ON UNIVERSAL TRANSFORMATION GROUPS

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I. Introduction

In this paper, we characterize minimal sets (X, T, π) where X is Tychonoff (see [6]) by algebras of continuous functions, study compactifications of a transformation group, and prove that there is a unique universal compactification up to isomorphism of transformation groups. We develop several algebraic-topology and Banach-algebra properties for the universal minimal set associated with a discrete group (see [3]). We define a universal almost periodic minimal set associated with any topological group and prove there is a unique universal almost periodic minimal set associated with a topological group up to homeomorphism of spaces. In particular, we show that the phase space of an almost periodic minimal set (X, T, π) with compact Hausdorff space X is homeomorphic to a quotient space of a topological group $L(T)$, which is the maximal ideal space of the algebra of all left almost periodic functions on T. In the last section, we define a universal minimal set associated with any topological group and prove there is a unique universal minimal set up to isomorphism, which is a generalization of a result of Professor R. Ellis (see [3]). As a general reference for the notions occurring here consult [6] and [9]. The author wishes to take this opportunity to express his indebtedness to Professor W. H. Gottschalk and Professor H. C. Wang for their encouragement and direction.

II. The general case

Let (X, T, π) be a transformation group with Tychonoff phase space X. Let $C^*(X, R)$ and $C^*(T, R)$ be the algebras of all bounded, continuous, real-valued functions on X and on T, respectively, with the uniform norm. For each $t \in T$, we define

$$(\pi^*)^t : C^*(X, R) \rightarrow C^*(X, R) \quad \text{by} \quad (x)(f(\pi^*)^t) = (x\pi^t)f$$

for $f \in C^*(X, R)$ and $x \in X$, and

$$(\rho^*)^t : C^*(T, R) \rightarrow C^*(T, R) \quad \text{by} \quad (s)(g(\rho^*)^t) = (st)g$$

for $g \in C^*(T, R)$ and $s \in T$, respectively. Then t is an algebra-isomorphism. Let T_d be the set of all these t, for $t \in T$, with the discrete topology. Then T_d is an automorphism group of $C^*(X, R)$ and $C^*(T, R)$, respectively. Thus, we have

LEMMA 1. (1) *These* $(C^*(X, R), T_d, \pi^*)$ *and* $(C^*(T, R), T_d, \rho^*)$ *are transformation groups.*

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(2) *The* $(C^*(T, R), T_d, \rho^*)$ *is always effective.*

(3) (X, T, π) *is effective if* $(C^*(X, R), T_d, \pi^*)$ *is effective.*

(4) *These two transformation groups, however, are never strongly effective.*

Proof. (1) follows from the fact that T_d is an automorphism group. (2) and (3) follow from the fact that the groups T and X are both Tychonoff spaces. (4) holds because $C^*(X, R)$ and $C^*(T, R)$ both contain constant functions.

DEFINITION 1. Let (X, T_{s_1}, π) and (Y, T_{s_2}, δ) be transformation groups with phase groups T_{s_1} and T_{s_2} respectively, such that they both have the same group structure T but they may have different topologies S_1 and S_2 respectively. We say (X, T_{s_1}, π) is *homomorphic into or onto* (Y, T_{s_2}, δ) by ϕ if there is a continuous map ϕ of X into or onto Y such that $\pi^t\phi = \phi\delta^t$ for $t \in T$, or $xt\phi = x\phi t$ for short. If this ϕ is continuous, one-to-one, into or onto, we say (X, T_{s_1}, π) *is continuously isomorphic into or onto* (X, T_{s_2}, π) *by* ϕ. If this ϕ is homeomorphic into or onto, we say (X, T_{s_1}, π) is *topologically isomorphic into or onto* (Y, T_{s_2}, δ).

Remark 1. By this new definition of homomorphism, the results in [4] are still valid if the topology of the phase group T is not involved. If the results in [4] involve the topology of the phase group T, they are still true, in almost all cases, if $S_1 \supset S_2$, i.e., every open set in S_2 is open in S_1.

