ON THE PROJECTIVE PLANE OF AN H-SPACE[1]

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1. Introduction

Let G be a topological group. Milnor [8] defines a sequence of principal G-bundles (E_n, B_n, G) $(1 \leqq n < \infty)$ such that

$$SG = B_1 \subset B_2 \subset \cdots \subset B_\infty = B_G,$$

where SG denotes the suspension of G and B_G is a classifying space for G. The work of Borel [3], [4] gives relations between the cohomology of G and that of B_G, whereas Rothenberg [10] investigates the cohomology of the spaces B_n.

Suppose now that X is an H-space, that is, X has a continuous multiplication with unit. One then may not be able to define a classifying space B_X, but Stasheff [11] has defined the *projective plane* of X, $P_2 X$, which has the homotopy type of the space B_2 in case X is actually a group. The purpose of this paper is to discuss the relationship between the cohomology of X and that of $P_2 X$.

Consider commutative, associative, and graded algebras A over a field k such that A_0 is isomorphic to k. We denote the ideal of positive-dimensional elements by $\bar{A}$ and set

$$D^1 A = \bar{A}, \qquad D^n A = D^{n-1} A \cdot \bar{A} \quad (n \geqq 2).$$

We call $D^2 A$ the ideal of decomposable elements. If A and B are two algebras, we define their tensor product $A \otimes B$ in the usual way with grading

$$(A \otimes B)_k = \textstyle\sum_{i+j=k} A_i \otimes B_j \qquad (k \geqq 0).$$

Let X be an H-space. For the rest of the paper we shall assume that X is arcwise connected and that the integral singular homology groups of X are finitely generated in each dimension. Now take singular cohomology with coefficients in a fixed field k. Recall that an element $u \in H^*(X)$ is called primitive if

$$m^* u = \pi_1^* u + \pi_2^* u,$$

where m^*, π_i^* $(i = 1, 2)$ are the homomorphisms induced by the maps from $X \times X$ to X given respectively by the multiplication and the projection on the i^{th} factor. In §3 we define a (group) homomorphism

$$\iota : H^{q+1}(P_2 X) \to H^q(X) \qquad (q > 0)$$

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such that

$$\text{Image } \iota = P(H^*(X)), \text{ the subspace of primitive classes in } H^*(X).$$

Let P^- and P^+ denote respectively the subspaces of odd- and even-dimensional primitive classes of $\bar{H}^*(X)$, where $\bar{H}^*(X)$ denotes the positive-dimensional cohomology of X. Let $\{u_i\}$, $\{v_j\}$ be bases for P^-, P^+ respectively, and choose classes $\{y_i\}$, $\{z_j\}$ in $\bar{H}^*(P_2 X)$ so that

$$\iota y_i = u_i, \qquad \iota z_j = v_j .$$

We shall prove

THEOREM (1.1). *Let X be an H-space such that the algebra $H^*(X)$ is primitively generated. Then there is an ideal S in $H^*(P_2 X)$ such that*

$$\iota S = 0, \qquad S \cdot \bar{H}^*(P_2 X) = 0,$$

and one has the following k-algebra splitting:

$$H^*(P_2 X) = (A/D^3A) \oplus S,$$

where

$$A = \otimes_i k[y_i] \otimes_j \Lambda(z_j) \quad \textit{if characteristic } k \neq 2,$$

$$A = \otimes_i k[y_i] \otimes_j k[z_j] \quad \textit{if characteristic } k = 2.$$

Moreover if $k = Z_p$ (p a prime), there is a group splitting

$$H^*(P_2 X) = (\tilde{A}/D^3\tilde{A}) \oplus \tilde{S},$$

where $\tilde{A} = \otimes_i k[y_i]$ and $\tilde{S}$ is an $\hat{\mathfrak{A}}_p$-module. Therefore $\tilde{A}/D^3\tilde{A}$ can be given the structure of an $\mathfrak{A}_p$-algebra.

Here $\mathfrak{A}_p$ denotes the mod p Steenrod algebra, and $\hat{\mathfrak{A}}_p$ the subalgebra of $\mathfrak{A}_p$ generated by the operations $\mathcal{P}^i$ ($i \geqq 0$). (Recall that $\mathcal{P}^i = \mathrm{Sq}^{2i}$, if $p = 2$).

