A NOTE ON THE GROUPS OF ORDER THIRTY-TWO

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The classification of the groups of order thirty-two has had an interesting and complicated history. In 1896, Professor G. A. Miller prepared a list of fifty-one groups of order thirty-two [1] formed by extending groups of order sixteen according to an automorphism of order two. Even though this method of constructing groups duplicates them in ways that are subtle and unpredictable, the list was considered to be correct as late as 1935 [2].[1] The next year, Professor Miller reconsidered this system of groups [3], describing only forty-seven groups, and charged the difference to duplicates in the first list.[2] In 1940, Philip Hall defined isoclinism and developed a procedure for the classification of prime-power groups [13]. J. K. Senior, using Hall's method, has prepared lists of groups of orders 2^a, $a \leqq 6$, which include the fifty-one groups of order thirty-two. In view of this confirmation of the early listing, attention turns quite naturally to the second list of Professor Miller. The general theorems he used are concerned with groups having five or fewer squares and are not sufficient to determine all the groups of order thirty-two since these may contain as many as eight squares. This paper gives a more careful consideration of groups having six, seven, or eight squares, completes the list by means of the general theorems, and, therefore, determines and corrects the oversight which caused the discrepancy between [1] and [3].

1. Summary of the general theorems which apply to groups of order thirty-two

Let H denote the subgroup generated by the elements that are squares in a group G of order thirty-two. Theorems concerning the relationship of H and G are sufficient for the determination of forty-one distinct groups. In the table, the groups described in [3] are matched with the numbers of the groups given in [1].

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[1] The discussion of this list was extended; cf. references [14] and [15].

[2] Reference [3] was brought to the attention of the editor of *The Collected Works of G. A. Miller* only after the death of Professor Miller. The editor's footnote, vol. V, p. 222, reads: "The reader might justifiably have expected a fuller consideration of this discrepancy especially in view of the discussions of these groups cited in the index of volume I. This set of groups will not be in a satisfactory state until the groups of the two determinations are matched against each other. H. R. Brahana." This paper gives a tribute to Professor G. A. Miller by showing that the list he first prepared is indeed correct even though it preceded the work of Hall and Senior by more than 40 years.

[4]	If the index of H in G is two, G is cyclic.	
[5]	If $H = \{1\}$, G is elementary abelian.	22
[6]	If H is of order two, there are six groups of order 32.	24, 26, 28, 29, 48, 50
[7]	If H is cyclic of index four in G, there are five groups.	1, 2, 3, 4, 5
[8]	If H is cyclic of order four, there are nine groups.	6, 8, 15, 20, 44, 45, 9, 17, 18
[9]	If G has exactly three squares, there are four groups.	30, 31, 35, 36
[10]	If G has four squares which constitute the four-group, there are eleven groups.	25, 34, 37, 27, 23, 33, 39, 40, 41, 42, 43
[11]	If G has four squares which generate an elementary abelian group of order eight, there is just one group of order thirty-two.	32
[12]	If G has exactly five squares, there are three groups.	13, 14, 19

There remain ten groups of the first list which inspection shows to contain six or eight squares, and which could not, therefore, be included under the general theorems. Six of them, numbers 49, 7, 16, 10, 12, and 38, are described in [3]. The other four are numbers 11, 21, 46, and 47.

2. Groups of order thirty-two having six, seven, or eight squares

If a group G of order thirty-two has exactly eight squares, they constitute an H of order eight which must be abelian by the theorem that fewer than nine squares in a group must be relatively commutative [5]. If H is of type (1,1,1), the subgroup generated by H and an element of order four must be a group of order sixteen, either H_{10} or H_{13}, number 10 or 13 in the list of these groups [1]. They contain two and three squares respectively. An element which extends either group of order sixteen must leave H invariant fixing at least four elements in H, and the new coset of sixteen elements, therefore, contains at most four distinct squares. No such G could have eight distinct squares.

If the eight squares constitute an H of type (2,1), G may be number 7 if it is abelian. A non-abelian G must involve a subgroup of order sixteen generated by H and an element of order eight. The subgroup is of type (3,1) if it is abelian, or if non-abelian, it is H_3, number 3 in the list of groups of order sixteen. No extension of H_3 gives a group of order thirty-two involving eight squares, and the single extension of the abelian group of type (3,1) having eight squares is number 16.

If G has only six squares, they generate an H of type (1,1,1) or (2,1) In the first case, G must contain H_{10} or H_{13}. Since any G containing H_{13} is homomorphic to the octic group and consequently contains H_{10} as well, all such groups may be found by examining possible extensions of H_{10}. There is only one, number 49.

A square of order four in an H generated by six squares requires that G contain an abelian subgroup of type (3,1) or H_3. All possible extensions of order thirty-two from these two groups give seven distinct groups having just six squares. These are numbers 10, 12, 38, 11, 21, 46, and 47.

No group of order thirty-two can have only seven squares, for if H is of type (2,1), G must contain a subgroup of type (3,1), and if H is of type (1,1,1), it is contained in a subgroup of type (2,1,1) or an H_{10}. Extensions of these groups having seven squares do not exist.

3. Clarification of the oversight

In paragraph two, p. 221 of the third reference, Professor Miller states that a group having H of type (2,1) must contain a subgroup of type (3,1) or type (2,2). It is not the group of type (2,2) which is necessary, but the H_3 generated by H and an element of order eight which interchanges two elements of order two in H. Extensions of H_3 introduce number 46 and number 47 which have no abelian subgroup of order sixteen. Groups number 11 and number 21 have a subgroup of type (3,1) but none of type (2,2). Their omission is likely due to overlooking certain automorphisms of the abelian subgroup of type (3,1).

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