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THE RANKIN-SELBERG METHOD ON CONGRUENCE SUBGROUPS

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ABSTRACT. A modified Rankin-Selberg integral is defined for the automorphic functions not of rapid decay on the congruence subgroup of $SL(2, Z)$. Analytic continuation and functional equations are obtained for this Rankin-Selberg integral.

1. Introduction

The Rankin-Selberg method was discovered independently by Rankin and Selberg. Rankin's paper [5] and Selberg's paper [6] both address the following situation. Briefly speaking, let $f(z) = \sum A(n)q^n$ and $g(z) = \sum B(n)q^n$ where $q = e^{2\pi iz}$ are modular forms; then, as the result of their "trick", which is now known as the Rankin-Selberg method, $\sum A(n)B(n)n^{-s}$ has an analytic continuation and a functional equation. Rankin and Selberg used these two basic facts to give bounds to the Fourier coefficients of the modular forms.

The idea of the Rankin-Selberg method is to seek the representation of an L-function as an integral of one or more automorphic forms against an Eisenstein series. The Eisenstein series has a functional equation and so if the L-function can be represented as such an integral, it inherits this functional equation.

Since then, the method have been greatly generalized, and many applications have been found for those generalization, such as the non-vanishing theorems, Doi-Naganuma lifting and the Shimura correspondence.

In 1981, Zagier [7] first extended the method and gave a clear formulation to the modular forms of the full modular group which are not of rapid decay.

Zagier's proof involves the application of the "folding-unfolding" trick to a truncated domain. His idea is particularly useful when applied to the metaplectic Eisenstein series and the theta series. These series are automorphic forms which are not of rapid decay. Their Fourier expansion often involves Dirichlet series. Hence we can obtain number theoretic information. For example, see [1], [3], [4].

In this work, we give a generalized theorem, an extension of Zagier's theorem to the congruence subgroup of $SL(2, Z)$. Our method of proof will not use the truncated domains and hence is more straightforward and hopefully this will help us enhance the understanding of the Rankin-Selberg method.

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2. The Main Theorem

Let $SL(2, R)$ operate on $\mathcal{H}$, the upper half plane, in the usual way. Let Γ be a congruence subgroup of $SL(2, Z)$ and

$$\Gamma_\infty = \{g \in \Gamma \mid g(\infty) = \infty\} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in Z \right\}.$$

In this case, Γ is said to be reduced at infinity.

Let $\kappa_1 = \infty, \kappa_2, \ldots, \kappa_h$ be the non equivalent cusps of Γ. Let $\Gamma_i = \{\sigma \in \Gamma \mid \sigma\kappa_i = \kappa_i\}$. Then there is an $\alpha_i \in SL(2, Q)$ for each cusp κ_i, such that $\alpha_i\infty = \kappa_i$ and $\alpha_i^{-1}\Gamma_i\alpha_i = \Gamma_\infty$. In particular we choose $\alpha_1 = I_{2\times 2}$.

Here are some basic facts of Eisenstein series. For each cusp κ_i, we may define an Eisenstein series as

$$E_i(z, s) = \sum_{\delta \in \Gamma_i \backslash \Gamma} y^s(\alpha_i^{-1}\delta z),$$

where $y(x + iy) = y$. Set

$$\vec{E}(z, s) = \begin{bmatrix} E_1(z, s) \\ E_2(z, s) \\ \vdots \\ E_h(z, s) \end{bmatrix}_{h\times 1}.$$

Then there is a matrix of functions, often called the scattering matrix, denoted by $\Phi(s)$, such that $\vec{E}(z, s)$ has the functional equation

$$\begin{aligned} \vec{E}(z, s) &= \Phi(s)\vec{E}(z, 1 - s), \\ \Phi(s)\Phi(1 - s) &= I_{h\times h} \end{aligned} \tag{1}$$

where

$$\Phi(s) = (\phi_{ij})_{h\times h},$$

$$\phi_{ij} = \pi^{1/2}\frac{\Gamma(s - 1/2)}{\Gamma(s)} \cdot \sum_{c>0} \frac{1}{|c|^{2s}} \sum_{\substack{d(\mathrm{mod}\, c) \\ (c\ d) \in \alpha_i^{-1}\Gamma\alpha_j}} 1.$$

Note that $E_1(z, s)$ is the Eisenstein series at the cusp ∞. We will write this as $E_\infty(z, s)$. For this reason, we will use the subindex ∞ instead of 1 for corresponding quantities, such as the constant term of its Fourier expansion. For more details concerning $\vec{E}(z, s)$ and $\Phi(s)$, see [2].