LEMMA 2. *Let* (X, T_{s_1}, π) *and* (Y, T_{s_2}, δ) *be two transformation groups with Tychonoff spaces X and Y respectively. If there is a continuous mapping* ϕ *from X into Y such that* $\overline{(X)\phi} = Y$ *and* (X, T_{s_1}, π) *is homomorphic into* (Y, T_{s_1}, δ), *then* $(C^*(Y, R), T_d, \delta^*)$ *is homomorphic into* $(C^*(X, R), T_d, \pi^*)$.

Proof. We define $\phi^* : C^*(Y, R) \rightarrow C^*(X, R)$ by $(f\phi^*)(x) = (x\phi)f$ for $f \in C^*(X, R)$ and $x \in X$. Then ϕ^* is an algebra-homomorphism. We show $(f\phi^*)(\pi^*)^t = (f(\delta^*)^t)\phi^*$ for $f \in C^*(Y, R)$ and $t \in T$. For each $x \in X$, we have

$$\begin{aligned}(x)[(f\phi^*)(\pi^*)^t] &= (x\pi^t)(f\phi^*) = (x\pi^t\phi)f \\ &= (x\phi\delta^t)f = (x\phi)(f(\delta^*)^t) \\ &= (x)[(f(\delta^*)^t)\phi^*].\end{aligned}$$

This proves that $(C^*(Y, R), T_d, \delta^*)$ is homomorphic into $(C^*(X, R), T_d, \pi^*)$.

LEMMA 3. *Let* (X, T_{s_1}, π) *and* (Y, T_{s_2}, δ) *be two transformation groups with Tychonoff spaces X and Y respectively. Let* (X, T_{s_1}, π) *be homomorphic into* (Y, T_{s_2}, δ) *by* ϕ. *Then* $\overline{(X)\phi} = Y$ *if and only if* $(C^*(Y, R), T_d, \delta^*)$ *is topologically isomorphic into* $(C^*(X, R), T_d, \pi^*)$ *by* ϕ^*.

Proof. Assume $\overline{(X)\phi} = Y$. By Lemma 2, we know ϕ^* is a homomorphism from $(C^*(Y, R), T_d, \delta^*)$ into $(C^*(X, R), T_d, \pi^*)$. We show it is a one-to-one mapping. For $f, g \in C^*(Y, R)$, if $f\phi^* = g\phi^*$, then $(x)(f\phi^*) = (x)(g\phi^*)$

or $(x\phi)f = (x\phi)g$ for all $x \in X$. Since $\overline{(X)\phi} = Y$ and Y is a Hausdorff space, we have $(y)f = (y)g$ for all $y \in Y$. It follows that $f = g$. We show $\| f\phi^* \| = \| f \|$, for $f \in C^*(Y, R)$. By the definition of the uniform norm, we have

$$\| f\phi^* \| = \sup \{ | (x\phi)f | \mid x \in X \} = \sup \{ | yf | \mid y \in Y \},$$

since $\overline{(X)\phi} = Y$. Consequently, the image of $C^*(Y, R)$, under ϕ^*, is a closed subalgebra of $C^*(X, R)$, and $(C^*(Y, R), T_d, \delta^*)$ is topologically isomorphic into $(C^*(X, R), T_d, \pi^*)$ by ϕ^*.

