ON THE THEORY OF DIFFERENTIAL BOUNDARY PROBLEMS[1]

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1. Introduction

In a previous publication [10] the author applied the method of negative norms to elliptic boundary value problems. It was shown how inequalities involving such norms led to solutions of boundary problems for distributions. Existence theorems were easily obtained together with powerful regularity results.

In this paper we extract the essential features of the method and apply them to general boundary problems for arbitrary partial differential equations. The problems are posed in such a way as to include all types of equations and boundary conditions. We seek necessary and sufficient conditions for the existence of classical, strong, and weak solutions. These conditions are usually expressed by means of inequalities (a priori estimates). In applying the theory one would have to show that a certain inequality is satisfied. Existence then follows automatically. In some particular cases (e.g., the Višik-Sobolev problems, cf. §5) it was discovered that the required inequalities were already known, giving the desired existence theorems immediately.

Our main tools are representation theorems for bounded linear functionals on the spaces $H^{t,p}(G)$ and their subspaces, where t is an arbitrary integer, p an arbitrary real number greater than one, and G an arbitrary domain in Euclidean n-space (cf. Theorems 2.1, 2.2, 6.2, 6.3, 7.1). For $t \geqq 0$, $H^{t,p}(G)$ is defined as the completion of $C^{\infty}(\mathrm{Cl}\, G)$ with respect to the norm

$$\| u \|_{t,p} = \left(\int_G \sum_{r \leqq t} | D^r u |^p \, dx \right)^{1/p},$$

where summation is taken over all derivatives D^r of order $r \leqq t$. For $t < 0$, $H^{t,p}(G)$ is the completion of $C^{\infty}(\mathrm{Cl}\, G)$ with respect to the norm

$$\| u \|_{t,p} = \text{l.u.b.}_{v \in C^{\infty}(\mathrm{Cl}\, G)} |(u, v)| / \| v \|_{-t,p'},$$

where

$$(u, v) = \int_G u\bar{v} \, dx \quad \text{and} \quad p' = p/(p-1).$$

Theorem 2.1 states that $H^{-t,p'}(G)$ is the dual of $H^{t,p}(G)$ and vice versa. Other theorems give representations for various types of subspaces of $H^{t,p}(G)$. They require the introduction of other negative norms.

In §3 we consider boundary problems for partial differential equations.

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Let V be a linear subspace of $C^{\infty}(\mathrm{Cl}\, G)$ which contains those functions with compact support in G. Typical result: Given a partial differential operator A and a distribution f, a necessary and sufficient condition that there exist a $u \in H^{t,p}(G)$ such that

$$(u, Av) = (f, v) \tag{1.1}$$

for all $v \in V$, is that

$$|(f, v)| \leqq c \| Av \|_{-t,p'} \tag{1.2}$$

for all $v \in V$. Moreover, there is such a u for each $f \in H^{r,q}(G)$ if and only if

$$\| v \|_{-r,q'} \leqq c \| Av \|_{-t,p'} \tag{1.3}$$

for all $v \in V$ (more refined statements will be found in §3). Moreover similar criteria are shown to hold when u is to be restricted to certain subspaces of $H^{t,p}(G)$. For these and related results we refer to §3.

In §4 the theorems of §3 are applied to elliptic problems. They give rise to new inequalities and existence and regularity theorems. For example if A is properly elliptic and V is determined by differential boundary conditions which cover A (cf. [2], [4], [9]), it follows from our results that for $s \geqq 0$,

$$\| v \|_{-s,p} \leqq c(\| Av \|_{-m-s,p} + \| v \|_{-m-s,p})$$

for all $v \in V$. Moreover, if h is a distribution and $|(h, Av)| \leqq c_0 \| v \|_{-s,p}$ for all $v \in V$, then $h \in H^{m+s,p'}(G)$, and there is a constant K independent of h such that $\| h \|_{m+s,p'} \leqq K(c_0 + \| h \|_{s,p'})$. Further results may be found in §4.

Another application is to the so-called Višik-Sobolev problems (cf. [5], [6], [11]). In §5 we embed these problems into a very general framework and then show how our theorems give the complete answers for them.

§6 is devoted to the spaces $H_{\mathrm{Cl}\,G}^{-s,p}$ which arose in connection with the Višik-Sobolev problems. We prove a representation theorem due to Lions [5], [6] and show that under mild assumptions on G that $H_{\mathrm{Cl}\,G}^{-s,p}$ and $H^{-s,p}(G)$ are isomorphic. The proofs of the theorems of §5 are also given.

Proofs of the theorems in §§3 and 4 are given in §7. We note that Theorem 7.1, Corollary 7.1, and Lemmas 7.1–7.4 are of interest in their own right. Proofs of the theorems of §2 are given in §8.

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2. Certain function spaces

Let G be a domain in Euclidean n-space E^n with boundary ∂G and closure $\mathrm{Cl}\, G$. Let $C^{\infty}(\mathrm{Cl}\, G)$ be the set of complex-valued functions infinitely differentiable on $\mathrm{Cl}\, G$ with compact support. If $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ is any multi-index of length $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$, set

$$D^{\mu} = \partial^{|\mu|}/(i\partial x_1)^{\mu_1}(i\partial x_2)^{\mu_2} \cdots (i\partial x_n)^{\mu_n}.$$

For any nonnegative integer s and any real number p greater than one, we define

$$(2.1) \qquad \| u \|_{s,p} = \left(\int_G \sum_{|\mu| \leqq s} | D^\mu u |^p \, dx \right)^{1/p},$$

$$(2.2) \qquad \| u \|_{-s,p} = \text{l.u.b.}_{v \in C^\infty(\mathrm{Cl}\, G)} |(u, v)| / \| v \|_{s,p'}, \qquad p' = p/(p-1)$$

for functions $u \in C^\infty(\mathrm{Cl}\, G)$, where

$$(u, v) = \int_G u\bar{v} \, dx.$$

Denote the completions of $C^\infty(\mathrm{Cl}\, G)$ with respect to these norms by $H^{s,p}(G)$ and $H^{-s,p}(G)$ respectively. They are Banach spaces.

Let $C_0^\infty(E^n)$ be the set of infinitely differentiable complex functions with compact support in E^n. For such functions we employ the norm

$$(2.3) \qquad \| \varphi \|_{s,p}^{E^n} = \left(\int_{E^n} \sum_{|\mu| \leqq s} | D^\mu \varphi |^p \, dx \right)^{1/p}.$$

Denote the closure of $C_0^\infty(E^n)$ with respect to this norm by $H^{s,p}(E^n)$. For any function $\psi \in H^{s,p}(E^n)$ we let ψ_G denote the restriction of ψ to G. Clearly $\psi_G \in H^{s,p}(G)$.