Let $F(z)$ be a continuous function invariant under the action of Γ. For each cusp κ of Γ, there is an α such that $\alpha\infty = \kappa$, $\alpha^{-1}\Gamma_\kappa\alpha = \Gamma_\infty$. For this α, define a new function

$$f(z) = F(\alpha z).$$

Then it is easily checked that $f(z)$ has period 1. Thus it has a Fourier expansion of the form

$$f(z) = \sum_{m \in Z} a_m^{(\kappa)}(y) e(mx).$$

We will define the above expansion of $f(z)$ to be the Fourier expansion of $F(z)$ at the cusp κ.

Further suppose that

$$f(z) = \psi_\kappa(y) + O(y^{-N}) \quad \forall N \text{ as } \operatorname{Im} z \to \infty, \tag{2}$$

where $\psi_\kappa(y)$ is a function of the form

$$\psi_\kappa(y) = \sum_{j=1}^{l} \frac{c_{ij}}{n_{ij}} y^{\alpha_{ij}} \log^{n_{ij}} y, \tag{3}$$

where $c_{ij},\ \alpha_{ij} \in C,\ n_{ij} \in N \geq 0, \kappa = \kappa_i$.

We will define the Rankin-Selberg transform of F at the cusp κ as

$$\begin{aligned} R_\kappa(F, s) &= \int_0^\infty \left(a_0^\kappa(y) - \psi_\kappa(y)\right) y^{s-2} dy \\ &= \int_0^\infty \int_0^1 \left(F(\alpha z) - \psi_\kappa(y)\right) y^s d\mu, \end{aligned} \tag{4}$$

where $d\mu = \frac{dx\,dy}{y^2}$. Thus $R_\kappa(F, s)$ converges absolutely for Re s sufficiently large.

We will state our main theorem now.

THEOREM. *If $R_\kappa(F, s)$ is defined as above, then $R_\kappa(F, s)$ has a meromorphic continuation to all s, the only possible poles being at $s = 0,\ 1,\ \alpha_{ij},\ 1 - \alpha_{ij}$ and $\rho/2$, where ρ 's are the nontrivial zeros of the Riemann zeta function.*

Further we have the following functional equation:

$$\vec{R}(F, s) = \begin{pmatrix} R_1(F, s) \\ R_2(F, s) \\ \vdots \\ R_h(F, s) \end{pmatrix} = \Phi(s) \begin{pmatrix} R_1(F, 1-s) \\ R_2(F, 1-s) \\ \vdots \\ R_h(F, 1-s) \end{pmatrix} = \Phi(s) \vec{R}(F, 1-s). \tag{5}$$

Proof. Let $\mathcal{D} = \{z = x + iy \in \mathcal{H} \mid |x| \leq 1/2,\ |z| \geq 1\}$ be the fundamental domain of the upper half plane under the action of $SL(2, Z)$.

For our convenience later, let

$$\tilde{\mathcal{D}} = \Gamma_\infty \backslash \mathcal{H} - \mathcal{D} = \bigcup_{\substack{\gamma \in \Gamma_\infty \backslash SL(2,Z) \\ \gamma \neq I}} \gamma \mathcal{D} = \{z = x + iy \in \mathcal{H} \mid |x| \leq 1/2,\ |z| < 1\}.$$

Let $\mathcal{D}_\Gamma$ be the fundamental domain of the upper half plane $\mathcal{H}$ under the action of Γ. Let $\alpha_i \in SL(2, Q)$ be defined as before for each cusp κ_i of Γ, that is $\alpha_i(\infty) = \kappa_i$, and $\alpha_i^{-1}\Gamma_i\alpha_i = \Gamma_\infty$ where $\Gamma_i = \{\sigma \in \Gamma \mid \sigma\kappa_i = \kappa_i\}$.

It is easy to show that $\mathcal{D}_\Gamma = \bigcup_{i=1}^h \alpha_i\mathcal{D}$ where $\alpha_i\mathcal{D}$ is the image of $\mathcal{D}$ under the action of α_i.