If X is a group and if $H^*(X)$ is an exterior algebra, then (1.1) is a special case of the results obtained in [10].

The theorem (for the case $k = Z_2$) has the following applications. In [13] the group splitting given at the end of (1.1) is used to study the behaviour of the Steenrod squares in the mod 2 cohomology of an H-space satisfying the hypotheses of (1.1). In particular it is shown that the primitive classes whose dimensions are one less than a power of two form a set of generators for $H^*(X)$ as an $\mathfrak{A}_2$-algebra. In [14] this result is combined with (1.1) to show that if an H-space (satisfying the hypotheses of (1.1)) has no 2-torsion, then its lowest positive-dimensional rational cohomology occurs in dimension 1, 3, or 7.

The remainder of the paper is devoted to the proof of (1.1).

2. The space $E_1 X$

Let X be an H-space with multiplication m. Denote by CX the (reduced) cone on X, which we think of as the space obtained from $[0, 1] \times X$ by identifying $\{0\} \times X$ and $[0, 1] \times \{e\}$ with a point $*$ (e denotes the unit of X). Following Stasheff [11] we define $E_1 X$ to be the space obtained from the disjoint union of $X \times CX$ and X by identifying $(x, (1, y))$ with $m(x, y)$: that is,

$$E_1 X = (X \times CX) \cup_m X.$$

Let SX denote the (reduced) suspension of X, obtained from CX by identifying $\{1\} \times X$ with a basepoint $*$. We define

$$p : E_1 X \to SX$$

by $p(x, (t, y)) = (t, y)$.

Now $E_1 X$ may be regarded as the total space of a proper triad [6] $(E_1 X, M_1, M_m)$, where M_1 and M_m denote respectively the mapping cylinders of π_1 and of m. That is,

$$M_1 = \{(x, (t, y)) \in E_1 X, 0 \leqq t \leqq \tfrac{1}{2}\},$$

$$M_m = \{(x, (t, y)) \in E_1 X, \tfrac{1}{2} \leqq t \leqq 1\},$$

so that

$$M_1 \cup M_m = E_1 X, \qquad M_1 \cap M_m = X \times X.$$

Take cohomology in the field k, and denote by Δ the Mayer-Vietoris coboundary [6, Chapter I, §15] from $H^q(X \times X)$ to $H^{q+1}(E_1 X)$ $(q > 0)$. From the exactness of the Mayer-Vietoris sequence it follows that the kernel of Δ is the subspace of $H^*(X \times X)$ spanned by $m^* \bar{H}^*(X)$ and $\pi_1^* \bar{H}^*(X)$. Since k is a field and X has homology of finite type, $H^*(X \times X) \approx H^*(X) \otimes H^*(X)$; and a simple argument shows that Δ restricted to $\bar{H}^*(X) \otimes \bar{H}^*(X)$ is an isomorphism:

$$\Delta : \bar{H}^*(X) \otimes \bar{H}^*(X) \approx \bar{H}^*(E_1 X). \tag{2.1}$$

Let $X \# X$ denote the collapsed product of X, which is obtained from the Cartesian product $X \times X$ by identifying the axes $X \vee X$ to a point. Let

$$\eta^* : H^*(X \# X) \to H^*(X \times X)$$

denote the homomorphism induced by the projection $X \times X \to X \# X$. By the Künneth formula, η^* is a monomorphism, and its image is a direct summand of $H^*(X \times X)$:

$$\eta^* H^q(X \# X) = \textstyle\sum_{i+j=q} \bar{H}^i(X) \otimes \bar{H}^j(X) \qquad (q > 0).$$

Thus

$$\bar{H}^*(X \# X) \approx \bar{H}^*(X) \otimes \bar{H}^*(X),$$

and therefore from (2.1) we obtain[2]

$$\Delta \circ \eta^* : \bar{H}^*(X \# X) \approx \bar{H}^*(E_1 X). \tag{2.2}$$

[2] The existence of such an isomorphism follows from the fact that $E_1 X$ has the homotopy type of $X * X$ (where $*$ denotes the join), but we shall find it convenient to have this specific form of the isomorphism.

