still having these disjointness properties, contradicting the maximality of A. The inclusion (2) actually proves (i) since each of the four quadrants of the rectangle $-N_1 \leq k \leq N_1$, $-N_2 \leq l \leq N_2$ contains the same number of points which are pairwise non-congruent mod (p, q). $\square$

Proof of Theorem 2. Assuming that σ is not a power of τ, we are going to construct a function $f \in L^\infty(X)$ which is a coboundary for σ, but not for τ. Although we will not be using the above mentioned Schmidt's criterion explicitly, the idea is very close to it in spirit: we construct a function f with "bounded behavior" of $s_n(f;\sigma)$ and "unbounded behavior" of $s_n(f;\tau)$. It turns out that we can use L^∞-boundedness instead of stochastic boundedness in our construction, and this allows us to keep the argument completely elementary. More precisely, a function $f \in L^\infty(X)$ will be constructed to satisfy the following properties:

(a) there exists a constant C such that

$$\|s_n(f;\sigma)\|_\infty \leq C, \qquad n = 1, 2, \dots; \tag{3}$$

(b) for every r there exist m_r and a measurable set D_r, $\mu(D_r) \geq \frac{1}{8}$, such that

$$|s_{m_r}(f;\tau)(x)| \geq r \tag{4}$$

for all $x \in D_r$.

It is easy to check that if f satisfies (3), then

$$f(x) = g(x) - g(\sigma x) \text{ a.e.},$$

with $g(x) = \sup_{n \geq 1} s_n(f;\sigma)$. Since $g \in L^\infty$, this shows that $f \in \mathrm{Cob}_\infty(\sigma)$.

On the other hand, property (4) guarantees that $f \notin \mathrm{Cob}(\tau)$. Indeed, if f were a measurable coboundary for τ with a transfer-function g, then (1) would imply that $s_n(f;\tau) = g - g \circ \tau^n$, and for any K with $\mu(\{x\colon\ |g(x)| \geq K\}) < \frac{1}{16}$, one would have

$$\mu(\{x\colon\ |s_n(f;\tau)(x)| \geq 2K\}) < \frac{1}{8}$$

for any n, contradicting (4).

Therefore, both statements of Theorem 2 follow from (3) and (4).

The function f will be constructed as the sum of an infinite series $f = \sum_{r=1}^{\infty} f_r$, in which every term f_r is associated with a certain Rokhlin tower $\mathcal{T}_r$ for the $\mathbb{Z}^2$-action generated by τ and σ.

Two cases should be considered separately, depending on whether this action is free or not. We will discuss the "non-free" case only since the other case is simpler and can be treated with the usual Rokhlin lemma, instead of its "non-free version". Hence we assume that the pair (τ, σ) is of type (p, q) for some fixed pair (p, q). Note that we can also assume that $p > 0$, since otherwise we can replace τ by τ^{-1}, and that $q > 1$, since if $q = 1$, then $\sigma = \tau^{-p}$, and we are done. In what follows we will consider the case when q is even; some minor modifications necessary for the case when q is odd, $q \geq 3$, will be given at the end of the proof.

In order to construct the functions f_r's, we will need to define two sequences: a sequence of natural numbers $\{n_r\} \uparrow \infty$ characterizing the sizes of the towers $\mathcal{T}_r$'s, and a sequence of real numbers $\{\alpha_r\} \downarrow 0$ determining what values the functions f_r's take. We will assume that

$$n_r \left(\sum_{s=r+1}^{\infty} \alpha_s \right) \leq 1; \qquad r = 1, 2, \ldots \tag{5}$$

and

$$\alpha_r n_r \geq 2 \left(r + \sum_{t=1}^{r-1} \alpha_t n_t \right); \qquad r = 2, 3, \ldots. \tag{6}$$

It is easy to see that $\{n_r\}$ and $\{\alpha_r\}$ satisfying (5) and (6) can be constructed inductively. Once these sequences are fixed, for each r we define the tower $\mathcal{T}_r$ by applying