THEOREM 2.1. *The spaces $H^{s,p}(G)$ and $H^{-s,p'}(G)$ are conjugate to one another.*

COROLLARY 2.1. *For $u \in H^{s,p}(G)$*

$$\| u \|_{s,p} = \text{l.u.b.}_{v \in C^\infty(\mathrm{Cl}\, G)} |(u, v)| / \| v \|_{-s,p'}.$$

Now let $C_0^\infty(G)$ denote the set of all $v \in C^\infty(\mathrm{Cl}\, G)$ which vanish near ∂G. Let V be any linear space of functions such that $C_0^\infty(G) \subseteq V \subseteq C^\infty(\mathrm{Cl}\, G)$. Corresponding to V we define the norm

$$(2.4) \qquad | u |_{-s,p} = \text{l.u.b.}_{v \in V} |(u, v)| / \| v \|_{s,p'}$$

for functions $u \in C^\infty(\mathrm{Cl}\, G)$. Complete $C^\infty(\mathrm{Cl}\, G)$ with respect to this norm, and call the resulting Banach space $V^{-s,p}(G)$. Let $V^{s,p}(G)$ be the closure of V in $H^{s,p}(G)$.

THEOREM 2.2. *$V^{s,p}(G)$ and $V^{-s,p'}(G)$ are dual spaces.*

COROLLARY 2.2. *For $u \in V^{s,p}(G)$*

$$\| u \|_{s,p} = \text{l.u.b.}_{v \in V} |(u, v)| / | v |_{-s,p'}.$$

3. Boundary value problems

Let $A = \sum_{|\mu| \leqq m} a_\mu(x) D^\mu$ be a partial differential operator of order $m \geqq 1$ with coefficients[2] $a_\mu(x) \in C^\infty(\mathrm{Cl}\, G)$. The formal adjoint A' of A is defined by

[2] Some assumptions are made for convenience only. Our results hold under less restrictive hypotheses.

integration by parts, i.e., by

$$(3.1) \qquad (Au, v) = (u, A'v),$$

holding when u or v belongs to $C_0^\infty(G)$. We define V' as the set of those $v \in C^\infty(\mathrm{Cl}\, G)$ which satisfy (3.1) for all $u \in V$. Clearly V' is a linear space containing $C_0^\infty(G)$. We set

$$(3.2) \qquad | u |'_{-s,p} = \text{l.u.b.}_{v \in V'} |(u, v)| / \| v \|_{s,p'}$$

and denote the completion of $C^\infty(\mathrm{Cl}\, G)$ with respect to $| \quad |'_{-s,p}$ by $V'^{-s,p}(G)$. The closure of V' in $H^{s,p}(G)$ is to be denoted by $V'^{s,p}(G)$.

Now by (2.2), (2.4), and (3.2) we have the following inequalities.

$$(3.3) \qquad |(u, v)| \leqq \| u \|_{-s,p} \| v \|_{s,p'}, \qquad u \in H^{-s,p}(G), \quad v \in H^{s,p'}(G),$$

$$(3.4) \qquad |(u, v)| \leqq | u |_{-s,p} \| v \|_{s,p'}, \qquad u \in V^{-s,p}(G), \quad v \in V^{s,p'}(G),$$

$$(3.5) \qquad |(u, v)| \leqq | u |'_{-s,p} \| v \|_{s,p'}, \qquad u \in V'^{-s,p}(G), \quad v \in V'^{s,p'}(G).$$

Let N (resp. N') be the set of those $u \in V$ (resp. $v \in V'$) such that $Au = 0$ (resp. $A'v = 0$). We shall assume that N' is finite-dimensional. As before, s will denote a nonnegative integer. The letters r, t will denote arbitrary integers, and p, q will denote real numbers greater than one, with

$$p' = p/(p-1), \qquad q' = q/(q-1).$$

If L and L' are subspaces of $H^{t,p}(G)$ and $H^{-t,p'}(G)$, respectively, we let L/L' represent the set of all $u \in L$ which satisfy $(u, L') = 0$ (i.e., $(u, v) = 0$ for all $v \in L'$).

When t is positive, we set $| u |_{t,p} = | u |'_{t,p} = \| u \|_{t,p}$. We write $u \in H^{-\infty}(G)$ (resp. $V^{-\infty}(G)$, $V'^{-\infty}(G)$) when $u \in H^{t,p}(G)$ (resp. $V^{t,p}(G)$, $V'^{t,p}(G)$) for some t and p.

THEOREM 3.1. *For $f \in V'^{-\infty}(G)$ there is a $u \in H^{s,p}(G)$ such that*

$$(3.6) \qquad (u, A'v) = (f, v) \qquad \textit{for all } v \in V'$$

if and only if

$$(3.7) \qquad |(f, v)| \leqq c \| A'v \|_{-s,p'} \qquad \textit{for all } v \in V'.$$

There is a $u \in V^{s,p}(G)$ satisfying (3.6) *if and only if*

$$(3.8) \qquad |(f, v)| \leqq c \,| A'v |_{-s,p'} \qquad \textit{for all } v \in V'.$$

In Theorems 3.2 and 3.4 below, when $t > 0$ the spaces $V'^{t,q}(G)/N'$ should be replaced by $H^{t,q}(G)/N'$.

THEOREM 3.2. *A necessary and sufficient condition that for each $f \in V'^{t,q}(G)/N'$ there exist a $u \in H^{s,p}(G)$ satisfying* (3.6) *is that*

$$(3.9) \qquad \| v \|_{-t,q'} \leqq c \| A'v \|_{-s,p'} \qquad \textit{for all } v \in V'/N'.$$

A necessary and sufficient condition that for each such f a $u \in V^{s,p}(G)$ satisfy

(3.6) *is that*

$$\| v \|_{-t,q'} \leqq c \, | A'v |_{-s,p'} \qquad \text{for all } v \in V'/N'. \tag{3.10}$$

THEOREM 3.3. *A* $u \in H^{-s,p}(G)$ *satisfies* (3.6) *if and only if*

$$|(f, v)| \leqq c \, \| A'v \|_{s,p'} \qquad \text{for all } v \in V'. \tag{3.11}$$

THEOREM 3.4. *A necessary and sufficient condition that for each* $f \in V'^{t,q}(G)/N'$ *there exist a* $u \in H^{-s,p}(G)$ *satisfying* (3.6) *is that*

$$\| v \|_{-t,q'} \leqq c \, \| A'v \|_{s,p'} \qquad \text{for all } v \in V'/N'. \tag{3.12}$$

THEOREM 3.5. *Let* $f \in V'^{-\infty}(G)$ *be given. Then the following statements are equivalent*:

(a) *There is a* $u \in V^{m+s,p}(G)$ *satisfying* (3.6).