We define a homomorphism

$$\phi : \bar{H}^*(X) \to \bar{H}^*(X \# X) \tag{2.3}$$

by requiring that $\eta^* \circ \phi = m^* - \pi_1^* - \pi_2^*$. Since

$$\text{Image } (m^* - \pi_1^* - \pi_2^*) \subset \text{Image } \eta^*,$$

and since η^* is a monomorphism, ϕ is well defined.

We also can regard the suspension SX as the total space of a proper triad (SX, C_0, C_1), where C_0 and C_1 denote respectively the upper and lower cones of SX. Thus,

$$C_0 \cup C_1 = SX, \qquad C_0 \cap C_1 = X.$$

We can take the map p to be a triad map $(E_1 X, X, M_1, M_m) \to (SX, C_0, C_1)$, and then

$$p \mid X \times X = \pi_2 : X \times X \to X.$$

Therefore we obtain the following commutative cohomology diagram, where σ denotes the suspension isomorphism and p^* is induced by p.

$$\begin{array}{ccc} H^q(X \times X) & \xrightarrow{\Delta} & H^{q+1}(E_1 X) \\ \uparrow{\scriptstyle \pi_2^*} & & \uparrow{\scriptstyle p^*} \\ H^q(X) & \xrightarrow{\sigma} & H^{q+1}(SX). \end{array} \tag{2.4}$$

Since $\eta^* \circ \phi = m^* - \pi_1^* - \pi_2^*$, and since $\Delta \circ (m^* - \pi_1^*) = 0$, we have

$$-\Delta \circ \eta^* \circ \phi = p^* \circ \sigma. \tag{2.5}$$

3. The projective plane

Let X be an H-space. Following Stasheff [11] we define the projective plane of X as

$$P_2 X = C(E_1 X) \cup_p SX.$$

That is, $P_2 X$ is the cone on $E_1 X$ attached to SX by p. Thus SX is a subspace of $P_2 X$, and by Adem [2, §3] we have an exact sequence $(q > 0)$

$$\cdots \to H^q(SX) \xrightarrow{p^*} H^q(E_1 X) \xrightarrow{\mu} H^{q+1}(P_2 X)$$
$$\xrightarrow{i^*} H^{q+1}(SX) \to \cdots,$$

where μ is a Mayer-Vietoris coboundary and i^* is the injection. We set

$$\lambda = \mu \circ \Delta \circ \eta^* : H^q(X \# X) \to H^{q+2}(P_2 X),$$

$$\iota = \sigma^{-1} \circ i^* : H^{q+1}(P_2 X) \to H^q(X).$$

Since $\Delta \circ \eta^*$ and σ are each isomorphisms, using (2.5) we obtain[3] the following exact sequence

$$(3.1) \qquad \cdots \to H^q(X) \xrightarrow{\phi} H^q(X \# X) \xrightarrow{\lambda} H^{q+2}(P_2 X) \xrightarrow{\iota} H^{q+1}(X) \to \cdots .$$

Notice that

(3.2)
(i) *ϕ is an algebra homomorphism (and an $\mathfrak{A}_p$-map when $k = Z_p$);*
(ii) *λ and ι are k-module homorphisms (and $\mathfrak{A}_p$-maps);*
(iii) $\iota(D) = 0$,

where $D = D^2H^*(P_2 X)$, since cup-prodcuts of positive-dimensional elements vanish in the cohomology of a suspension.

For cohomology classes $x_1, x_2 \in \bar{H}^*(X)$ we denote by $x_1 \otimes x_2$ both the element in $\bar{H}^*(X) \otimes \bar{H}^*(X)$ and its counterimage under η^* in $\bar{H}^*(X \# X)$. Suppose now that x_1 and x_2 are primitive classes. Then

$$m^* x_i = \pi_1^* x_i + \pi_2^* x_i \qquad (i = 1, 2),$$

and so

$$\phi(x_1) = \phi(x_2) = 0.$$

Therefore by the exactness of (3.1) there are classes $y_1, y_2 \in \bar{H}^*(P_2 X)$ such that $\iota y_i = x_i$ $(i = 1, 2)$. In [12, (2.4)] it is shown that

$$(3.3) \qquad \lambda(x_1 \otimes x_2) = (-1)^{r+1} y_1 y_2 ,$$

where $r = \dim x_1$, and where the product on the right is the cup-product in $H^*(P_2 X)$.