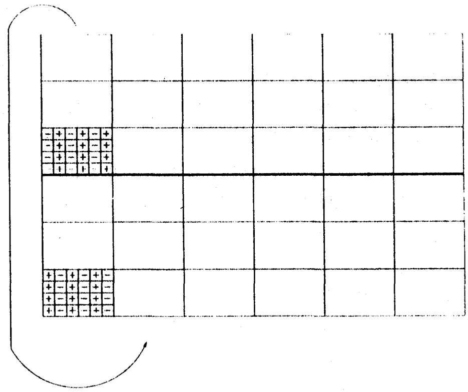

Figure 1

Lemma 1 with $N_1 = N_1^{(r)} = 2pn_r$, $N_2 = N_2^{(r)} = 2qn_r$, and getting the set A_r, the "base" of the tower $\mathcal{T}_r$. This tower can be visualized as a rectangle of size $2pn_r \times 2qn_r$ consisting of $2n_r \times 2n_r = 4n_r^2$ *blocks*, each of size $p \times q$; every block consists of $p \cdot q$ *squares* representing sets of the form $\tau^k \sigma^l A_r$ (see Figure 1). The horizontal direction in Figure 1 corresponds to the transformation σ, while the vertical direction corresponds to τ.

The function f_r is defined to be zero outside the tower $\mathcal{T}_r$. On the tower $\mathcal{T}_r$ it is defined to be constant on each square (i.e., on each set $\tau^k \sigma^l A_r \in \mathcal{T}_r$). Moreover, this constant for each square is always chosen to be either $+\alpha_r$, or $-\alpha_r$, where the choice of the sign, plus or minus, depends on the square. Once we describe this choice, the function f_r will be defined completely. This choice is illustrated in Figure 1 and is explained below.

First, we observe that by Lemma 1 it suffices to define f_r on all blocks lying in the bottommost row of blocks or in the leftmost column of blocks (this L-shaped area is shaded on Figure 1). Note also that by Lemma 1 all blocks in this area represent disjoint sets, and the sets represented by different squares in any given block in this area are also disjoint. For every other block in $\mathcal{T}_r$ there is a block in the L-shaped shaded area representing the same set.

Consider the leftmost column of blocks in $\mathcal{T}_r$ (the vertical part of the L-shaped area). It consists of $2n_r$ blocks half of which are below the thick line, and the other half are above it. For each of the n_r blocks in the lower half (below the thick line) we choose the following pattern of pluses and minuses: in every row of squares in the block the plus and minus signs are assigned in an alternative way, starting with the plus on the left. For each of the n_r blocks in the leftmost column which are above the thick line we also assign alternating pluses and minuses in each row, but we start with the minus on the left. Note that, since q is even, every row has the same number of pluses and minuses.

After f_r is defined on the leftmost column of blocks, we define it on the second from the left block in the bottommost row by copying the pattern of pluses and minuses from the topmost block in the leftmost column. On the other blocks in the second from the left column f_r is actually already defined because of (p, q)-periodicity. This allows one to define f_r on the third from the left block in the bottommost row by copying the pattern from the topmost block in the second from the left column of blocks, and so on. This completes the definition of f_r.

Our next step is to estimate the sums $s_n(f_r; \sigma)$ and $s_n(f_r; \tau)$ for fixed r. The sums $s_n(f_r; \sigma)$ will be estimated from above in the uniform norm, while the sums $s_n(f_r; \tau)$ will be estimated from below "in measure". Roughly speaking, these estimates will be obtained from the following feature of the pattern of pluses and minuses in Figure 1. In every row of $\mathcal{T}_r$ the plus signs and the minus signs are put alternatively within each block, while every column of $\mathcal{T}_r$ contains long string of elements of the same sign.

The estimate from above is straightforward:

$$|s_n(f_r; \sigma)| \leq 2\alpha_r, \qquad n = 1, 2, \ldots \tag{7}$$

To get an estimate from below, let $m_r = \left[\frac{n_r}{2}\right]$, and let

$$D_r = \cup_{k=0}^{p-1} \cup_{l=0}^{m_r-1} \tau^k \sigma^l A_r.$$

The set D_r is the set represented by the first m_r bottommost blocks of the leftmost column in Figure 1.