(b) $f \in H^{s,p}(G)/N'$, *and there are a* $u \in V^{m+s,p}(G)$ *and a sequence* $\{u_k\} \subset V$ *such that* $\| u_k - u \|_{m+s,p} \to 0$ *and* $\| Au_k - f \|_{s,p} \to 0$ *as* $k \to \infty$.

(c) $|(f, v)| \leqq c \, | A'v |_{-m-s,p'}$ *for all* $v \in V'$.

THEOREM 3.6. *The following statements are equivalent*:

(a) *For every* $f \in H^{s,p}(G)/N'$ *there is a* $u \in V^{m+s,p}(G)$ *satisfying* (3.6).

(b) *For every* $f \in H^{s,p}(G)/N'$ *there are a* $u \in V^{m+s,p}(G)$ *and a sequence* $\{u_k\} \subset V$ *such that* $\| u_k - u \|_{m+s,p} \to 0$ *and* $\| Au_k - f \|_{s,p} \to 0$ *as* $k \to \infty$.

(c) $\| v \|_{-s,p'} \leqq c \, | A'v |_{-m-s,p'}$ *for all* $v \in V'/N'$.

THEOREM 3.7. *Let* $f \in V'^{-\infty}(G)$ *be given. The following statements are equivalent*:

(a) *There is a* $u \in H^{-s,p}(G)$ *satisfying* (3.6).

(b) $f \in V'^{-m-s,p}(G)/N'$, *and there are a* $u \in H^{-s,p}(G)$ *and a sequence* $\{u_k\} \subset C_0^\infty(G)$ *such that* $\| u_k - u \|_{-s,p} \to 0$, $| Au_k - f |'_{-m-s,p} \to 0$ *as* $k \to \infty$.

(c) $|(f, v)| \leqq c \, \| A'v \|_{s,p'}$ *for all* $v \in V'$.

THEOREM 3.8. *The following are equivalent*:

(a) *For every* $f \in V'^{-m-s,p}(G)/N'$ *there is a* $u \in H^{-s,p}(G)$ *satisfying* (3.6).

(b) *For every* $f \in V'^{-m-s,p}(G)/N'$ *there are a* $u \in H^{-s,p}(G)$ *and a sequence* $\{u_k\} \subset C_0^\infty(G)$ *such that* $\| u_k - u \|_{-s,p} \to 0$, $| Au_k - f |'_{-m-s,p} \to 0$ *as* $k \to \infty$.

(c) $\| v \|_{m+s,p'} \leqq c \, \| A'v \|_{s,p'}$ *for all* $v \in V'/N'$.

4. Elliptic problems

We shall say that A is elliptic on V if

(a) N and N' are finite-dimensional,

(b) for every $f \in C^\infty(\mathrm{Cl}\, G)/N'$ there is a $u \in V$ such that $Au = f$,

(c) for some s and p,

$$\| u \|_{m+s,p'} \leqq c_0 \, \| Au \|_{s,p'} \qquad \text{for all } u \in V/N.$$

It can be shown that a properly elliptic operator A is elliptic on a wide variety of spaces V satisfying homogeneous boundary data (cf. [2], [4], [9]). In this section we shall assume that (a)–(c) hold and consider some consequences.

THEOREM 4.1. *For each* $f \in V^{-m-s,p}(G)/N$ *there are a* $u \in H^{-s,p}(G)$ *and a sequence* $\{u_k\} \subset C_0^\infty(G)$ *such that*

$$\| u_k - u \|_{-s,p} \to 0 \quad \textit{and} \quad | A'u_k - f |_{-m-s,p} \to 0.$$

THEOREM 4.2. *There are constants* K *and* K' *such that*

$$\| v \|_{-s,p} \leqq K \, | A'v |_{-m-s,p} \qquad \textit{for all } v \in V'/N',$$

$$\| v \|_{-s,p} \leqq K'(| A'v |_{-m-s,p} + \| v \|_{-m-s,p}) \qquad \textit{for all } v \in V'.$$

THEOREM 4.3. *If* $h \in H^{-\infty}(G)$ *and* $|(h, Au)| \leqq c_1 \| u \|_{m+s,p'}$ *for all* $u \in V/N$, *then* $h \in H^{-s,p}(G)$, *and there is a constant* K_1 *depending only on* N', m, s, *and* p *such that* $\| h \|_{-s,p} \leqq K_1(c_1 c_0 + \| h \|_{-m-s,p})$.

THEOREM 4.4. *Assume, in addition, that for every* $g \in C^\infty(\mathrm{Cl}\, G)/N$ *there is a* $v \in V'$ *such that* $A'v = g$. *Then if* $h \in H^{-\infty}(G)$ *and* $|(h, A'v)| \leqq c_2 \| v \|_{-s,p}$ *for all* $v \in V'/N'$, *we have* $h \in V^{m+s,p'}(G)$, *and there is a constant* K_2 *depending only on* N, m, s, *and* p *such that* $\| h \|_{m+s,p'} \leqq K_2(c_2 c_0 + \| h \|_{s,p'})$.

5. Višik-Sobolev problems

We introduce negative norms on E^n as follows. Let

$$\langle \varphi, \psi \rangle = \int_{E^n} \varphi \bar{\psi} \, dx$$

be the $L^2(E^n)$ inner product. Set

$$\| \varphi \|_{-s,p}^{E^n} = \text{l.u.b.}_{\psi \in C_0^\infty(E^n)} |\langle \varphi, \psi \rangle| / \| \psi \|_{s,p'}^{E^n} .$$

Denote the completion of $C_0^\infty(E^n)$ with respect to the norm $\| \quad \|_{-s,p}^{E^n}$ by $H^{-s,p}(E^n)$. This space may be defined in other ways (cf. [5], [7]).

We shall say that the domain G satisfies Assumption (s, p) if there is a linear mapping $\theta_{s,p}$ of $H^{s,p}(G)$ into $H^{s,p}(E^n)$ such that

(a) $(\theta_{s,p} v)_G = v$ a.e.,

(b) $\| \theta_{s,p} v \|_{s,p}^{E^n} \leqq c_{s,p} \| v \|_{s,p}$

for all $v \in H^{s,p}(G)$.

An element $u \in H^{-s,p}(E^n)$ is said to be in $H_{\mathrm{Cl}\,G}^{-s,p}$ if $\langle u, v \rangle = 0$ for all v in $H^{s,p'}(E^n)$ with $v_G = 0$. We shall show in §6 that $H_{\mathrm{Cl}\,G}^{-s,p}$ is isomorphic to $H^{-s,p}(G)$ provided G satisfies Assumption (s, p'). We let $M^{-s,p}$ be the set of all $h \in H^{-s,p}(E^n)$ which satisfy $\langle h, v \rangle = 0$ whenever $v \in H^{s,p'}(E^n)$ and $v_G \in V'^{s,p'}(G)$. Clearly, $M^{-s,p}$ is a subspace of $H_{\mathrm{Cl}\,G}^{-s,p}$.

