A REPRESENTATION THEOREM FOR TRANSFORMATIONS OF RINGS OF CONTINUOUS FUNCTIONS[1]

BY

CARL W. KOHLS

1. Introduction

Let X and Y be compact Hausdorff spaces, and let T be a ring isomorphism[2] of $C(X)$ onto $C(Y)$. The transformation T can be represented by means of a homeomorphism ϕ of Y onto X; the representation has the simple form $Tf(y) = f(\phi(y))$, $y \in Y$, $f \in C(X)$. A slightly more complicated result was actually obtained first: It was observed by Banach [1, p. 172] that if X and Y are compact metric spaces, and T is an isometry of $C(X)$ onto $C(Y)$ such that $T0 = 0$, then we have $Tf(y) = \varepsilon(y)f(\phi(y))$, where $\varepsilon \in C(Y)$ and $|\varepsilon(y)| = 1$ for all $y \in Y$. Later, Stone [5, Theorem 83] generalized this to arbitrary compact spaces and arbitrary isometries, obtaining the equation $Tf(y) = T1(y)f(\phi(y)) + T0(y)$. (Thus, ε can be specified— it is $T1$.) A significant generalization of all these ideas was subsequently made by Kaplansky [3, Theorem 1], who considered those lattice isomorphisms of $C(X)$ onto $C(Y)$ that are also homeomorphisms in the topology of uniform convergence. His representation has the form $Tf(y) = \omega(f(\phi(y)), y)$, where ω is a continuous mapping from $\mathbf{R} \times Y$ onto $\mathbf{R}$ such that $\omega(\cdot, y)$ is a homeomorphism for each $y \in Y$; ω may be defined by $\omega(r, y) = Tr(y)$. In a recent paper, Whittaker [6, Theorem 6] obtained the same representation for a class of transformations that does not coincide with any of the familiar ones.[3] We shall present here a similar result, for a wider class of transformations; in fact, we characterize the class of transformations having a representation of this type. Actually, we shall consider a somewhat more general situation, in which some $\omega(\cdot, y)$ are constant mappings; this possibility was suggested by [2, 10.8]. It turns out, as we shall see, that the requirement that $\omega(\cdot, y)$ be either one-one or constant for each $y \in Y$ is equivalent to a condition on T relating it to the ring operations in $C(X)$ (condition (a)), one that is not unnatural as a candidate for a weaker requirement than that T be a ring homomorphism.

Theorem 1 is our representation theorem, while Theorem 2 is a converse. The idea of the proof of Theorem 1 is the following: If T has a representa-

[2] All special symbols and terms are defined in [2]. For general background, the reader is also referred to [2], especially Chapter 10.

[3] The representation for ring isomorphisms stated above has since been generalized to one that is valid when X and Y are arbitrary completely regular spaces [2, 10.8]. Whittaker assumes that X is normal and that Y is locally compact Hausdorff. In both cases, ϕ is also of a more general type.

tion of the desired type, then there can be associated with T in a natural way a ring homomorphism of $C(X)$ into $C(Y)$, namely, the one induced by the mapping ϕ of Y into vX. But there is a duality between homomorphisms and continuous mappings [2, Chap. 10], so if this homomorphism can be constructed *directly* from T, it can be used to obtain the mapping ϕ. Because of the way ϕ is related to the homomorphism, it is sufficient to construct the set of homomorphisms onto $\mathbf{R}$ obtained by composing this homomorphism with the evaluation mappings associated with the points of Y; the original homomorphism need not (and does not) appear at all.

2. The results[4]

We begin with a lemma that seems interesting in itself.

LEMMA. *Let E be any topological space, and let f be a function from $\mathbf{R} \times E$ into $\mathbf{R}$, such that $f(r, \cdot)$ is continuous for each $r \in \mathbf{R}$, and $f(\cdot, y)$ is a homeomorphism into $\mathbf{R}$ for each $y \in E$. Define g to be the function from a subset of $\mathbf{R} \times E$ onto $\mathbf{R}$ such that $g(\cdot, y)$ is the inverse of $f(\cdot, y)$ for each $y \in E$. Then both f and g are jointly continuous.*

Proof. To show that f is jointly continuous, let $(r, y) \in \mathbf{R} \times E$ and $\varepsilon > 0$ be given. Select $\delta > 0$ so that $|r - s| \leqq \delta$ implies $|f(r, y) - f(s, y)| < \varepsilon/2$, and then let U be a neighborhood of y such that $z \in U$ implies both $|f(r + \delta, y) - f(r + \delta, z)| < \varepsilon/2$ and $|f(r - \delta, y) - f(r - \delta, z)| < \varepsilon/2$. Now, if $|r - s| < \delta$, we know that $f(s, z)$ is between $f(r - \delta, z)$ and $f(r + \delta, z)$ for each $z \in U$; it follows quickly that $|f(r, y) - f(s, z)| < \varepsilon$.