Let us consider first the Rankin-Selberg transform of F at the cusp ∞. We have

$$
\begin{aligned}
(6) \qquad R_\infty(F, s) &= \int_0^\infty \int_0^1 [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \\
&= \int\int_{\Gamma_\infty\backslash\mathcal{H}} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \\
&= \int\int_{\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{D}_\Gamma} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \\
&= \int\int_{\bigcup_{i=1}^h\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\alpha_i\mathcal{D})} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \\
&= \sum_{i=1}^h \left(\int\int_{\bigcup_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \alpha_i\neq I \text{ or } \gamma\neq I}}\gamma(\alpha_i\mathcal{D})} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \right) \\
&\quad + \int\int_{\mathcal{D}} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \\
&= I_2 + I_1
\end{aligned}
$$

where $I_1 = \int\int_{\mathcal{D}} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2}$ converges for any s, since $F(z) - \psi_\infty(y)$ is of rapid decay when $y \to \infty$.

For I_2, we have

$$
\begin{aligned}
(7) \quad I_2 &= \sum_{i=1}^h \left(\int\int_{\bigcup_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \alpha_i\neq I \text{ or } \gamma\neq I}}\gamma(\alpha_i\mathcal{D})} [F(z) - \psi_\infty(y)]\, y^s \frac{dx\,dy}{y^2} \right) \\
&= \sum_{i=1}^h \left(\int\int_{\bigcup_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \alpha_i\neq I \text{ or } \gamma\neq I}}\gamma(\alpha_i\mathcal{D})} F(z) y^s \frac{dx\,dy}{y^2} \right) \\
&\quad - \sum_{i=1}^h \left(\int\int_{\bigcup_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \alpha_i\neq I \text{ or } \gamma\neq I}}\gamma(\alpha_i\mathcal{D})} \psi_\infty(y) y^s \frac{dx\,dy}{y^2} \right) \\
&= \sum_{i=1}^h \int\int_{\mathcal{D}} F(\alpha_i z)\left(E_\infty(\alpha_i z, s) - \delta_{i\infty} y^s\right) d\mu - \int\int_{\tilde{\mathcal{D}}} \psi_\infty(y) y^s \frac{dx\,dy}{y^2}.
\end{aligned}
$$

Let $e_{i\infty}$ be the constant term of the expansion of $E_\infty(z, s)$ at the cusp κ_i, which is defined as

$$e_{i\infty} = \int_0^1 E_\infty(\alpha_i z, s)dx.$$

It is easily verified that for $s \neq 0, 1/2, 1$,

$$e_{i\infty} = \delta_{i\infty} y^s + \phi_{i\infty} y^{1-s}, \tag{8}$$

where $\phi_{i\infty}$ are entries of the scattering matrix $\Phi(s)$ defined as before.

For for each i, $E_\infty(\alpha_i z, s) - e_{i\infty}$ is of rapid decay, as $y \to +\infty$. And

$$E_\infty(\alpha_i z, s) - \delta_{i\infty} y^s$$

is of slow decay for Re s sufficiently large.

Also $\psi_\infty(y)$ is of the form as in (1.3), hence both integrals in I_2 converges for Re s sufficiently large.

Now let us rearrange the first term of (7), so that it has a functional equation. We have

$$\begin{aligned} I_2 = &\sum_{i=1}^h \int\int_{\mathcal{D}} F(\alpha_i z)(E_\infty(\alpha_i z, s) - e_{i\infty})d\mu \\ &+ \sum_{i=1}^h \int\int_{\mathcal{D}} F(\alpha_i z)\phi_{i\infty} y^{1-s} d\mu - \int\int_{\tilde{\mathcal{D}}} \psi_\infty(y) y^s \frac{dx\,dy}{y^2}. \end{aligned} \tag{9}$$

Let $I_{\infty,F}(s)$ be the first term in the above sum for I_2, which converges absolutely for all s. Now let us write the second term of the above sum as

$$\begin{aligned} &\sum_{i=1}^h \int\int_{\mathcal{D}} (F(\alpha_i z) - \psi_i(y))\phi_{i\infty} y^{1-s} d\mu + \sum_{i=1}^h \int\int_{\mathcal{D}} \psi_i(y)\phi_{i\infty} y^{1-s} d\mu \\ &= \sum_{i=1}^h \int\int_{\mathcal{D}} (F(\alpha_i z) - \psi_i(y)) e_{i\infty} d\mu \\ &\quad - \int\int_{\mathcal{D}} (F(z) - \psi_\infty(y))\, y^s d\mu \\ &\quad + \sum_{i=1}^h \int\int_{\mathcal{D}} \psi_i(y)\phi_{i\infty} y^{1-s} d\mu. \end{aligned} \tag{10}$$

Substituting (10) into (9), we have

$$\begin{aligned} I_2 = &I_{\infty,F}(s) + I_{\infty,F,\psi}(s) - \int\int_{\tilde{\mathcal{D}}} \psi_\infty(y) y^s \frac{dx\,dy}{y^2} \\ &+ \sum_{i=1}^h \int\int_{\mathcal{D}} \psi_i(y)\phi_{i\infty} y^{1-s} d\mu - \int\int_{\mathcal{D}} (F(z) - \psi_\infty(y)) y^s d\mu. \end{aligned} \tag{11}$$

where $I_{\infty,F,\psi}(s)$ is the first term on the RHS of (10), which converges absolutely for all s.