It is easy to check that $\mu(D_r) \geq \frac{1}{8}$, and, for any $x \in D_r$,

$$|s_{m_r}(f_r; \tau)(x)| \geq m_r \alpha_r. \tag{8}$$

Property (3) follows immediately from (7):

$$|s_n(f; \sigma)| \leq \sum_{r=1}^{\infty} |s_n(f_r; \sigma)| \leq 2 \sum_{r=1}^{\infty} \alpha_r =: C.$$

To get (4), we first write, for fixed r,

$$|s_{m_r}(f; \tau)| \geq S - \Sigma_1 - \Sigma_2,$$

where

$$S = |s_{m_r}(f_r; \tau)|,$$

$$\Sigma_1 = \sum_{t=1}^{r-1} |s_{m_r}(f_t; \tau)|,$$

and

$$\Sigma_2 = \sum_{s=r+1}^{\infty} |s_{m_r}(f_s; \tau)|.$$

The sum S has already been estimated in (8). The sum Σ_2 can be estimated in a trivial way:

$$\Sigma_2 \le m_r \sum_{s=r+1}^{\infty} \alpha_s. \tag{9}$$

To estimate Σ_1, one should be a little more careful. Fix t, $1 \le t < r$, and consider the sum

$$s_{m_r}(f_t; \tau)(x) = f_t(x) + \ldots + f_t(\tau^{m_r - 1} x).$$

The orbit segment $\Delta = \{x, \tau x, \ldots \tau^{m_r-1}x\}$ corresponding to this sum can be split into subsegments of three different types. The subsegments of the first type are of the form $\{y, \tau y, \ldots \tau^a y\}$, all of whose points $\tau^i y$, $0 \le i \le a$, are outside the tower $\mathcal{T}_t$. The subsegments of the second type are of the form $\{z, \tau z, \ldots \tau^{N_1^t - 1} z\}$ with the first point z belonging to the bottommost row of $\mathcal{T}_t$, and the last point $\tau^{N_1^t-1}z$ belonging to the topmost row of $\mathcal{T}_t$. In other words, a point in a segment of the second type runs vertically up through an entire column of the tower $\mathcal{T}_t$, from the bottom to the top.

Finally, the segments of the third type (there may be at most two of them) correspond to the situation when a point runs vertically up through *a part* of a column of the tower $\mathcal{T}_t$. This can happen only in the very beginning of the orbit segment Δ (if $x \in \mathcal{T}_t$), or in the very end (if $\tau^{m_r-1}x \in \mathcal{T}_t$).

It follows from the construction of the function f_t that the part of the sum $s_{m_r}(f_t; \tau)$ corresponding to any subsegment of the first type is zero, since $f_t(x) = 0$ if $x \notin \mathcal{T}_t$. A part of this sum corresponding to any subsegment of the second type is also zero, because any column of $\mathcal{T}_t$ contains the same number of pluses and minuses. Therefore, the only contribution to the sum comes from the two segments of the third type, and we get

$$|s_{m_r}(f_t; \tau)| \le 2m_t\alpha_t.$$

This yields

$$\Sigma_1 \le 2 \sum_{t=1}^{r-1} m_t \alpha_t. \tag{10}$$

Combining (8), (9), (10), and taking (5), (6) into account, for every $x \in D_r$ we get

$$|s_{m_r}(f_r; \tau)(x)| \ge (m_r\alpha_r - 2\sum_{t=1}^{r-1} m_t\alpha_t) - m_r \sum_{s=r+1}^{\infty} \alpha_s \ge 2r - 1 \ge r.$$

This proves (4) and completes the proof in the case when q is even.

It remains to consider the case q is odd, $q \geq 3$. In this case the sequences $\{n_r\}$, $\{\alpha_r\}$, as well as the patterns of pluses and minuses in the tower $\mathcal{T}_r$, are defined in the same way. The only change should be made in the definition of the functions $\{f_r\}$.

Fix $r \geq 1$. We still set $f_r = 0$ outside the tower $\mathcal{T}_r$, and the sign of f_r is still determined by the same pattern of pluses and minuses in Figure 1. But we no longer want the absolute value of f_r to be α_r on the entire tower $\mathcal{T}_r$.

Fix a block in $\mathcal{T}_r$, and fix a row in this block. Since q is odd, either this row both starts and ends with the plus sign, or it both starts and ends with the minus sign. In either case we define f_r to be $(\operatorname{sign} f_r) \cdot \frac{\alpha_r}{2}$ on the leftmost and on the rightmost square of the row; on all other squares we define f_r to be $(\operatorname{sign} f_r) \cdot \alpha_r$, as before. It is easy to check that with this definition of f_r our proof of the properties (3) and (4) can be repeated verbatim. □

Proof of Theorem 3. Since, as was earlier mentioned, the proof is essentially the same as the proof of Theorem 2, we only indicate the necessary modifications without repeating the argument.