For $u \in H^*(X)$ and $c \in H_*(X)$ we denote by $\langle u, c \rangle \in k$ the Kronecker index of u and c, which gives us a dual pairing [7], $H^*(X) \otimes H_*(X) \to k$. If $u_1, u_2 \in \bar{H}^*(X)$ and $c_1, c_2 \in \bar{H}_*(X)$, then

$$(3.4) \qquad \langle u_1 \otimes u_2 , c_1 \otimes c_2 \rangle = \langle u_1 , c_1 \rangle \langle u_2 , c_2 \rangle.$$

Moreover,

$$(3.5) \qquad \langle \phi(u), c_1 \otimes c_2 \rangle = \langle u, c_1 \cdot c_2 \rangle,$$

where $c_1 \cdot c_2$ denotes the Pontrjagin product of the homology classes c_1 and c_2.

We now use these various facts to obtain the proof of our theorem. As in §1 take cohomology with coefficients in the field k. Again denote by P^- and P^+ the respective subspaces of $\bar{H}^*(X)$ spanned by the odd- and even-dimensional primitive classes. Let $\tilde{D}$ denote the subspace of $\bar{H}^*(X)$ spanned by P^+ and the decomposable elements. Assume now that the algebra $H^*(X)$ is primitively generated. Then,

$$\bar{H}^*(X) = P^- \oplus \tilde{D},$$

[3] A special case of this is considered by Adams in [1].

as a vector space over k. Choose a complementary subspace, $\hat{D}$, to P^+ in $\tilde{D}$. Thus, $\bar{H}^*(X) = P \oplus \hat{D}$, where $P = P(H^*(X)) = P^- \oplus P^+$. Define

$$\hat{S} = \hat{D} \otimes \hat{D} \oplus \hat{D} \otimes P \oplus P \otimes \hat{D}$$

in $\bar{H}^*(X \# X)$, and set

$$S = \lambda(\hat{S}) \quad \text{in} \quad \bar{H}^*(P_2 X).$$

Let $\{u_i\}$, $\{v_j\}$ be respective bases for P^-, P^+, and choose classes $\{y_i\}$, $\{z_j\}$ in $H^*(P_2 X)$ such that

$$\iota y_i = u_i, \qquad \iota z_j = v_j .$$

Each z_j is odd-dimensional, and consequently,

$$z_r \cdot z_s + z_s \cdot z_r = 0 \qquad \text{(all } r, s).$$

In particular, if characteristic $k \neq 2$, $z_j^2 = 0$.

Denote by N the subalgebra of $H^*(P_2 X)$ spanned by the classes $\{y_i\}$ and $\{z_j\}$. Since $P_2 X$ has category three (it is formed from the suspension by attaching a cone), the classes

(3.6) $\{y_i\}$, $\{z_j\}$, $\{y_i z_j\}$, $\{y_p y_q\}$ $(p \leqq q)$, $\{z_r z_s\}$ $(r \leqq s)$,

span N, as a k-module. We show

LEMMA (3.7). *Suppose that the algebra $H^*(X)$ is primitively generated. Set*

$$x = \sum_i a_i y_i + \sum_j b_j z_j + \sum_i \sum_j c_{ij} y_i z_j + \sum_{p \leqq q} d_{pq} y_p y_q + \sum_{r \leqq s} e_{rs} z_r z_s ,$$

where $a_i, \cdots, e_{rs} \in k$, and where each $e_{ss} = 0$ if characteristic $k \neq 2$. If $x \in S$, then all the coefficients $a_i, \cdots, e_{rs}$ are zero.