Now, substituting (11) into (6), and cancelling out I_1, we have

$$R_\infty(F,s) = I_{\infty,F}(s) + I_{\infty,F,\psi}(s) + I_{\infty,\psi}(s) \tag{12}$$

where

$$I_{\infty,\psi}(s) = \sum_{i=1}^{h} \int\int_{\mathcal{D}} \psi_i(y)\phi_{i\infty}y^{1-s}d\mu - \int\int_{\tilde{\mathcal{D}}} \psi_\infty(y)y^s\frac{dx\,dy}{y^2}$$

which converges absolutely for Re s sufficiently large.

Now let us consider the situation at a general cusp κ. By definition, the Fourier expansion of $F(z)$ at κ is the same as the Fourier expansion of $f(z) = F(\alpha z)$ at ∞ where $\alpha\infty = \kappa$, $\alpha^{-1}\Gamma_\kappa\alpha = \Gamma_\infty$. Let the constant term in the expansion be denoted by $a_0^\kappa(y)$. We have

$$\begin{aligned} R_\infty(f,s) = R_\kappa(F,s) &= \int_0^\infty\int_0^1 (F(\alpha_\kappa z) - \psi_\kappa(y))\, y^s d\mu \\ &= \int_0^\infty (a_0^\kappa(y) - \psi_\kappa(y))y^s\frac{dy}{y^2}. \end{aligned}$$

The function $f(z) = F(\alpha z)$ is invariant under the action of $\alpha^{-1}\Gamma\alpha = \tilde{\Gamma}$.

Let $\mathcal{D}_{\tilde{\Gamma}} = \mathcal{D}_{\alpha^{-1}\Gamma\alpha} = \alpha^{-1}\mathcal{D}_\Gamma$. Thus the cusps of $\tilde{\Gamma}$ are $\{\tilde{\kappa_i}\} = \{\alpha^{-1}\kappa_i\}$. Thus $\tilde{\alpha_i}\infty = \tilde{\kappa_i}$, where $\tilde{\alpha_i} = \alpha^{-1}\alpha_i$. Consider

$$\begin{aligned} \tilde{E_\infty}(\tilde{\alpha_i}z,s) &= \sum_{\gamma\in\Gamma_\infty\backslash\alpha^{-1}\Gamma\alpha} \operatorname{Im}(\gamma\tilde{\alpha_i}z)^s \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\alpha^{-1}\Gamma\alpha} \operatorname{Im}(\alpha^{-1}\alpha\gamma\alpha^{-1}\alpha_i z)^s \\ &= \sum_{\gamma\in\Gamma_\kappa\backslash\Gamma} \operatorname{Im}(\alpha^{-1}\gamma\alpha_i z)^s \\ &= E_\kappa(\alpha_i z, s). \end{aligned} \tag{13}$$

And $f(\tilde{\alpha_i}z) = F(\alpha\tilde{\alpha_i}z) = F(\alpha\alpha^{-1}\alpha_i z) = F(\alpha_i z)$.
Hence

$$\begin{aligned} \tilde{\psi_i}(y) &= \text{polynomial part of the constant term of } f(\tilde{\alpha_i}z) \\ &= \text{polynomial part of the constant term of } F(\alpha_i z) \\ &= \psi_i(y). \end{aligned}$$

Also

$$\begin{aligned} \tilde{e}_{i\infty} &= \text{constant term of } \tilde{E_\infty}(\tilde{\alpha}_i z) \\ &= \text{constant term of } E_\infty(\alpha_i z) \\ &= e_{i\kappa}. \end{aligned}$$