The proof of the joint continuity of g requires several preliminary results.

First, we observe that for each $y \in E$, there is a neighborhood U of y such that either $z \in U$ implies that $f(\cdot, z)$ is increasing, or $z \in U$ implies that $f(\cdot, z)$ is decreasing. For, otherwise there would be a set of points z having y as a limit point for which $f(0, z) < f(1, z)$, and another set for which $f(0, z) > f(1, z)$; this implies that $f(0, y) = f(1, y)$, a contradiction.

Next, it will be shown that for any $r \in \mathbf{R}$, $g(r, \cdot)$ is continuous. Given $\varepsilon > 0$, and y in the domain of $g(r, \cdot)$, set $g(r, y) = t$. Choose a neighborhood U of y satisfying the condition stated in the previous paragraph, and such that if $z \in U$ and (r, z) is in the domain of g, then

$$|f(t + \varepsilon, y) - f(t + \varepsilon, z)| < |f(t + \varepsilon, y) - f(t, y)| \quad \text{and}$$

$$|f(t - \varepsilon, y) - f(t - \varepsilon, z)| < |f(t - \varepsilon, y) - f(t, y)|.$$

If $f(\cdot, z)$ is increasing for all $z \in U$, it follows that $f(t - \varepsilon, z) < f(t, y) < f(t + \varepsilon, z)$, or, equivalently, $f(t - \varepsilon, z) < f(g(r, z), z) < f(t + \varepsilon, z)$. Thus

$t - \varepsilon < g(r, z) < t + \varepsilon$, that is, $| g(r, y) - g(r, z) | < \varepsilon$. The proof is similar in the other case.

The last preliminary result required is that for any $r \in \mathbf{R}$, the domain of $g(r, \cdot)$ is open in E. Again let y be in the domain of $g(r, \cdot)$, and set $g(r, y) = t$. Assume, without loss of generality, that $f(\cdot, y)$ is increasing. Then $f(t - 1, z) < r < f(t + 1, z)$ for all z in some neighborhood U of y. Hence, for each $z \in U$, there exists $s \in \mathbf{R}$ such that $f(s, z) = r$, because the range of $f(\cdot, z)$ is connected; so $g(r, z)$ is defined. Thus, U is contained in the domain of $g(r, \cdot)$.

To show that g is jointly continuous, we may now employ an argument similar to that used for f. We need only select δ so that $g(s, y)$ is defined, and then observe that, since $g(r + \delta, \cdot)$ and $g(r - \delta, \cdot)$ have open domains, and the domain of each $g(\cdot, z)$ is connected, we can find a neighborhood U of y such that $| r - s | \leq \delta$ and $z \in U$ imply that $g(s, z)$ is defined.

THEOREM 1. *Let X and Y be completely regular Hausdorff spaces, and let T be a transformation of $C(X)$ into $C(Y)$ such that:*

(a) *For each $y \in Y$, if f and g are any functions in $C(X)$ such that $Tf(y) = Tg(y)$, then $T(f + h)(y) = T(g + h)(y)$ and $T(fh)(y) = T(gh)(y)$ for all $h \in C(X)$.*

(b) *For each $y \in Y$, the function $r \to Tr(y)$ from $\mathbf{R}$ into $\mathbf{R}$ is a continuous mapping onto $T[C(X)](y)$.*

Set $E = \{y \in Y : T[C(X)](y)$ contains more than one point\}. Then there exist a continuous mapping ϕ of E into υX, and a continuous mapping ω from $\mathbf{R} \times E$ into $\mathbf{R}$ such that $\omega(\cdot, y)$ is a homeomorphism onto $T[C(X)](y)$ for each $y \in E$, satisfying

$$
(1) \qquad
\begin{aligned}
Tf(y) &= \omega(f^{\upsilon}(\phi(y)), y), & \text{if} \quad y \in E, \\
&= T\mathbf{0}(y), & \text{if} \quad y \in Y - E, \qquad f \in C(X).
\end{aligned}
$$

Furthermore, E is an open subset of Y, and $\phi[E]$ is dense in υX if and only if T is one-one.