Note that in our construction of the towers $\mathcal{T}_r$'s we can assume, without loss of generality, that for each r the set A_r (the base of the tower $\mathcal{T}_r$) is closed. Otherwise, using the regularity of the Borel measure μ, we can slightly shrink A_r to make it closed. For $\gamma_r > 0$ let $\tilde{A}_r$ denote the open γ_r-neighborhood of A_r. If γ_r is sufficiently small, the sets $\{\tau^m \sigma^n \tilde{A}_r\}$ satisfy the same disjointness properties as the sets $\{\tau^m \sigma^n A_r\}$, and we can consider the "outer" tower $\tilde{\mathcal{T}}_r = \bigcup_{k=0}^{N_1} \bigcup_{l=0}^{N_2} \tau^k \sigma^l \tilde{A}_r$.

The function f_r is defined on $\mathcal{T}_r = \bigcup_{k=0}^{N_1} \bigcup_{l=0}^{N_2} \tau^k \sigma^l A_r$ exactly as before. Outside the outer tower $\tilde{\mathcal{T}}_r$ it is defined to be zero, and we use the Urysohn theorem to extend f_r to the entire space X so that $\|f_r\|_C \leq \alpha_r$. Then for $f = \sum_{r=1}^{\infty} f_r$ we get (3) and (4) as before, and (4) implies that $f \notin \operatorname{Cob}(\tau)$. By the Gottschalk-Hedlund theorem [GoHed], (3) implies that $f \in \operatorname{Cob}_C(\sigma)$.

Therefore, by argument of the proof of Theorem 2, if $\operatorname{Cob}_C(\sigma) \subseteq \operatorname{Cob}_C(\tau)$, then for some n we have $\sigma x = \tau^n x$ for μ-a.e. $x \in X$. Since the support of an invariant measure of a minimal homeomorphism must coincide with the entire space X, the equality $\sigma = \tau^n$ holds everywhere. □

3. Concluding remarks and open questions

The assumption of commutativity of τ and σ in Theorems 1–3 cannot be dropped. In the measurable case (Theorems 1 and 2) this is an immediate corollary of the following result of K. Dajani, whose proof is based on Schmidt's criterion for coboundaries.

THEOREM [D]. *Suppose τ and σ are two ergodic measure preserving transformations of a Lebesgue probability space X, and σ is a generalized power of τ, i.e., $\sigma x = \tau^{m(x)} x$, $x \in X$, where $m = m(x)$ is measurable. If $m \in L^1(X)$, then* $\operatorname{Cob}(\sigma) \subseteq \operatorname{Cob}(\tau)$.

By ergodicity, the transformations τ and $\sigma = \tau^{m(x)}$ can commute only if $m(x)$ is constant a.e. This gives many examples of non-commuting transformations τ, σ having the same coboundaries (τ can be taken arbitrary, and σ of the form $\tau^{m(x)}$ with, say, bounded and non-constant function m).

This argument admits a natural topological version for minimal homeomorphisms of the Cantor set C. Using this version, one can produce examples of non-commuting minimal homeomorphisms τ and σ of the set C preserving the same Borel probability measure on it and such that $\mathrm{Cob}_C(\tau) = \mathrm{Cob}_C(\sigma)$. Therefore, the condition of commuting cannot be dropped in Theorem 3 either.

I am grateful to D. Volny for attracting my attention to the paper [D]. Actually, without using the result of [D], D. Volny and T. de la Rue gave a simple example showing that Theorem 1 cannot be true without the assumption of commutativity.

Question 1. Suppose τ and σ are two ergodic transformations of a probability space, and $\mathrm{Cob}(\sigma) \subseteq \mathrm{Cob}(\tau)$. Are there any algebraic conditions weaker than commutativity (amenability, solvability, nilpotency, etc) on the group generated by τ and σ, under which one must have $\sigma = \tau^n$ for some $n \in \mathbb{Z}$?

Another direction for further work is to establish to what extent the sets $\mathrm{Cob}(\tau)$ can be different for different τ's. To formulate this question more precisely, we first make the following simple observation.