By definition, if $x \in S$, then $x = \lambda(\hat{s})$, for some $\hat{s} \in \hat{S}$, and therefore $\iota(x) = 0$, by the exactness of (3.1). Now

$$\iota(y_i z_j) = \iota(y_p y_q) = \iota(z_r z_s) = 0,$$

by (3.2)(iii), and by hypothesis

$$\iota y_i = u_i, \qquad \iota z_j = v_j .$$

Thus, assuming $x \in S$ we have

$$0 = \iota(x) = \sum_i a_i u_i + \sum_j b_j v_j ,$$

and hence each a_i, b_j is zero since $\{u_i\}$ and $\{v_j\}$ are respective bases for P^-, P^+ and $P^- \cap P^+ = 0$.

Set

$$\{w_m\} = \{u_i\} \cup \{v_j\}.$$

Thus $\{w_m\}$ is a basis for P $(= P^- \oplus P^+)$, and the classes $\{w_m \otimes w_n\}$ form a k-basis for $P \otimes P$. Let $\{\bar{w}_m\}$ denote a set of dual homology classes to $\{w_m\}$. That is,

$$\langle w_m , \bar{w}_n \rangle = \delta_{mn} , \quad \text{the Kronecker delta.}$$

Since $\bar{H}^*(X) = P \oplus \hat{D}$, and since $H^q(X)$ is finite-dimensional $(q \geqq 0)$, we may choose the classes $\{\bar{w}_m\}$ so that for all m,

$$\langle \hat{D}, \bar{w}_m \rangle = 0.$$

Thus by (3.4) and the definition of $\hat{S}$,

$$\langle \hat{S}, \bar{w}_m \otimes \bar{w}_n \rangle = 0 \qquad (\text{all} \quad m, n). \tag{3.8}$$

We define

$$A = \textstyle\sum_i \sum_j c_{ij}\, u_i \otimes v_j + \sum_{p \leqq q} d_{pq}\, u_p \otimes u_q - \sum_{r \leqq s} e_{rs}\, v_r \otimes v_s$$

in $\bar{H}^*(X \# X)$ and obtain by (3.3) that

$$\lambda(A) = \textstyle\sum_i \sum_j c_{ij}\, y_i z_j + \sum_{p \leqq q} d_{pq}\, y_p y_q + \sum_{r \leqq s} e_{rs}\, z_r z_s = x.$$

Consequently $\lambda(A - \hat{s}) = 0$, and hence by exactness there is an element $f \in \bar{H}^*(X)$ such that

$$\phi(f) = A - \hat{s}.$$

To show that each coefficient c_{ij} is zero, we observe that

$$\langle A, \bar{u}_i \otimes \bar{v}_j \rangle = c_{ij} , \qquad \langle A, \bar{v}_j \otimes \bar{u}_i \rangle = 0,$$

by the choice of the dual classes $\bar{u}_i$ and $\bar{v}_j$. Therefore if we define

$$\bar{g} = \bar{u}_i \cdot \bar{v}_j - \bar{v}_j \cdot \bar{u}_i ,$$

we obtain, by (3.5) and (3.8), that

$$\begin{aligned} \langle f, \bar{g} \rangle &= \langle \phi(f), \bar{u}_i \otimes \bar{v}_j - \bar{v}_j \otimes \bar{u}_i \rangle \\ &= \langle A - \hat{s}, \bar{u}_i \otimes \bar{v}_j - \bar{v}_j \otimes \bar{u}_i \rangle = c_{ij} . \end{aligned}$$

Hence if $\bar{g} = 0$, then $c_{ij} = 0$. Now $H^*(X)$ is a primitively generated Hopf algebra, and therefore its dual algebra, $H_*(X)$, is commutative by (4.8) and (4.9) of [9]. Consequently since $\bar{g}$ is the commutator of $\bar{u}_i$ and $\bar{v}_j$, it must vanish, and hence so must c_{ij}. An entirely similar argument shows that the coefficients d_{pq} and e_{rs} are zero, when $p < q$ and $r < s$.