Now assume that the coefficients[3] of A are in $C_0^\infty(E^n)$. For every $h \in H^{-s,p}(E^n)$ there is an element $g \in H^{-m-s,p}(E^n)$ such that

$$\langle h, A'v \rangle = \langle g, v \rangle \tag{5.1}$$

for all $v \in H^{m+s,p'}(E^n)$. We define Ah to be g. Let $f \in H_{\mathrm{Cl}\,G}^{-m-s,p}$ be given. We shall say that an element $u \in H_{\mathrm{Cl}\,G}^{-s,p}$ is a solution of the Višik-Sobolev

[3] Cf. Footnote 2.

problem if

$$\langle u, A'v\rangle = \langle f, v\rangle \tag{5.2}$$

for all $v \in H^{m+s,p'}(E^n)$ such that $v_G \in V'^{m+s,p'}(G)$. If (5.2) holds, then by (5.1), $\langle Au - f, v\rangle = 0$ for all $v \in H^{m+s,p'}(E)$ such that $v_G \in V'^{m+s,p'}(G)$. Hence $Au - f \in M^{-m-s,p}$. Conversely, if $Au - f \in M^{-m-s,p}$, then (5.2) holds. Therefore we may say that u is a solution of the Višik-Sobolev problem if $Au - f \in M^{-m-s,p}$. For previous results on such problems, we refer to Lions [5], [6] and Višik-Sobolev [11].

THEOREM 5.1. *A sufficient condition that there exist a solution to the Višik-Sobolev problem is that*

$$|\langle f, v\rangle| \leqq c \, \| A'v_G\|_{s,p'} \tag{5.3}$$

for all $v \in H^{m+s,p'}(E^n)$ *such that* $v_G \in V'^{m+s,p'}(G)$. *Under Assumption* (s, p') *on* G, *this is also necessary.*

THEOREM 5.2. *If* G *satisfies Assumptions* (s, p') *and* $(m + s, p')$, *then the Višik-Sobolev problem has a solution for every*[4] $f \in H_{\text{Cl}G}^{-m-s,p}/N'$ *if, and only if,*

$$\| v \|_{m+s,p'} \leqq c \, \| A'v \|_{s,p'} \qquad \textit{for all } v \in V'/N'.$$

COROLLARY 5.1. *If* A *is elliptic on* V, *and* G *satisfies Assumptions* (s, p') *and* $(m + s, p')$, *then the Višik-Sobolev problem has a solution for each* $f \in H_{\text{Cl}G}^{-m-s,p}/N'$.

6. The spaces $H_{\text{Cl}G}^{-s,p}$

In this section we shall give proofs for the theorems of §5. We shall first discuss some properties of the spaces $H_{\text{Cl}G}^{-s,p}$.

THEOREM 6.1 (Lions [5], [6]). *If* $F(u)$ *is a bounded linear functional on* $H^{s,p}(G)$, *then there is a unique* $f \in H_{\text{Cl}G}^{-s,p'}$ *such that* $F(v_G) = \langle v, f\rangle$ *for all* $v \in H^{s,p}(E^n)$.

Proof. For $v \in H^{s,p}(E^n)$, set $F_1(v) = F(v_G)$. Then $| F_1(v)| = | F(v_G)| \leqq c \, \| v_G\|_{s,p} \leqq c \, \| v \|_{s,p}^{E^n}$. Thus $F_1(v)$ is a bounded linear functional on $H^{s,p}(E^n)$. By Theorem 2.1, there is an $f \in H^{-s,p'}(E^n)$ such that $F_1(v) = \langle v, f\rangle$. Moreover, if $\varphi \in H^{s,p}(E^n)$ and $\varphi_G = 0$, then $\langle \varphi, f\rangle = F_1(\varphi) = F(\varphi_G) = 0$. Hence $f \in H_{\text{Cl}G}^{-s,p'}$, and the proof is complete.

THEOREM 6.2. *Given any bounded linear functional* $F(w)$ *on* $H_{\text{Cl}G}^{-s,p}$, *there is an* $f \in H^{s,p'}(E^n)$ *such that* $F(w) = \langle w, f\rangle$. *Moreover, another element* f' *has this property if and only if* $f'_G = f_G$.

Proof. By hypothesis there is a constant c such that $| F(w)| \leqq c \, \| w \|_{-s,p}^{E^n}$ for all $w \in H_{\text{Cl}G}^{-s,p}$. Extend $F(w)$ to be a bounded linear functional on the

[4] Here we mean that $\langle f, \psi\rangle = 0$ for every $\psi \in H^{m+s,p'}(E^n)$ such that $\psi_G \in N'$.

whole of $H^{-s,p}(E^n)$. Then by Theorem 2.1, there is an $f \in H^{s,p'}(E^n)$ such that $F(w) = \langle w, f\rangle$ for all $w \in H^{-s,p}(E^n)$ and a fortiori for all $w \in H^{-s,p}_{\mathrm{Cl}G}$. If $f' \in H^{s,p'}(E^n)$ and $f'_G = f_G$, then $\langle f' - f, w\rangle = 0$ for all $w \in H^{-s,p}_{\mathrm{Cl}G}$. Hence f' can also be employed to represent $F(w)$. Conversely, if $f'' \in H^{s,p'}(E^n)$ and $F(w) = \langle w, f''\rangle$ for all $w \in H^{-s,p}_{\mathrm{Cl}G}$, then $\langle f - f'', w\rangle = 0$ and $(f - f'')_G = 0$, since $C_0^\infty(G) \subset H^{-s,p}_{\mathrm{Cl}G}$.