If we assume in addition that T maps $C(X)$ onto $C(Y)$, then ϕ is a homeomorphism of Y onto a C-embedded subset of υX.

Proof. Let ω be the function from $\mathbf{R} \times E$ into $\mathbf{R}$ defined by $\omega(r, y) = Tr(y)$. First we observe that (a) implies that $\omega(\cdot, y)$ is one-one for each $y \in E$: If $Tr(y) = Ts(y)$ for some $r \neq s$, we have $T(r - s)(y) = T\mathbf{0}(y)$, $T\mathbf{1}(y) = T\mathbf{0}(y)$, and finally $Th(y) = T\mathbf{0}(y)$ for all $h \in C(X)$, so that $y \notin E$. It follows immediately from (b) that $\omega(\cdot, y)$ is a homeomorphism onto $T[C(X)](y)$ for each $y \in E$. Of course, $\omega(r, \cdot) = Tr$ is continuous for each $r \in \mathbf{R}$.

Let α be the function from a subset of $\mathbf{R} \times E$ onto $\mathbf{R}$ such that $\alpha(\cdot, y)$ is the inverse of $\omega(\cdot, y)$ for each $y \in E$. It follows from the lemma that both ω and α are jointly continuous.

For each $y \in E$, define a mapping H_y of $C(X)$ onto $\mathbf{R}$ by

$$H_y(f) = \alpha(Tf(y), y).$$

We prove that H_y is a homomorphism.   Given any $f, g \in C(X)$, let $H_y(f) = r$ and $H_y(g) = s$, so that $Tf(y) = \omega(r, y) = Tr(y)$ and $Tg(y) = \omega(s, y) = Ts(y)$.   Then $T(f + g)(y) = T(r + g)(y) = T(r + s)(y) = \omega(r + s, y)$, whence $H_y(f + g) = r + s = H_y(f) + H_y(g)$.   The result for products follows similarly.

By [2, 10.5(c)], there is a mapping ϕ of E into vX defined as follows: For each $y \in E$, $\phi(y)$ is the unique point of vX such that $H_y(f) = f^v(\phi(y))$ for all $f \in C(X)$.   Since we have

(2) $$f^v(\phi(y)) = \alpha(Tf(y), y),$$

it follows that $Tf(y) = \omega(f^v(\phi(y)), y)$, if $y \in E$.   And it is immediate that $T1(y) = T0(y)$ if $y \in Y - E$.   Thus, equation (1) holds.   Now, for any $f \in C(X)$, the function K_f defined by

$$K_f(y) = \alpha(Tf(y), y)$$

is in $C(E)$; so by [2, 3.8], equation (2) implies that ϕ is continuous.

From the definition of E, and the discussion in the first paragraph of the proof, we conclude that $\{y \in Y : T1(y) = T0(y)\}$, which is obviously closed, coincides with $Y - E$; so E is an open subset of Y.   Also, it is easy to see that the following statements are equivalent: $\phi[E]$ is not dense in vX; there exist $f^v, g^v \in C(vX)$ that coincide on $\phi[E]$ but not on vX; there exist $f, g \in C(X)$ such that $f \neq g$ but $\omega(f^v(\phi(y)), y) = \omega(g^v(\phi(y)), y)$ for all $y \in E$; T is not one-one.

Now consider the special case in which T is onto.   Clearly $T[C(X)](y) = \mathbf{R}$ for each $y \in Y$, so that $E = Y$.

If $\phi(y) = \phi(z)$, then $H_y = H_z$, which implies that

$$T(\cdot)(y) = \omega(\alpha(T(\cdot)(z), z), y).$$

Since T is onto, we must have $y = z$.   Thus, ϕ is one-one.

For each $g \in C(Y)$, the function defined by $y \to \omega(g(y), y)$ is in $C(Y)$; it follows that $\{K_f : f \in C(X)\} = C(Y)$.   Hence, writing (2) in the form $f^v(x) = \alpha(Tf(\phi^\leftarrow(x)), \phi^\leftarrow(x)) = K_f(\phi^\leftarrow(x))$, we see that $\phi^\leftarrow$ is continuous, by [2, 3.8].