To show that the remaining coefficients d_{qq} and e_{ss} are all zero, we use the fact that

$$\langle \phi(f), \bar{u}_q \otimes \bar{u}_q \rangle = d_{qq} , \qquad \langle \phi(f), \bar{v}_s \otimes \bar{v}_s \rangle = e_{ss} . \tag{3.9}$$

Suppose that characteristic $k = 2$. Then by (4.9) of [9], since $H^*(X)$ is primitively generated, we have $x^2 = 0$ for all classes $x \in \bar{H}_*(X)$. Thus,

$$\bar{u}_q^2 = 0, \qquad \bar{v}_s^2 = 0,$$

and hence by (3.5), $d_{qq} = e_{ss} = 0$. On the other hand, if characteristic $k \neq 2$, then by hypothesis, $e_{ss} = 0$ (all s). But since $\bar{u}_q$ is odd-dimensional and $H_*(X)$ is commutative, we must have $\bar{u}_q^2 = 0$, which shows by (3.9) that $d_{qq} = 0$, completing the proof of (3.7).

4. Proof of (1.1)

It follows from (3.7), by taking $x = 0$, that the classes given in (3.6) form a k-basis for the subalgebra N, and thus $N = (\bar{A}/D^3A)$, as given in (1.1). Moreover, $N \cap S = 0$, again by (3.7), and since $S = \lambda(\hat{S})$, it follows from (3.1) that $\iota S = 0$. We show in an appendix (§6) that $S \cdot \bar{H}^*(P_2 X) = 0$. Thus S is an ideal, and we complete the proof of the splitting given in (1.1) by showing that

$$\bar{H}^*(P_2 X) = N + S.$$

Let $x \in \bar{H}^*(P_2 X)$. Since $\iota(x)$ is primitive, we may write

$$\iota x = \textstyle\sum_i a_i u_i + \sum_j b_j v_j .$$

where a_i, $b_j \in k$. Set

$$y = \textstyle\sum_i a_i y_i + \sum_j b_j z_j \in N.$$

Then $\iota(x - y) = 0$, and therefore by the exactness of (3.1) there is a class $w \in \bar{H}^*(X \# X)$ such that $\lambda(w) = x - y$. However,

$$\bar{H}^*(X \# X) = \hat{S} \oplus (P \otimes P),$$

and since $\lambda(\hat{S}) = S$ and $\lambda(P \otimes P) \subset N$ (by (3.3)), we have $x \in N + S$, completing the proof of the splitting.

Assume now that $k = Z_p$, p a prime. As above let N denote the subalgebra of $\bar{H}^*(P_2 X)$ generated by the classes given in (3.6), and define $\tilde{N}$ to be the subalgebra of N generated by the classes $\{y_i\}$. Let I denote the ideal of N generated by $\{z_j\}$. Then

$$N = \tilde{N} \oplus I,$$

as a k-module. Define $\tilde{S} = I \oplus S$, which is an ideal in $\bar{H}^*(P_2 X)$. Then by the splitting obtained above we have

$$\bar{H}^*(P_2 X) = \tilde{N} \oplus \tilde{S},$$

as a group. In §3 we defined $\tilde{D}$ to be the subspace of $\bar{H}^*(X)$ spanned by P^+ and the decomposable elements. Define

$$\hat{S}_1 = \tilde{D} \otimes \tilde{D} \oplus \tilde{D} \otimes P^- \oplus P^- \otimes \tilde{D}$$

in $\bar{H}^*(X \# X)$, and set

$$L = \text{the linear subspace of } \quad H^*(P_2 X) \quad \text{spanned by} \quad \{z_j\}.$$

Then by (3.3) and the definition of the subspace S (see §3), we have

$$\tilde{S} = I \oplus S = L \oplus \lambda(\hat{S}_1).$$

Since the elements of $\hat{\mathcal{a}}_p$ all have even degree, $\hat{\mathcal{a}}_p(P^+) \subset P^+$ and $\hat{\mathcal{a}}_p(P^-) \subset P^-$. Moreover by the Cartan product formula, $\hat{\mathcal{a}}_p(D^2) \subset D^2$, where $D^2 = D^2H^*(X)$. Thus $\hat{\mathcal{a}}_p(\tilde{D}) \subset \tilde{D}$, and therefore (again by the Cartan formula), $\hat{\mathcal{a}}_p(\hat{S}_1) \subset \hat{S}_1$. Thus, by (3.2)(ii), $\hat{\mathcal{a}}_p(\lambda\hat{S}_1) \subset \lambda\hat{S}_1$. Since the elements of $\tilde{N}$ all have even degree and those of L have odd degree, $\hat{\mathcal{a}}_p(L) \subset \tilde{S}$, and therefore $\hat{\mathcal{a}}_p(\tilde{S}) \subset \tilde{S}$, as required.