THEOREM 6.3. *There is a linear mapping $\tau_{-s,p}$ of $H^{-s,p}(G)$ into $H^{-s,p}_{\mathrm{Cl}G}$ such that*

$$(w, v_G) = \langle \tau_{-s,p} w, v\rangle, \qquad \| \tau_{-s,p} w \|^{E^n}_{-s,p} \leqq \| w \|_{-s,p}$$

for all $w \in H^{-s,p}(G)$ and $\varphi \in H^{s,p'}(E^n)$. If G satisfies Assumption (s, p'), then $\tau_{-s,p}$ is one-to-one, onto, and also satisfies

$$\| w \|_{-s,p} \leqq c_{s,p'} \| \tau_{-s,p} w\|^{E^n}_{-s,p} .$$

Proof. For each $w \in H^{-s,p}(G)$, (v, w) is a bounded linear functional on $H^{s,p'}(G)$. Hence by Theorem 6.1, there is an $f \in H^{-s,p}_{\mathrm{Cl}G}$ such that

$$(v_G , w) = \langle v, f\rangle$$

for all $v \in H^{s,p'}(E^n)$. Thus

$$\| f \|^{E^n}_{-s,p} = \operatorname*{l.u.b.}_{v\in C_0^\infty(E^n)} \frac{|\langle f, v\rangle|}{\| v \|^{E^n}_{s,p'}} \leqq \operatorname*{l.u.b.}_{v\in C_0^\infty(E^n)} \frac{|(w, v_G)|}{\| v \|_{s,p'}} \leqq \| w \|_{-s,p} ,$$

which also shows that f is unique. Moreover, if G satisfies Assumption (s, p'),

$$\| w \|_{-s,p} = \operatorname*{l.u.b.}_{g\in C^\infty(\mathrm{Cl}G)} \frac{|(w, g)|}{\| g \|_{s,p'}} = \operatorname*{l.u.b.}_{g\in C^\infty(\mathrm{Cl}G)} \frac{|\langle f, \theta_{s,p'} g\rangle|}{\| \theta_{s,p'} g \|_{s,p'}}$$

$$\leqq c_{s,p'} \operatorname*{l.u.b.}_{g\in C^\infty(\mathrm{Cl}G)} \frac{|\langle f, \theta_{s,p'} g\rangle|}{\| \theta_{s,p'} g \|^{E^n}_{s,p'}} \leqq \| f \|^{E^n}_{-s,p} .$$

Set $\tau_{-s,p} w = f$. The last inequality shows that $\tau_{-s,p}$ is one-to-one. To show that it is onto, we note that for any $f \in H^{-s,p}_{\mathrm{Cl}G}$ and $v \in H^{s,p'}(E^n)$

$$|\langle v, f\rangle| \leqq \| v \|^{E^n}_{s,p'} \| f \|^{E^n}_{-s,p} \leqq c_{s,p'} \| f \|^{E^n}_{-s,p} \| v \|_{s,p'} .$$

Hence $\langle v, f\rangle$ is a bounded linear functional on $H^{s,p'}(G)$ (since, under Assumption (s, p'), every function in $H^{s,p'}(G)$ is the restriction to G of a function in $H^{s,p'}(E^n)$). Thus by Theorem 2.1 there is a $w \in H^{-s,p}(G)$ such that

$$\langle v, f\rangle = (v_G , w),$$

and hence $f = \tau_{-s,p} w$. This completes the proof.

Proof of Theorem 5.1. *Necessity.* By Theorem 6.3,

$$|\langle f, v\rangle| = |\langle u, A'v\rangle| = |(\tau^{-1}_{-s,p} u, A'v_G)| \leqq \| \tau^{-1}_{-s,p} u \|_{-s,p} \| A'v_G \|_{s,p'}$$

for all $v \in H^{m+s,p'}(E^n)$ such that $v_G \in V'^{m+s,p'}(G)$.

Sufficiency. Let Y be the set of all $w \in H^{s,p'}(G)$ such that there is a $\psi \in H^{m+s,p'}(E^n)$ satisfying $\psi_G \in V'^{m+s,p'}(G)$ and $A'\psi_G = w$. Clearly Y is a linear subspace of $H^{s,p'}(G)$. Set $F(w) = \langle \psi, f\rangle$. Then $F(w)$ is a bounded linear functional on the subspace Y of $H^{s,p'}(G)$. Extending it to be bounded on the whole of $H^{s,p'}(G)$, we see that there is a $\varphi \in H^{-s,p}_{\mathrm{Cl}\,G}$ such that $F(\zeta_G) = \langle \zeta, \varphi\rangle$ for all $\zeta \in H^{s,p'}(G)$ (Theorem 6.1). In particular, this holds if $\psi \in H^{m+s,p'}(E^n)$, $\psi_G \in V'^{m+s,p'}(G)$, and $A'\psi = \zeta$. Hence $\langle \psi, f\rangle = \langle A'\psi, \varphi\rangle$ for all such ψ. Thus φ is a solution of the Višik-Sobolev problem.

Proof of Theorem 5.2. Clearly $f \in H^{-m-s,p}_{\mathrm{Cl}\,G}/N'$ if and only if

$$\tau^{-1}_{-m-s,p} f \in H^{-m-s,p}(G)/N'.$$

Moreover, $u \in H^{-s,p}_{\mathrm{Cl}\,G}$ satisfies (5.2) if and only if $(\tau^{-1}_{-s,p}\, u, A'u) = (\tau^{-1}_{-m-s,p}\, f, v)$ for all $v \in V'$. The theorem now follows immediately from Theorem 3.8.

Corollary 5.1 is an immediate consequence of Theorem 5.2 and the definition of an elliptic problem.

7. Further considerations and the remaining proofs

Before giving the proofs of the theorems of §§3 and 4, we shall discuss some results of interest in themselves.

LEMMA 7.1. *Let S be a finite-dimensional subspace of $H^{s,p}(G)$ with $s > 0$. Then there is a constant $C_{s,p}$ such that $\| v \|_{s,p} \leqq C_{s,p} \| v \|_{-s,p'}$ for all $v \in S$.*

Proof. We first prove that $\| v \|_{0,2} \leqq c \| v \|_{-s,p'}$ for all $v \in S$. If this were not so, there would be a sequence $\{v_k\} \subset S$ such that $\| v_k \|_{0,2} = 1$ and $\| v_k \|_{-s,p'} \to 0$. This means that for each $w \in S$,

$$|(v_k, w)| \leqq \| v_k \|_{-s,p'} \| w \|_{s,p} \to 0.$$

Since S is a finite-dimensional subspace of $L^2(G)$, weak convergence is equivalent to strong convergence. Hence $\| v_k \|_{0,2} \to 0$, contradicting the hypothesis. We next show that $\| v \|_{s,p} \leqq c \| v \|_{0,2}$ for all $v \in S$. Let $w_1, \cdots, w_N$ be a basis for S which satisfies $(w_j, w_k) = \delta_{jk}$, where δ_{jk} is the Kronecker delta. If our second assertion were not true, there would be a sequence $\{v_k\} \subset S$ such that $\| v_k \|_{s,p} = 1$ while $\| v_k \|_{0,2} \to 0$. But $v_k = \sum_l \alpha_{kl} w_l$, and hence $\| v_k \|^2_{0,2} = \sum_l | \alpha_{kl} |^2$. Therefore $\alpha_{kl} \to 0$ as $k \to \infty$ for each l. But this means that

$$\| v_k \|_{s,p} \leqq \sum_l | \alpha_{kl} | \; \| w_l \|_{s,p} \leqq K \sum_l | \alpha_{kl} | \to 0,$$

where $K = \max_l \| w_l \|_{s,p}$. This contradicts the second hypothesis, and the proof is complete.