Finally, $\phi[Y]$ is C-embedded in vX: If $g \in C(\phi[Y])$, then the function defined by $y \to \omega(g(\phi(y)), y)$ is in $C(Y)$.   Thus for some $f \in C(X)$, $\omega(g(\phi(y)), y) = \omega(f^v(\phi(y)), y)$ for each $y \in Y$, whence $g(\phi(y)) = f^v(\phi(y))$.   Therefore f^v is an extension of g.

COROLLARY.   *Let X and Y be realcompact spaces.   If there exists a one-one transformation T of $C(X)$ onto $C(Y)$ satisfying conditions* (a) *and* (b), *then X and Y are homeomorphic.*

Proof. We know from Theorem 1 that ϕ is a homeomorphism of Y onto a dense, C-embedded subset of $vX = X$. By [2, 8.6], $vX = v(\phi[Y])$. But $\phi[Y]$ is also realcompact, so $\phi[Y] = X$.

THEOREM 2. *Let X and Y be completely regular Hausdorff spaces, and let ϕ be a continuous mapping from an open subset E of Y into vX, and ω a continuous mapping from $\mathbf{R} \times Y$ into $\mathbf{R}$ such that $\omega(\cdot, y)$ is a homeomorphism into $\mathbf{R}$ for each $y \in E$, and a constant mapping for each $y \in Y - E$. Assume[5] that the transformation T defined for each $f \in C(X)$ by*

$$Tf(y) = \omega(f^v(\phi(y)), y) \qquad if \quad y \in E,$$

$$= the\ range\ of\ \omega(\cdot, y) \quad if \quad y \in Y - E,$$

is into $C(Y)$. Then T satisfies conditions (a) and (b) of Theorem 1. Furthermore, T is one-one if and only if $\phi[E]$ is dense in vX.

If we assume in addition that ϕ is a homeomorphism of Y onto a C-embedded subset of vX, then T maps $C(X)$ onto $C(Y)$.

Proof. For each $y \in Y - E$, it is obvious that both (a) and (b) are satisfied by T. For $y \in E$, it follows easily from the hypothesis that $\omega(\cdot, y)$ is one-one, and from the pointwise definition of the ring operations in $C(X)$ that T satisfies (a); and we have $Tr(y) = \omega(r, y)$, so that the fact that T satisfies (b) is a consequence of the continuity of $\omega(\cdot, y)$ and the definition of T. The proof that T is one-one if and only if $\phi[E]$ is dense in vX is the same as in Theorem 1.

Now assume that ϕ is a homeomorphism of Y onto a C-embedded subset of vX. Let α be the function from a subset of $\mathbf{R} \times Y$ onto $\mathbf{R}$ such that $\alpha(\cdot, y)$ is the inverse of $\omega(\cdot, y)$ for each $y \in Y$. It follows from the lemma that α is jointly continuous. Given $g \in C(Y)$, define f by

$$f(x) = \alpha(g(\phi^{\leftarrow}(x)), \phi^{\leftarrow}(x)) \qquad\qquad for\ x \in \phi[Y].$$

Then $f \in C(\phi[Y])$; so there exists $h \in C(X)$ such that $h^v \mid \phi[Y] = f$. We have $Th(y) = \omega(h^v(\phi(y)), y) = \omega(f(\phi(y)), \phi^{\leftarrow}(\phi(y))) = g(\phi^{\leftarrow}(\phi(y))) = g(y)$ for each $y \in Y$. Hence T maps $C(X)$ onto $C(Y)$.

3. Remarks

Each of the hypotheses in Theorem 1 (including the special hypothesis that T is onto) is independent of the remaining ones. That T need not be onto is clear. Examples showing that each hypothesis in (b) is independent may be easily constructed using spaces of one or two points; details are left to the reader. The examples below show that each hypothesis in (a) is independent.

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[5] Obviously the assumption is superfluous if the boundary of E is empty (in particular, if $E = Y$). However, there seems to be no nice *necessary* condition that implies that every Tf is continuous.