Assuming $(C^*(Y, R), T_d, \delta^*)$ is topologically isomorphic into $(C^*(X, R), T_d, \pi^*)$ by ϕ^*, we show $\overline{(X)\phi} = Y$. Suppose $\overline{(X)\phi} \neq Y$; there is $y_0 \in Y$ such that $y_0 \notin \overline{(X)\phi}$. Since Y is Tychonoff, there exists a continuous function

$$f : Y \to [0, 1]$$

such that $(y)f = 0$ for $y \in \overline{(X)\phi}$ and $(y_0)f = 1$. Then $f \in C^*(Y, R)$ and $f \neq 0$. However, $f\phi^* \in C^*(X, R)$, and $f\phi^* = 0$. This shows that ϕ^* is not an isomorphism. It is a contradiction to the hypothesis. Therefore

$$\overline{(X)\phi} = Y.$$

THEOREM 1. *Let (X, T, π) be a transformation group with Tychonoff space X. Then (X, T, π) is a minimal set if and only if for each $x \in X$, $(C^*(X, R), T_d, \pi^*)$ is topologically isomorphic into $(C^*(T, R), T_d, \rho^*)$ by x^*.*

Proof. For $s, t \in T$, define $s\rho^t = st$. Then (T, T, ρ) is a transformation group, and (T, T, ρ) is homomorphic into (X, T, π) by each $x \in X$. Thus this theorem is a direct consequence of Lemma 3.

COROLLARY 1. *Let (X, T, π) be a transformation group with a compact Hausdorff phase space X and an Abelian phase group T. If (X, T, π) is an almost periodic minimal set, then for $x, y \in X$, $(C^*(X, R))x^* = (C^*(X, R))y^*$.*

Proof. For $x, y \in X$, $f \in C^*(X, R)$, and for $\varepsilon > 0$, there exists $\alpha \in U$, where U is the uniformity of X, such that

$$| (xt)f - (z)f | < \varepsilon/2 \qquad \text{for } z \in (xt)\alpha \text{ and } t \in T.$$

This statement is true, because f is uniformly continuous. Since X is a compact Hausdorff minimal set, it is known that T is equicontinuous on X. It follows that there exists $\beta \in U$ such that

$$(x)\beta t \subset (xt)\alpha \qquad \text{for } t \in T.$$

Since $x \in \overline{yT}$, there exists $s \in T$ such that $ys \in (x)\beta$ and

$$| (xt)f - (yst)f | < \varepsilon/2 \qquad \text{for } t \in T,$$

where y and s are independent of the choice of t. Since T is Abelian we have

$$(yst)f = (yts)f = (yt)[f(\pi^*)^s] = (t)[(f(\pi^*)^s)y^*].$$

Consequently $\| fx^* - (f(\pi^*)^s)y^* \| < \varepsilon$. From the facts that $f(\pi^*)^s \in C^*(X, R)$ and $(C^*(X, R))y^*$ is closed in $C^*(T, R)$, it follows that $fx^* \in (C^*(X, R))y^*$. Similarly, we can show that $fy^* \in (C^*(X, R))x^*$ for $f \in C^*(X, R)$. Consequently $(C^*(X, R))x^* = (C^*(X, R))y^*$.

Remark 2. Corollary 1 shows that if (X, T, π) is a compact Hausdorff almost periodic minimal set, then for each pair $x, y \in X$, $x^*(y^*)^{-1}$ is an automorphism of $C^*(X, R)$.

LEMMA 4. *Let (X, T_{s_1}, π) and (Y, T_{s_1}, δ) be two transformation groups with compact Hausdorff spaces X and Y respectively. Let (X, T_{s_1}, π) be homomorphic to (Y, T_{s_2}, δ) by ϕ. Then,*

(1) *ϕ is onto if and only if $(C^*(Y, R), T_d, \delta^*)$ is continuously isomorphic into $(C^*(X, R), T_d, \pi^*)$,*

(2) *ϕ is one-to-one if and only if $(C^*(Y, R), T_d, \delta^*)$ is homomorphic onto $(C^*(X, R), T_d, \pi^*)$.*

Proof. It is a consequence of Lemma 2 and known facts that ϕ is onto if and only if ϕ^* is one-to-one, and ϕ is one-to-one if and only if ϕ^* is onto.