LEMMA 7.2. *Under the same hypotheses, every $f \in H^{-s,p'}(G)$ can be written in the form $f = f' + f''$, where $f'' \in S$ and $(f', S) = 0$.*

Proof. If $f \in H^{-s,p'}(G)$, there is a sequence $\{f_k\} \subset C^\infty(\mathrm{Cl}\, G)$ such that $\| f_k - f \|_{-s,p'} \to 0$. Since S is closed in $L^2(G)$, $f_k = f_k' + f_k''$, where $f_k'' \in S$ and $(f_k', S) = 0$ (projection theorem). But this means that

$$(f_k'' - f_l'', S) = (f_k - f_l, S) \to 0 \qquad \text{as } k, l \to \infty.$$

The equivalence of weak and strong convergence now tells us that there is an $f'' \in S$ such that $\| f_k'' - f'' \|_{0,2} \to 0$. Set $f' = f - f''$. Then

$$(f', S) = \lim (f_k - f'', S) = \lim (f_k'' - f'', S) = 0,$$

and the lemma is proved.

LEMMA 7.3. *In addition to the above, for each t, q such that*

$$H^{s,p}(G) \subseteq H^{t,q}(G) \subseteq H^{-s,p'}(G)$$

there is a constant $c^{t,q}$ such that

$$\| f' \|_{t,q} \leqq c^{t,q} \| f \|_{t,q} \qquad \textit{for all } f \in H^{t,q}(G).$$

Proof. Otherwise there would exist a sequence $\{f_k\} \subset C^\infty(\mathrm{Cl}\, G)$ such that $\| f' \|_{t,q} = 1$ and $\| f_k \|_{t,q} \to 0$ as $k \to \infty$. In this case, $\| f_k'' \|_{t,q} \to 1$, and hence $\| f_k'' \|_{t,q} \leqq M$ for some constant M. Since S is finite-dimensional, there are a subsequence of $\{f_k''\}$ (also denoted by $\{f_k''\}$) and an $f'' \in S$ such that $\| f_k'' - f'' \|_{t,q} \to 0$. Now $f_k' + f'' = f_k' + f_k'' - f_k'' + f'' = f_k - (f_k'' - f'')$, and hence $\| f_k' + f'' \|_{t,q} \leqq \| f_k \|_{t,q} + \| f_k'' - f'' \|_{t,q} \to 0$. But $(f'', S) = (f_k' + f'', S) \to 0$ which shows that $f'' = 0$. Therefore $\| f_k' \|_{t,q} \to 0$, which contradicts the hypothesis. The lemma now follows.

LEMMA 7.4. *Under the same hypotheses*

$$\| v \|_{-t,q'} \leqq c^{t,q} \operatorname{l.u.b.}_{g \in H^{t,q}(G)/S} |(v, g)| / \| g \|_{t,q}$$

for all $v \in H^{-t,q'}(G)/S$.

Proof. For $w \in H^{t,q}(G)$,

$$|(v, w)| / \| w \|_{t,q} = |(v, w')| / \| w \|_{t,q} \leqq c^{t,q} |(v, w')| / \| w' \|_{t,q}$$

by Lemma 7.3. If $t \geqq 0$, our result follows from the definition of $\| v \|_{-t,q'}$. Otherwise, we apply Corollary 2.1.

THEOREM 7.1. *Under the same hypotheses, every bounded linear functional $F(w)$ on $H^{t,q}(G)/S$ can be written in the form*

$$F(w) = (w, f),$$

where $f \in H^{-t,q'}(G)/S$.

Proof. For each $v \in H^{t,q}(G)$, $v = v' + v''$ where $v'' \in S$ and $(v', S) = 0$ (Lemma 7.2). Let $G(v)$ be the linear functional on $H^{t,q}(G)$ defined by $G(v) = F(v')$. Then $G(v)$ is bounded, since, by Lemma 7.3,

$$| G(v) | = | F(v') | \leqq c \| v' \|_{t,q} \leqq cc^{t,q} \| v \|_{t,q}.$$

Now by Theorem 2.1, there is an $f \in H^{-t,q'}(G)$ such that $G(v) = (v, f)$ for all $v \in H^{t,q}(G)$. Moreover, if $v \in S$, $(v, f) = G(v) = 0$. Hence $f \in H^{-t,q'}(G)/S$ and the proof is complete.

COROLLARY 7.1. *Under the same hypotheses, if* $f \in H^{-s,p'}(G)$ *and* $|(f, v)| \leqq c_0 \| v \|_{t,q}$ *for all* $v \in C^\infty(\text{Cl } G)/S$, *then* $f \in H^{-t,q'}(G)$. *If* $(f, S) = 0$, *then* $\| f \|_{-t,q'} \leqq c^{t,q} c_0$. *In any event, there is a constant* K_0 *independent of* f *such that* $\| f \|_{-t,q'} \leqq K_0(c_0 + \| f \|_{-s,p'})$.

Proof. By Lemma 7.2, $f = f' + f''$, where $f'' \in S$ and $(f', S) = 0$. Since $S \subset H^{-t,q'}(G)$, we must show that $f' \in H^{-t,q'}(G)$. Now $|(f', v)| = |(f, v)| \leqq c \| v \|_{t,q}$ for all $v \in C^\infty(\text{Cl } G)/S$. Let $F(v) = (v, f')$ for all such v, and extend $F(v)$ to be a bounded linear functional on $H^{t,q}(G)/S$. By Theorem 7.1, there is an $f_0 \in H^{-t,q'}(G)/S$ such that $F(v) = (v, f_0)$. Hence $(f' - f_0, v) = 0$ for all $v \in H^{-s,p'}(G)/S$. Lemma 7.4 now shows us that $\| f' - f_0 \|_{-s,p'} = 0$. Hence $f' = f_0 \in H^{-t,q'}(G)$, and our first assertion is proved. Now if $(f, S) = 0$, then $f'' = 0$ and $f = f_0$. But by completion, $|(f, v)| \leqq c_0 \| v \|_{t,q}$ for all $v \in H^{t,q}(G)/S$. Hence, by Lemma 7.4 $\| f \|_{-t,q'} \leqq c_0 c^{t,q}$, giving the second assertion. If $f'' \neq 0$, then

$$\begin{aligned} \| f \|_{-t,q'} &\leqq \| f' \|_{-t,q'} + \| f'' \|_{-t,q'} \leqq \| f' \|_{-t,q'} + C_{s,p} \| f'' \|_{-s,p'} \\ &\leqq (1 + C_{s,p}) \| f' \|_{-t,q'} + C_{s,p} \| f \|_{-s,p'} \\ &\leqq (1 + C_{s,p}) c^{t,q} c_0 + C_{s,p} \| f \|_{-s,p'}, \end{aligned}$$

by Lemma 7.1. This completes the proof.