Example 1.   Let X be a space of two points, and let Y be a space of one point.   For any $(a, b) \in C(X)$, define $T(a, b)$ to be ab if $a \geqq 0$, and $-ab$ if $a < 0$.   We assert that T satisfies all of the hypotheses except the first one in (a).   The only point that may need explanation is the proof that the second hypothesis in (a) is satisfied by T.   Indeed, assume that $T(a, b) = T(c, d)$, and let $(e, f) \in C(X)$ be arbitrary.   If either a or c is zero, the desired result is obvious.   If a and c have the same (opposite) sign, then $ab = cd$ $(ab = -cd)$, whence $aebf = cedf$ $(aebf = -cedf)$.   But, either $e = 0$, or ae and ce also have the same (opposite) sign; so we have $T(ae, bf) = T(ce, df)$.

Example 2.   Let X be a compact interval in $\mathbf{R}$, and let Y be the union of two disjoint compact intervals in $\mathbf{R}$.   Then $C(X)$ and $C(Y)$ have the same dimension as vector spaces over $\mathbf{R}$, so there exists a vector space isomorphism T from $C(X)$ onto $C(Y)$.   Now, any transformation S that satisfies (a) and is the identity on the constant functions is a ring homomorphism, because the proof that H_y is a homomorphism remains valid for S, and $H_y(f) = Sf(y)$ for each $y \in Y$.   Thus if T satisfied (a), it would be a ring isomorphism of $C(X)$ onto $C(Y)$, which is impossible.   It is therefore clear that T satisfies all of the hypotheses except the second one in (a).

Our remaining remarks relate the type of transformation postulated in Theorem 1 to more familiar types.

In the first place, it is easy to show that every ring homomorphism satisfies both (a) and (b), and that, in fact, Theorem 1 may be regarded as a generalization of [2, 10.8] (as well as part of [2, 10.3]); for, as was observed in [6], if T is a ring homomorphism, each $\omega(\cdot, y)$ $(y \in E)$ is a nonzero ring homomorphism of $\mathbf{R}$ into $\mathbf{R}$, and hence is the identity [2, 0.22].

The class of lattice homomorphisms, which is more general [2, 1.6], contains mappings that are quite different.   One can see quickly that a lattice homomorphism need not satisfy either part of (a), by taking X and Y to be one-point spaces, and T to be a suitable monotone function that is not one-one.   Kaplansky [3, p. 629] gave an example of a lattice automorphism that is not continuous in the topology of uniform convergence, from which it follows that it cannot be represented in terms of a continuous ω.   The following example, which is almost the same as his, is given to show that a lattice automorphism need not satisfy any of the four hypotheses in (a) and (b), and in fact has no representation of the type obtained in Theorem 1 even if ω is allowed to be discontinuous.[6]

Example 3.   First, for each $r \in \mathbf{R}$, we define $\psi(r, \cdot)$ on $\mathbf{R}$ as follows: If $r \leqq 0$, $\psi(r, \cdot) = r$; if $r > 0$, $\psi(r, \cdot)$ coincides with r to the left of r^{-1}, and

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[6] When we say that a transformation T "has no representation", we mean that for at least one $y \in Y$ and one $f \in C(X)$, $Tf(y)$ cannot be found by evaluating a fixed function at $f^v(\phi(y))$, where ϕ is a continuous mapping of a subset of Y into vX that is determined in some natural way from T (for example, by means of maximal ring ideals when T is a ring isomorphism).

with $1 + r - (ri)^{-1}$ to the right. Let Σ be the space defined in [2, 4M]. Given $f \in C(\Sigma)$, Tf is defined to be the extension to Σ of the function $\psi(f(n), n)$ on the subspace $\mathbf{N}$; this is possible because $\mathbf{N}$ is C^*-embedded in Σ [2, 4M.5]. It is not hard to verify that T is a lattice automorphism of $C(\Sigma)$. To see that (a) fails to hold for T, one may take, for example, the extensions to Σ of the following functions on $\mathbf{N}$: for the first part, $f = \mathbf{i}^{-(1/2)} - \mathbf{1}$, $g = -\mathbf{1}$, $h = \mathbf{1}$; for the second part, $f = (2\mathbf{i})^{-1}$, $g = \mathbf{i}^{-1}$, $h = \mathbf{2}$. That (b) fails to hold is trivial. Finally, T has no representation. For, any homeomorphism ϕ must take σ to σ, and if $Tf \in C(\Sigma)$ and $0 < Tf(\sigma) \leqq 1$, then $f(\sigma) = 0$; thus, if there did exist a representation, we should have $Tf(\sigma) = \omega(0, \sigma)$ for each such f, which is impossible.