III. Compactification

DEFINITION 2. *Let (X, T, π) be a transformation group with Tychonoff phase space X.* We say a transformation group (Y, T_x, ρ) is a *compactification of (X, T, π) by ϕ* if T_s is a topological group with the same group structure as T and with a topology s, and there is a homeomorphism ϕ from X into Y such that (X, T, π) is isomorphic into (Y, T_s, ρ) by ϕ. A compactification (Y, T_s, δ) of (X, T, π) by ϕ is called *universal* if for any other compactification (Z, T_v, δ) of (X, T, π) by f there is a continuous mapping g from Y onto Z such that $\phi \circ g = f$ on X and (Y, T_s, ρ) is homomorphic onto (Z, T_v, δ) by g.

LEMMA 5. *There is a universal compactification of (X, T, π) with Tychonoff phase space X.*

Proof. Let $\beta(X)$ be the Čech-Stone compactification of the space X. Then, for every $t \in T$, there is a unique extension $(\pi^*)^t$ of π^t such that $(\pi^*)^t$ is also a homeomorphism of $\beta(X)$. Let $T_d = \{t \mid t \in T\}$ with the discrete topology. Then $(\beta(X), T_d, \pi^*)$ is a transformation group. It is easy to see that this is a compactification of (X, T, π). We show it is universal. Let (Z, T_v, δ) be a compactification of (X, T, π) by f. Then there is a continuous mapping $\tilde{f} : \beta(X) \to Z$ which is an extension of $f : X \to Z$. Since $\overline{(X)f} = Z$ and $\beta(X)$ is compact, we have $(\beta(X)\tilde{f} = Z$, or $\tilde{f}$ is onto. We show $(\beta(X), T_d, \pi^*)$ is homomorphic onto (Z, T_v, δ) by $\tilde{f}$. It is enough to show that $((y)(\pi^*)^t)\tilde{f} = ((y)\tilde{f})(\delta)^t$, for $y \in \beta(X)$ and $t \in T$. Suppose there are $y \in \beta(X)$ and $t \in T$ and $((y)(\pi^*)^t)\tilde{f} \neq ((y)\tilde{f})(\delta)^t$. By continuity and the fact that Z is Hausdorff, there exists $\alpha \in V$, where V is the uniformity of $\beta(X)$, such that

$$((y)\alpha(\pi^*)^t)\tilde{f} \cap ((y)\alpha\tilde{f})(\delta)^t = \phi.$$

Since X is dense in $\beta(X)$, there exists $x \in X \cap (y)\alpha$ such that

$$(x\pi^*)f \neq (xf)\delta^t.$$

It is a contradiction to the hypothesis that (X, T, π) is isomorphic into (Z, T_v, δ) by f. Hence $(\beta(X), T_d, \pi^*)$ is homomorphic onto (Z, T_v, δ) by f. It is clear that $e \circ \tilde{f} = f$ where e is the evaluation map of X into $\beta(X)$. It follows that $(\beta(X), T_d, \pi^*)$ is a universal compactification of (X, T, π).

THEOREM 2. *There is a universal compactification of (X, T, π) with Tychonoff phase space X, and any two universal compactifications of (X, T, π) are topologically isomorphic.*

Proof. The first statement is Lemma 5. We show the second statement. Let (Y, T_s, ρ) be another universal compactification of (X, T, π) by f. We show (Y, T_s, ρ) and $(\beta(X), T_d, \pi^*)$ are topologically isomorphic onto. Since (Y, T_s, ρ) is universal, there exists a continuous mapping $g : Y \to \beta(X)$ such that (Y, T_s, ρ) is homomorphic onto $(\beta(X), T_d, \pi^*)$, and $f \circ g = e$ where e is the evaluation map of X into $\beta(X)$. Let $\tilde{f} : \beta(X) \to Y$ be the continuous extension of $f : X \to Y$. Let $\tilde{e} : \beta(X) \to \beta(X)$ be the continuous extension of e. Then $\tilde{e}$ is a homeomorphism and $\tilde{f} \circ g = \tilde{e}$ on $\beta(X)$. Hence $\tilde{f}$ is a homeomorphism of $\beta(X)$ onto Y, and $(\beta(X), T_d, \pi^*)$ is topologically isomorphic onto (Y, T_s, ρ) by $\tilde{f}$. The uniqueness is proved.