Proof of Theorem 3.1. Inequality (3.7) is clearly necessary for (3.6) to hold. For $|(f, v)| = |(u, A'v)| \leqq \| u \|_{s,p} \| A'v \|_{-s,p'}$, by (3.3). To show that it is sufficient, set $F(A'v) = (v, f)$. Then $F(w)$ is a properly defined linear functional on the set R' of all w such that there is a $v \in V'$ satisfying $A'v = w$. For if v_1 and v_2 are in V' and $A'v_1 = A'v_2 = w$, then

$$|(v_1 - v_2, f)| \leqq c \| A'(v_1 - v_2) \|_{-s,p} = 0$$

and $(v_1, f) = (v_2, f)$. Moreover, (3.7) implies that $F(w)$ is bounded on R' considered as a subspace of $H^{-s,p'}(G)$. Extend $F(w)$ to be bounded on the whole of $H^{-s,p'}(G)$. Then by Theorem 2.1, there is a $u \in H^{s,p}(G)$ such that $F(w) = (w, u)$. In particular, this holds if $v \in V'$ and $A'v = w$. Thus (3.6) holds. The proof of (3.8) is almost identical employing $V^{-s,p'}(G)$ and Theorem 2.2 in place of $H^{-s,p'}(G)$ and Theorem 2.1, respectively.

Proof of Theorem 3.2. Clearly, (3.9) is sufficient. For if $f \in V'^{t,q}(G)/N'$,

$$|(f, v)| \leqq | f |'_{t,q} \| v \|_{-t,q'} \leqq c | f |'_{t,q} \| A'v \|_{-s,p'}$$

for all $v \in V'/N'$. Now by Lemma 7.2, each $v \in V'$ can be written in the form $v = v' + v''$, where $v' \in V'/N'$ and $v'' \in N'$. Hence

$$|(f, v)| = |(f, v')| \leqq c | f |'_{t,q} \| A'v' \|_{-\ ,p'} = c | f |'_{t,q} \| A'v \|_{-s,p'}$$

for all $v \in V'$. Thus, by Theorem 3.1, there is a $u \in H^{s,p}(G)$ such that (3.6) holds.

Now assume that for each $f \in V'^{t,q}(G)/N'$ there is a $u \in H^{s,p}(G)$ satisfying (3.6). Let $\hat{N}^{s,p}(G)$ be the set of those $u \in H^{s,p}(G)$ satisfying $(u, A'v) = 0$

for all $v \in V'$. $\hat{N}^{s,p}(G)$ is a closed subspace of $H^{s,p}(G)$, and for each f there is at most one[5] $u \in H^{s,p}(G)/\hat{N}^{s,p}(G)$ satisfying (3.6). Write $u = Tf$. If $| f_k - g |'_{t,q} \to 0$ and $\| Tf_k - w \|_{s,p} \to 0$, then

$$(w, A'v) = \lim (Tf_k, A'v) = \lim (f_k, v) = (g, v)$$

for all $v \in V'$, and hence $w = Tg$ This means that T has a closed graph. By the closed graph theorem, $\| Tf \|_{s,p} \leqq c_0 | f |'_{t,q}$ for all $f \in V'^{t,q}(G)/N'$. Now if (3.6) holds for every such f, we have

$$|(f, v)| = |(Tf, A'v)| \leqq \| Tf \|_{s,p} \| A'v \|_{-s,p'} \leqq c\, | f |'_{t,q} \| A'v \|_{-s,p'}$$

for all $v \in V'$ and $f \in V'^{t,q}(G)/N'$. Hence

$$|(f, v)| / | f |'_{t,q} \leqq c_0 \| A'v \|_{-s,p'} .$$

But by Lemma 7.4,

$$\| v \|_{-t,q'} \leqq c^{t,q} \text{l.u.b.}_{f \in C^\infty(\mathrm{Cl}\, G)/N'} |(v, f)| / | f |'_{t,q}$$

for all $v \in V'/N'$. (Here we have made use of the fact that for $f \in C^\infty(\mathrm{Cl}\, G)$, $| f |'_{t,q} \leqq \| f \|_{t,q}$.) Since c_0 is independent of f, (3.9) follows immediately. The proof of (3.10) is similar and is omitted.

Proof of Theorem 3.3. The method of the proof of Theorem 3.1 is employed. It should be noted that now (v, f) is considered a functional on $H^{s,p'}(G)$.

Proof of Theorem 3.4. The sufficiency follows from Theorem 3.3 as in the proof of Theorem 3.2. The necessity proof is almost identical to that of Theorem 3.2.

Proof of Theorem 3.5. (a) $\Rightarrow$ (b). Since $u \in V^{m+s,p}(G)$, there is a sequence $\{u_k\} \subset V$ such that $\| u_k - u \|_{m+s,p} \to 0$. Then

$$\| A(u_k - u_l) \|_{s,p} \leqq c \| u_k - u_l \|_{m+s,p} \to 0 \quad \text{as } k, l \to \infty .$$

Hence there is a $g \in H^{s,p}(G)$ such that $\| Au_k - g \|_{s,p} \to 0$. But

$$(g, v) = \lim (Au_k, v) = \lim (u_k, A'v) = (u, A'v) = (f, v)$$

for all $v \in V'$. Hence $f = g \in H^{s,p}(G)$, and $\| Au_k - f \|_{s,p} \to 0$.

(b) $\Rightarrow$ (a). Clearly, $(f, v) = \lim (Au_k, v) = \lim (u_k, A'v) = (u, A'v)$ for all $v \in V'$.

That (a) and (c) are equivalent follows from Theorem 3.1.

Theorem 3.6 is an immediate consequence of Theorems 3.2 and 3.5.

Proof of Theorem 3.7. (a) $\Rightarrow$ (b). Since $u \in H^{-s,p}(G)$, there is a sequence $\{u_k\} \subset C^\infty(G)$ such that $\| u_k - u \|_{-s,p} \to 0$ (Lemma 6.2). Hence

$$| A(u_k - u_l) |'_{-m-s,p} \leqq c \| u_k - u_l \|_{-s,p} \to 0 \quad \text{as } k, l \to \infty .$$

[5] Here we mean the actual quotient space consisting of the cosets of $\hat{N}^{s,p}(G)$. If an element u of this space is represented by $u \in H^{s,p}(G)$, then

$$\| u \|_{s,p} = \text{g.l.b.}_h \| w + h \|_{s,p} ,$$

where $h \in \hat{N}^{s,p}(G)$.