Suppose that two ergodic transformations, τ and σ, commute. Then the set $\mathrm{Cob}_p(\tau) \cap \mathrm{Cob}_p(\sigma)$ consisting of their joint L^p-coboundaries is dense in L_0^p, for $1 \leq p < \infty$. Indeed, any function $f \in L^p$ with $\int f = 0$ can be approximated in L^p by a function f_1 which is a coboundary for σ, $f_1 = g - g \circ \sigma$, with $g \in L_0^p$. Approximating g, in turn, by a function g_1 which is a coboundary for τ, $g_1 = h - h \circ \tau$, we get a function f_2, $f_2 = h - h \circ \sigma - h \circ \tau + h \circ \tau \circ \sigma$, which is L^p-close to f, and which is obviously a coboundary for both τ and σ (due to commutativity). It is clear that the same argument can be applied to any finite collection of commuting transformations.

Question 2. Suppose τ and σ are two ergodic, not necessarily commuting, transformations of a probability space, and p is fixed, $1 \leq p < \infty$. Is it true that the set $\mathrm{Cob}_p(\tau) \cap \mathrm{Cob}_p(\sigma)$ is dense in L_0^p?

If the answer to Question 2 is negative, it would be natural to ask how small the intersection $\mathrm{Cob}(\tau) \cap \mathrm{Cob}(\sigma)$ can be.

Question 3. Are there two ergodic transformations of a probability space such that $\mathrm{Cob}(\tau) \cap \mathrm{Cob}(\sigma)$ consists of the zero function only?

It is known that for a single ergodic transformation τ of a probability space, the set $\mathrm{Cob}(\tau) \cap L_0^\infty$ is dense in the set L_0^∞ *with the* L^∞*-topology* ([K], [OSm], see also

[KoKr]). Let now τ and σ be two ergodic transformations of the same probability space. The answer to the following question seems to be unknown even if τ and σ commute.

Question 4. Is it true that $\mathrm{Cob}(\tau) \cap \mathrm{Cob}(\sigma) \cap L_0^\infty$ is dense in the set L_0^∞ with the L^∞-topology?

Finally, let us consider a "more invariant" situation when τ and σ are not necessarily defined on the same space.

Using the theorem of H. Dye on orbit equivalence, A. M. Stepin [St] showed that for any two approximately finite countable ergodic measure preserving groups of automorphisms of non-atomic Lebesgue spaces their cohomology groups are isomorphic (as abstract groups). In particular, the cohomology groups of any two invertible ergodic transformations τ and σ, acting on the Lebesgue spaces (X, μ) and (Y, ν) respectively, are isomorphic. A natural linear correspondence $\mathcal{L}$: $\mathcal{M}(X) \to \mathcal{M}(Y)$ between the spaces of measurable functions on X and Y (coming from the orbit equivalence of τ and σ) was defined in [St], under which $\mathcal{L}(\mathrm{Cob}(\tau)) = \mathrm{Cob}(\sigma)$.

The operator $\mathcal{L}$, however, need not be associated with any measure preserving isomorphism between the spaces X and Y (and it cannot be of such form if, say, τ is weak mixing, but σ is not). Let us say that the transformations τ and σ acting on (X, μ) and (Y, ν) respectively, are *CB-equivalent* (CB stands for coboundaries) if there exists a measure preserving isomorphism θ: $X \to Y$ taking τ-coboundaries onto σ-coboundaries, i.e., $f \in \mathrm{Cob}(\sigma)$ if and only if $f \circ \theta \in \mathrm{Cob}(\tau)$.

It seems interesting to understand the relationship between the notion of CB-equivalence and other notions of equivalence of measure preserving transformations. Following [GPS], let us call τ and σ *flip conjugate* if σ is isomorphic either to τ, or to τ^{-1} (or to both). If τ and σ are flip conjugate, then they are obviously CB-equivalent.

Question 5. Suppose τ and σ are CB-equivalent. Is it true that they are flip conjugate? Is it true that they are spectrally isomorphic?

It is known ([Ha], proof of Theorem 2) that the eigenvalues of a transformation can be expressed in terms of the set of its coboundaries (taking two values mod 0). This means that for the transformations with pure point spectrum the answer to Question 5 is positive. Moreover, in this case, if the space X is Lebesgue, the von Neumann pure point spectrum theorem implies that τ and σ are isomorphic. This observation concerning the pure point spectrum was also made by I. Assani (personal communication).

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