COROLLARY 2. *Let (X, T_{s_1}, π) and (Y, T_{s_2}, ρ) be homomorphic. Then their universal compactifications are also homomorphic.*

Proof. Let (X, T_{s_1}, π) and (Y, T_{s_2}, ρ) be homomorphic by f. Then $(\beta(X), T_d, \pi^*)$ is homomorphic to $(\beta(Y), T_d, \rho^*)$ by $\tilde{f}$, where $\tilde{f}$ is the continuous extension of $f : X \to Y$.

IV. Minimal sets

Let T be a topological group. There exists at least one minimal set M in the transformation group $(\beta(T), T_d, \tilde{\rho})$, where $\beta(T)$ is the Čech-Stone compactification of T. Then $(M, T_d, \tilde{\rho})$ is a transformation group such that $\overline{xT} = M$ for $x \in M$. If T is discrete, Professor R. Ellis called, in [2], $(M, T_d, \tilde{\rho})$ a universal minimal set associated with T.

LEMMA 6. *The transformation group $(M, T_d, \tilde{\rho})$ is homomorphic onto any compact Hausdorff minimal set (X, T, π).*

Proof. By the proof of Theorem 1, we know (T, T, ρ) is homomorphic into (X, T, π) by $x \in X$. By Corollary 2 to Theorem 2, we know that $(\beta(T), T_d, \tilde{\rho})$ is homomorphic to (X, T, π) by $\tilde{x}$, where $\tilde{x}$ is the extension of x. Since $\overline{xT} = X$, we know $\beta(T)\tilde{x} = X$ or $\tilde{x}$ is onto. Choose a minimal set $(M, T_d, \tilde{\rho})$ from $(\beta(T), T_d, \tilde{\rho})$. Since (X, T, π) is minimal, it follows that $(M, T_d, \tilde{\rho})$ is homomorphic onto (X, T, π) by $\tilde{x}$.

THEOREM 3. *Let T be a topological group as well as a normal space, and its Čech groups $H_f^q(T; G) = H_q^f(T; G) = 0$ for $q \geqq n + 1$. Let $(M, T_d, \tilde{\rho})$ be* *a minimal set chosen from* $(\beta(T), T_d, \mathfrak{p})$. *Then*

(1) $H_{q+1}(\beta(T), M; G) \cong H_q(M; G)$ *for* $q \geqq n+1$ *and* $H_{n+1}(\beta(T), M; G)$ *is isomorphic into* $H_n(M; G)$, *where* G *is a compact group or a vector space over a field.*

(2) $H^q(M; G) \cong H^{q+1}(\beta(T), M; G)$ *for* $q \geqq n+1$ *and* $H^n(M; G)$ *is homomorphic onto* $H^{n+1}(\beta(T), M; G)$, *where* G *is a* K*-module over any ring* K.

If T *is of covering dimension* n, *then* $H_q(M; G) = 0$, $H^q(M; G) = 0$ *for* $q \geqq n+1$.

Proof. By using the facts that

$$H_q(\beta(T); G) \cong H_q^f(T; G) = 0 \quad \text{and} \quad H^q(\beta(T); G) \cong H_f^q(T; G) = 0$$

for $q \geqq n+1$, and exact sequences of pair $(\beta(T), M)$, (1) and (2) follow. If T is of covering dimension n, then it is known that the covering dimension of $\beta(T)$ is also n. Consequently

$$H_q(M; G) = 0 \quad \text{and} \quad H^q(M; G) = 0$$

for all $q \geqq n+1$. In particular, if T is discrete, then

$$H_q(M; G) = H^q(M; G) = 0$$

for $q \neq 0$, and if $T = R$, then

$$H_q(M; G) = H^q(M; G) = 0$$

for $q \neq 0, 1$.