Since

$$\tilde{A}/D^3\tilde{A} \approx H^*(P_2 X)/\tilde{S},$$

we can regard $\tilde{A}/D^3\tilde{A}$ as an $\hat{\mathcal{a}}_p$-algebra. Define $\mathcal{B}_p$ to be the ideal of $\mathcal{a}_p$ generated by the Bockstein operator β_p. Then $\mathcal{a}_p = \mathcal{B}_p \oplus \hat{\mathcal{a}}_p$, as a Z_p-vector space. Since the elements of $\tilde{A}/D^3\tilde{A}$ all have even degree, we then can regard $\tilde{A}/D^3\tilde{A}$ as an algebra over all of $\mathcal{a}_p$ by setting $\mathcal{B}_p(\tilde{A}/D^3\tilde{A}) = 0$. This completes the proof of (1.1).

5. Remarks

The hypotheses of (1.1) can be altered in various ways. For example let k denote either the rational numbers Q or the field Z_p, p a prime; and suppose that the algebra $H^*(X)$ is not primitively generated. If, instead, one has that $H^*(X)$ is finite-dimensional (as a vector space), then the splitting given in (1.1) is still obtained, but one can no longer assert that $\tilde{S}$ is an $\tilde{\mathcal{a}}_p$-module. Since $H^*(X)$ is not primitively generated, we can no longer use [9] to obtain the appropriate lemma analogous to (3.7). Instead one now applies the results of [5], especially (6.8).

Another change is to use the integers for coefficients, rather than a field k. If one assumes that X has no torsion, and that $H^*(X; Z)$ is primitively generated and of finite rank, then a splitting analogous to that given in (1.1) is obtained. One uses the fact that $H^*(X; Z)$ is an exterior algebra on odd-dimensional, primitive generators. Thus, only the polynomial part (generated by the classes $\{y_i\}$) is obtained in the algebra N.

6. Appendix

Let X and Y be spaces, and f a map $X \to Y$. Denote by C_f the cone on X attached to Y by means of f. Then we have a proper triad (C_f, CX, M_f), where CX is the cone on X and M_f is the mapping cylinder of f (see [2, §§2, 3] for details). Moreover,

$$CX \cup M_f = C_f, \qquad CX \cap M_f = X,$$

and hence one has a Mayer-Vietoris coboundary

$$\mu : H^q(X) \to H^{q+1}(C_f).$$

We prove

LEMMA (6.1). *Let $x \in \bar{H}^*(X)$ and $y \in \bar{H}^*(C_f)$. Then*

$$\mu(x) \smile y = 0.$$

This clearly implies the result needed in (1.1)—that $S \cdot \bar{H}^*(P_2 X) = 0$—since $S = \lambda(\hat{S})$ and λ is the composition of μ with other homomorphisms.

To prove (6.1) we recall the definition of μ ([2, §3]). This is given by the following commutative diagram, where δ is the coboundary, m^* is the excision isomorphism, and n^* is induced by the inclusion:

$$\begin{array}{ccc} H^q(X) & \xrightarrow{\mu} & H^{q+1}(C_f) \\ \Big\downarrow{\scriptstyle\delta} & & \Big\uparrow{\scriptstyle n^*} \\ H^{q+1}(M_f, X) & \xrightarrow[m^{*-1}]{} & H^{q+1}(C_f, CX). \end{array}$$

Since CX is acyclic, there is a class $v \in \bar{H}^*(C_f, CX)$ such that $n^*v = y$. Choose $u \in \bar{H}^*(C_f, CX)$ such that $m^*u = \delta x$. Then,

$$\mu(x) \smile y = n^*(u \smile v) = n^*m^{*-1}(\delta x \smile m^*v).$$

Let l^* denote the homomorphism induced by the inclusion $M_f \subset (M_f, X)$. Then, by the naturality of the cup-product and the exactness of the cohomology sequence of a pair, one has

$$\delta x \smile m^*v = (l^*\delta x) \smile m^*v = 0 \smile m^*v = 0,$$

completing the proof of the lemma.

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