Thus there is a $g \in V'^{-m-s,p}(G)$ such that $| Au_k - g |'_{-m-s,p} \to 0$. But

$$(g, v) = \lim (Au_k, v) = \lim (u_k, A'v) = (u, A'v) = (f, v)$$

for all $v \in V'$. Hence $f = g$.

That (b) implies (a) follows as in the proof of Theorem 3.5. The equivalence of (a) and (c) follows from Theorem 3.3.

Theorem 3.8 follows immediately from Theorems 3.4 and 3.7. Theorem 4.1 is an obvious consequence of Theorem 3.8.

Proof of Theorem 4.2. Let v be any function in V'. Then $(v, Au) = (u, A'v)$ for all $u \in V$. Hence

$$|(v, Au)| \leqq \| u \|_{m+s,p'} \, | A'v |_{-m-s,p} \leqq c_0 \, | A'v |_{-m-s,p} \, \| Au \|_{s,p'}$$

for all $u \in V/N$ (by (c)). Moreover, by (b) for each $f \in C^\infty(\mathrm{Cl}\, G)/N'$ there is a $u \in V$ such that $Au = f$. By the finite-dimensionality of N and Lemma 7.2, we may assume that $u \in V/N$. Hence for every such f,

$$|(v, f)| \leqq c_0 \, | A'v |_{-m-s,p} \, \| f \|_{s,p'} \, .$$

The theorem now follows readily from Lemmas 7.4 and 7.1 (cf. the proof of Corollary 7.1).

Proof of Theorem 4.3. By (c), $|(h, Au)| \leqq c_1 \| u \|_{m+s,p'} \leqq c_1 c_0 \| Au \|_{s,p'}$ for all $u \in V/N$. By (b) and the reasoning above, it follows from this that $|(h, f)| \leqq c_1 c_0 \| f \|_{s,p'}$ for all $f \in C^\infty(\mathrm{Cl}\, G)/N'$. Our result now follows immediately from Corollary 7.1.

Proof of Theorem 4.4. Following the procedure of the last proof, we see that $|(h, A'v)| \leqq c_2 K \, | A'v |_{-m-s,p}$ for all $v \in V'/N'$ (Theorem 4.2). Hence $|(h, g)| \leqq c_2 K \, | g |_{-m-s,p} \leqq c_2 K \, \| g \|_{-m-s,p}$ for all $g \in C^\infty(\mathrm{Cl}\, G)/N$. Thus by Corollary 7.1, $h \in H^{m+s,p'}(G)$, and $\| h \|_{m+s,p'} \leqq K_0(c_2 K + \| h \|_{s,p'})$. Moreover, $(h, g) = 0$ for all $g \in C^\infty(\mathrm{Cl}\, G)$ such that $(g, V) = 0$. Hence h must be in the closure of V in $H^{m+s,p'}(G)$. Thus $h \in V^{m+s,p'}(G)$, and the proof is complete.

8. The remaining proofs

It remains to prove the theorems of §2. Clearly, Theorem 2.1 is a special case of Theorem 2.2.

Proof of Theorem 2.2. We first note that $V^{s,p}(G)$ is reflexive, since it is a closed subspace of the reflexive space $L^p(G) \times L^p(G) \times \cdots \times L^p(G)$. Next, we let Z denote the set of all bounded linear functionals $F(v)$ on $V^{s,p}(G)$ which can be written in the form $F(v) = (v, u)$, where $u \in V^{-s,p'}(G)$. (That (v, u) is a bounded linear functional on $V^{s,p}(G)$ for $u \in V^{-s,p'}(G)$ follows from (3.4).) Now the set Z is a complete set of functionals, i.e., their simultaneous vanishing for any element $v \in V^{s,p}(G)$ implies that $v = 0$ (this follows from the fact that $C_0^\infty(G) \subset V^{-s,p'}(G)$). Hence Z is dense in the set of all bounded linear functionals on $V^{s,p}(G)$.

In addition, the norm of $F(v) = (v, u)$ as a bounded linear functional on

$V^{s,p}(G)$ is $| u |_{-s,p}$. Since $V^{-s,p'}(G)$ is complete, this means that Z is also closed in the set of all bounded linear functionals on $V^{s,p}(G)$. Thus Z is both dense and closed which makes it equal to the set of bounded linear functionals on $V^{s,p}(G)$. Since $V^{s,p}(G)$ is reflexive, $(V^{-s,p'}(G))' = V^{s,p}(G)$, and the proof is complete.

Proof of Corollary 2.2. We know that for $v \in V^{s,p}(G)$

$$\| v \|_{s,p} = \text{l.u.b.} \, | F(v)|/\| F \| ,$$

where the least upper bound is taken over all bounded linear functionals on $V^{s,p}(G)$. By Theorem 2.2, every such F can be written in the form $F(v) = (v, u)$, where $u \in V^{-s,p'}(G)$. Moreover, $\| F \| = | u |_{-s,p'}$, and the result is immediate.

Corollary 2.1 is a special case of Corollary 2.2.

Bibliography

1. SHMUEL AGMON, *The L_p approach to the Dirichlet problem. I*, Technical Note No. 7, Jerusalem, The Hebrew University, 1959.
2. S. AGMON, A. DOUGLIS, AND L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., vol. 12 (1959), pp. 623–727.
3. N. BOURBAKI, *Espaces vectoriels topologiques*, Éléments de Mathématique, Livre V, Paris, Hermann, 1955.
4. F. E. BROWDER, *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Acad. Sci. U. S. A., vol. 45 (1959), pp. 365–372.
5. J. L. LIONS, *Conditions aux limites de Visik-Soboleff et problèmes mixtes*, C. R. Acad. Sci. Paris, vol. 244 (1957), pp. 1126–1128.
6. ———, *Lectures on elliptic partial differential equations*, Bombay, Tata Institute of Fundamental Research, 1957.
7. ENRICO MAGENES AND GUIDO STAMPACCHIA, *I problemi al contorno per le equazioni differenziali di tipo ellittico*, Ann. Scuola Norm. Sup. Pisa (3), vol. 12 (1958), pp. 247–358.
8. MARTIN SCHECHTER, *General boundary value problems for elliptic partial differential equations*, Comm. Pure Appl. Math., vol. 12 (1959), pp. 457–486.
9. ———, *Remarks on elliptic boundary value problems*, Comm. Pure Appl. Math., vol. 12 (1959), pp. 561–578.
10. ———, *Negative norms and boundary problems*, Ann. of Math. (2), vol. 72 (1960), pp. 581–593.
11. M. I. VIŠIK AND S. L. SOBOLEV, *General formulation of certain boundary problems for elliptic partial differential equations*, Dokl. Akad. Nauk SSSR (N.S.), vol. 111 (1956), pp. 521–523 (in Russian).

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