LEMMA 7. *Let* $(M, T_d, \mathfrak{p})$ *be a universal minimal set associated with a directed group* T_d. *Then* M *is a retract of* $\beta(T_d)$.

Proof. Let $x \in M$; then $f_x : T_d \to M$ by $(t)f_x = xt$ is continuous, and $\overline{(T_d)f_x} = \overline{xT_d} = M$. Consequently, there exists an extension $\tilde{f}_x$ of f_x such that $\tilde{f}_x : \beta(T_d) \to M$ is a continuous onto mapping, and by Corollary 2 to Theorem 2, $(\beta(T_d), T_d, \mathfrak{p})$ is homomorphic onto $(M, T_d, \mathfrak{p})$ by $\tilde{f}_x$. Since $(M, T_d, \mathfrak{p})$ is universal minimal, the mapping, $\tilde{f}_{x|M} : M \to M$ is homeomorphic onto (see [2]). Then $r_x = f_x \circ (\tilde{f}_{x|M})^{-1}$ is a retraction, namely,

$$r_x : \beta(T_d) \to M.$$

Remark 3. The proof of this lemma shows that we can consider the points of M as a set of homeomorphisms of M.

THEOREM 4. *Let* r_x *be the retraction of* $\beta(T_d)$ *onto* M *as we state in Lemma* 7. *Let* $i : M \to \beta(T_d)$ *be the inclusion mapping. Then*

$$C^*(\beta(T_d), R) = \text{image } (r_x^*) + \text{kernel } (i^*),$$

where

$$r_x^* : C^*(M, R) \to C^*(\beta(T_d), R) \quad by \quad fr_x^* = r_x f$$

for $f \in C^*(M, R)$, *and*

$$i^* : C^*(\beta(T_d), R) \to C^*(M, R) \quad by \quad gi^* = ig$$

for $g \in C^*(\beta(T), R)$, *and* image (r^*) *is a closed subalgebra, and* kernel (i^*) *is a closed ideal.*

Proof. Since

$$\begin{array}{ccc} & \beta(T_d) & \\ {\scriptstyle i}\nearrow & & \searrow{\scriptstyle r_x} \\ M & \xrightarrow[i_1]{} & M \end{array}$$

is commutative where i_1 is the identity mapping, we have

$$\begin{array}{ccc} & C^*(\beta(T_d), R) & \\ {\scriptstyle i^*}\swarrow & & \nwarrow{\scriptstyle r_x^*} \\ C^*(M, R) & \xleftarrow[i_1^*]{} & C^*(M, R) \end{array}$$

is commutative, and by Lemma 4, we know i_1^* is isomorphic onto, i^* is homomorphic onto, and r_x^* is isomorphic into. By Theorem 1, we know that image (r_x^*) is a closed subalgebra of $C^*(\beta(T_d), R)$. By this commutative diagram, we have the desired results

V. Universal almost periodic minimal sets

Let T be a topological group. Let $L^*(T, R)$ be the algebra of all real-valued, left almost periodic functions (see [9]) on this topological group T with the uniform norm.

LEMMA 8. *Let* (X, T, π) *be an almost periodic, minimal set with compact Hausdorff space* X. *For each* $x \in X$,

$$x^* : C^*(X, R) \to L^*(T, R) \quad by \quad (t)fx^* = (xt)f,$$

for $f \in C^*(X, R)$ *and* $t \in T$, *is isomorphic into, and the image under* x^* *is closed in* $L^*(T, R)$.