Example 3 can be modified to give a transformation of $C(\Sigma)$ into (but not onto) $C(\Sigma)$ that is an isomorphism with respect to the multiplicative semigroup structure of $C(\Sigma)$, but again has no representation even if ω is allowed to be discontinuous. This time we define $\psi(r, n)$, for $n \in \mathbf{N}$, to be $(\text{sgn } r) |r|^{1/n}$. To see what ϕ is for the transformation T defined by means of ψ, one may note that in the general definition [4, 3.1], which is given only for compact spaces and isomorphisms that are onto but applies here also, the sets O_p, $p \in Y$, are mapped onto sets O_p, $p \in X$. By considering specific functions, it is easy to see that ϕ must be the identity. Thus, as before, T has no representation.

Milgram [4] has discussed the representation of multiplicative semigroup isomorphisms that are onto, for the case in which X and Y are compact. He shows that every transformation of this type does have a representation, but with $\omega(\cdot, y)$ possibly discontinuous at a finite number of isolated points. The natural setting for this question is with X and Y realcompact, since the isomorphism still induces a homeomorphism of Y onto X (see [2, p. 271]). The existence of a representation in the compact case is proved in [4, 4.4]. Now, the entire discussion of Milgram leading to this result is valid in the general setting, except for the extension of a function in the proof of 4.1; since the function in question is bounded, one simply works with the Stone-Čech compactification in this proof.[7] (The results about the precise nature of ω, on the other hand, obviously will not carry over completely.) Finally, we observe that discontinuities can actually occur (this is not stated specifically by Milgram): Simply take X and Y to be one-point spaces, and let S be any discontinuous one-one additive function from $\mathbf{R}$ onto $\mathbf{R}$. Then define $T(r) = (\text{sgn } r) \exp(S(\log |r|))$ for $r \neq 0$, and $T(0) = 0$. Clearly T is also discontinuous, so that in the trivial representation that is the only one possible, ω must be discontinuous.

In the converse direction, how can a transformation T of the type postu-

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[7] There is a slight gap in the original proof of 4.1: It is not shown that $f'(H(x_0))$ cannot be -1 if $f(x_0) = 1$. However, by considering the square root of $f \vee 2^{-1}$, one can see quickly that $f'(H(x_0))$ is nonnegative.

lated in Theorem 1 be described in terms of familiar types?   To provide such a description, we start with the representation of T given by Theorem 1, and set

$$A = \{y \in E : \omega(\cdot, y) \text{ is increasing}\} \quad \text{and} \quad B = \{y \in E : \omega(\cdot, y) \text{ is decreasing}\}.$$

It follows from the proof of the lemma that A and B are open subsets of Y. Now, we may view T as made up of mappings from $C(X)$ into the three sets $C(A)$, $C(B)$, and $C(Y - E)$.   Because the lattice operations are defined pointwise, these mappings are, respectively, a lattice homomorphism, a lattice anti-homomorphism, and a constant mapping.   Notice, incidentally, that A is open-and-closed when $E = Y$; so if Y is connected, every transformation with $E = Y$ is either a lattice homomorphism or a lattice anti-homomorphism.

Thus, of the many classes of transformations preserving some structure on $C(X)$ that have been studied, the class of lattice homomorphisms seems to be closest to the class we have described.   Roughly speaking, the lattice homomorphisms are not quite right because they may be discontinuous, and they do not take into account the possibility of both preserving and reversing the order.   The isometries (in the cases where they are defined) do not have either of these defects, but do not allow for changes of distance in **R**.

Finally, we note that one may give a simple example of a transformation of the type postulated in Theorem 1 that, simultaneously, fails to be an isomorphism with respect to the multiplicative semigroup, lattice (and hence ring), and partially ordered additive group (and hence lattice-ordered group) structures of $C(X)$ and $C(Y)$, and also fails to be an isometry.   Let X and Y be two-point spaces, and let T be defined by $T(a, b) = (-a, 2b)$.   It is easy to see that T satisfies our requirements.

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UNIVERSITY OF ILLINOIS
    URBANA, ILLINOIS
UNIVERSITY OF ROCHESTER
    ROCHESTER, NEW YORK