Thus for any cusp κ, by (12) we have

$$R_\kappa(F,s) = I_{\kappa,F}(s) + I_{\kappa,F,\psi}(s) + I_{\kappa,\psi}(s), \tag{14}$$

where

$$\begin{aligned} R_\kappa(F,s) &= \int_0^\infty \int_0^1 [F(\alpha z) - \psi_\kappa(y)] y^s \frac{dx\,dy}{y^2}, \\ I_{\kappa,F}(s) &= \sum_{i=1}^h \int\int_{\mathcal{D}} F(\alpha_i z)(E_\kappa(\alpha_i z, s) - e_{i\kappa}) d\mu, \\ I_{\kappa,F,\psi}(s) &= \sum_{i=1}^h \int\int_{\mathcal{D}} F(\alpha_i z) - \psi_i(y)) e_{i\kappa} d\mu, \\ I_{\kappa,\psi}(s) &= \sum_{i=1}^h \int\int_{\mathcal{D}} \psi_i(y)\phi_{i\kappa} y^{1-s} d\mu - \int\int_{\tilde{\mathcal{D}}} \psi_\kappa(y) y^s \frac{dx\,dy}{y^2}. \end{aligned}$$

Now to convert to a matrix form, let

$$\begin{aligned} \vec{R}(F,s) &= (R_\kappa(F,s))_{h\times 1}, \\ \vec{I}_F(s) &= \left(I_{\kappa,F}(s)\right)_{h\times 1}, \\ \vec{I}_{F,\psi}(s) &= \left(I_{\kappa,\psi}(s)\right)_{h\times 1}, \\ \vec{I}_\psi(s) &= \left(I_{\kappa,\psi}(s)\right)_{h\times 1} \\ \vec{e}_i &= (e_{i\kappa})_{h\times 1}. \end{aligned}$$

Then the equation (14) becomes

$$\vec{R}(F,s) = \vec{I}_F(s) + \vec{I}_{F,\psi}(s) + \vec{I}_\psi(s). \tag{15}$$

To prove the functional equation of $\vec{R}(F,s)$, it is sufficient to do so for each term on the righthand side of (15).

Note that

$$\vec{e}_i(y,s) = (e_{i\kappa})_{h\times 1} = (y^s + \Phi(s)y^{1-s})\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}.$$

Thus

$$\begin{aligned}\Phi(s)\vec{e}_i(y,1-s) &= \Phi(s)(y^{1-s} + \Phi(1-s)y^s)\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\\ &= (y^s + \Phi(s)y^{1-s})\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\\ &= \vec{e}_i(y,s).\end{aligned}$$

Thus, each term in the sum for $\vec{I}_{F,\psi}(s)$ has the functional equation. Since, $\Phi(s)\vec{E}(z,1-s) = \vec{E}(z,s)$, the function $\vec{I}_F(s)$ has the same functional equation too. Both $\vec{I}_{F\psi}(s)$ and $\vec{F}(s)$ converges absolutely for all s.

To complete the proof, write

$$\vec{I}_\psi(s) = \sum_{i=1}^{h} \vec{I}_{\psi_i}(s)$$

where

$$\vec{I}_{\psi_i}(s) = \int\int_{\mathcal{D}} \psi_i(y)\begin{pmatrix}\vdots\\ \phi_{i\kappa}\\ \vdots\end{pmatrix}_{h\times 1} y^{1-s}d\mu - \int\int_{\tilde{\mathcal{D}}} \psi_i(y)\begin{pmatrix}\vdots\\ \delta_{i\kappa}\\ \vdots\end{pmatrix}_{h\times 1} y^s d\mu$$

which converges absolutely for Re s sufficiently large.

Now consider

$$-\int\int_{\tilde{\mathcal{D}}} \psi_i(y)\begin{pmatrix}\vdots\\ \phi_{i\kappa}\\ \vdots\end{pmatrix}_{h\times 1} y^{1-s}d\mu + \int\int_{\tilde{\mathcal{D}}} \psi_i(y)\begin{pmatrix}\vdots\\ \delta_{i\kappa}\\ \vdots\end{pmatrix}_{h\times 1} y^s d\mu$$

which converges absolutely for Re s sufficiently small. It is easy to see its meromorphic continuation equals the meromorphic continuation of $\vec{I}_{\psi_i}(s)$. Further, as

$$\Phi(s)\begin{pmatrix} \vdots \\ \phi_{i\kappa} \\ \vdots \end{pmatrix}_{h\times 1}(1-s) = \Phi(s)\Phi(1-s)\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = I_{h\times h}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \vdots \\ \delta_{i\kappa} \\ \vdots \end{pmatrix}_{h\times 1}$$

for each i, we see that

$$\Phi(s)\vec{I}_{\psi_i}(1-s) = \vec{I}_{\psi_i}(s).$$

Thus we have meromorphic continuation and functional equations for $\vec{I}_{\psi}(s)$. To obtain the claims about the poles, we combine (3) and (14).

Thus the proof is complete.

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