Proof. By Theorem 1, we know x^* is an isomorphism from $C^*(X, R)$ into $C^*(T, R)$. However, (X, T, π) is an almost periodic minimal set, and X is compact Hausdorff; it is not hard to see that for every $f \in C^*(X, R)$, fx^* is left almost periodic. Since $L^*(T, R)$ is a subalgebra of $C^*(T, R)$, we know

$$x^* : C^*(X, R) \to L^*(T, R)$$

is isomorphic into. That the image of $C^*(X, R)$ under x^* is closed follows from Lemma 3.

DEFINITION 3. Let T be a topological group. A transformation group (X, T, π) is called a *universal almost periodic minimal set associated with* T if (1) (X, T, π) is an almost periodic minimal set with compact Hausdorff phase space X, (2) there is a continuous mapping $\alpha: T \rightarrow X$, with $\overline{(T)\alpha} = X$ such that $\alpha^* : C^*(X, R) \rightarrow L^*(T, R)$ induced by α is an isometric and isomorphic onto mapping, and (3) for any almost periodic minimal set (Y, T, δ) with compact Hausdorff phase space Y, Y is a continuous image of X.

LEMMA 9. *For every topological group T, there is a universal almost periodic minimal set associated with T.*

Proof. Let $L^*(T, R)$ be the algebra of all real-valued, left almost periodic functions on T. Then it is known (see [9]) that the maximal ideal space, with the hull-kernel topology, of $L^*(T, R)$ is a compact group $L(T)$, and there is a continous homomorphism $\alpha: T \rightarrow L(T)$ such that $\overline{(T)\alpha} = L(T)$ and $\alpha^*: C^*(L(T), R) \rightarrow C^*(T, R)$ induced by α is an isometric and isomorphic onto mapping. Define $\pi : L(T) \times T \rightarrow L(T)$ by $(x, t)\pi = x \cdot \alpha(t)$ for $x \in L(T)$ and $t \in T$. Then $(L(T), T, \pi)$ is a transformation group, and it is an almost periodic minimal set (see [6]). We show it is universal. Let (Y, T, δ) be an almost periodic minimal set with Y as compact Hausdorff phase space. For each $y \in Y$

$$y : T \rightarrow Y \quad \text{is continuous,}$$

and by Lemma 8, we know $y^* : C^*(X, R) \rightarrow L^*(T, R)$ is isomorphic into. Hence $y^*(\alpha^*)^{-1} : C^*(X, R) \rightarrow C^*(L(T), R)$ is isomorphic into. It is known there is $f : L(T) \rightarrow X$ which is a continuous and onto mapping such that $f^* = y^*(\alpha^*)^{-1}$. This shows $(L(T), T, \pi)$ is a universal almost periodic minimal set associated with the given T.

THEOREM 5. *For every topological group T, there is a universal almost periodic minimal set associated with T. Let (X, T, π) and (Y, T, δ) be any two universal almost periodic minimal sets associated with T. Then X and Y are homeomorphic to each other.*

Proof. The first statement is Lemma 9. We show the second statement. Since, by definition, $C^*(X, R) \cong L^*(T, R)$ and $C^*(Y, R) \cong L^*(T, R)$, we have $C^*(X, R) \cong C^*(Y, R)$. Hence there is a homeomorphism α such that $\alpha: X \rightarrow Y$ is a homeomorphic onto mapping.

COROLLARY 3. *For every almost periodic minimal set (Y, T, π) with compact Hausdorff phase space, this space Y is homeomorphic to a quotient space of the compact group $L(T)$.*

Proof. By Theorem 5, we know there is a continuous mapping $\phi: L(T) \rightarrow Y$ from $L(T)$ onto Y. For each $s, t \in L(T)$, define $s\,R\,t$ if and only if $(s)\phi = (t)\phi$. Then R is a closed equivalence relation, and the quotient space $L(T)/R$ is homeomorphic with Y.

VI. Universal minimal sets associated with a topological group

DEFINITION 4. Let T be a topological group. We say a transformation group (X, T, π) with compact Hausdorff phase space X is a *universal minimal set associated with a topological group T* if any other compact Hausdorff minimal set (Y, T, ρ) associated with the same topological group T is its homomorphic image.

LEMMA 10. *Let T be a topological group. There is a universal minimal set associated with T.*

Proof. Let F be a set of compact Hausdorff minimal sets $(X_\alpha, T, \pi_\alpha)$, $\alpha \in \Gamma$, associated with T, where Γ is the index set corresponding to F. By the preceding theorem, we know F is not empty. Let PX_α be the Tychonoff product of X_α, for $\alpha \in \Gamma$. Define

$$P\pi_\alpha : PX_\alpha \times T \to PX_\alpha \quad \text{by} \quad \{x_\alpha \mid \alpha \in \Gamma\}(P\pi_\alpha)^t = \{x_\alpha \pi_\alpha^t \mid \alpha \in \Gamma\}$$

for $\{x_\alpha \mid \alpha \in \Gamma\} \in PX_\alpha$ and $t \in T$. Then $(PX_\alpha, T, P\pi_\alpha)$ is a transformation group with the compact Hausdorff phase space PX_α. It is known that there is a minimal set M in PX_α. Define

$$P_\alpha : PX_\alpha \to X_\alpha \quad \text{by} \quad P_\alpha\{x_\alpha \mid \alpha \in \Gamma\} = x_\alpha ,$$

for $\{x_\alpha \mid \alpha \in \Gamma\} \in PX_\alpha$ and $t \in T$, to be the α^{th} projection of PX_α onto X_α. Then $(X, T, P\pi_\alpha)$ is homomorphic to $(X_\alpha, T, \pi_\alpha)$ by P_α, for each $\alpha \in \Gamma$. Since $(M, T, P\pi_\alpha)$ and $(X_\alpha, T, \pi_\alpha)$ are minimal sets, the mapping P_α is onto. This shows $(M)_{P_\alpha} = X_\alpha$ for all $\alpha \in \Gamma$. Hence $(M, T, P\pi_\alpha)$ is a minimal set associated with T. Complete the proof by Zorn's Lemma.

THEOREM 6. *Let T be a topological group. There is a unique compact Hausdorff universal minimal set associated with T, up to isomorphism.*

Proof. By the preceding lemma, we know there exists a compact Hausdorff minimal set M, which we choose from $(PX_\alpha, T, P\pi_\alpha)$, $\alpha \in \Gamma$. It is enough to show that any other universal minimal set $(X_\gamma, T, \pi_\gamma)$ associated with T, with compact Hausdorff phase space is isomorphic onto (M, T). Let $E(M, T)$ be the enveloping semigroup (see [4]) of (M, T), and let I be its minimal right ideal. Then (I, T) is a transformation group. For $x \in M$,

$$\pi_x : (I, T) \to (M, T) \quad \text{by} \quad p\pi_x = xp,$$

for $p \in I$, is a homomorphism. Since (M, T) is onto (X, T), and (X, T) is universal, there exists a continuous mapping $g:(X, T) \to (I, T)$ which is homomorphic onto. Consequently, $\pi_x fg:(I, T) \to (I, T)$ is homomorphic onto. By a known result (see Lemma 5, [3]), $\pi_x fg$ is isomorphic onto. Since π_x, f, and g are onto mappings, and M and X are compact Hausdorff, it follows that $f:(M, T) \to (X, T)$ is isomorphic onto. The theorem is proved.

Remark 4. Our universal minimal sets generalize those of [2]. In that paper, Professor Ellis defines universal minimal sets for discrete groups only, and he constructs a universal minimal set associated with a discrete group T from the Čech-Stone compactification of T. In our results, we did not use the Čech-Stone compactification of T. By Theorem 6, however, these two are isomorphic.

COROLLARY 4. *Let T be a maximally almost periodic group (e.g., T is a locally compact Abelian group, a free group of several generators with the discrete topology, etc.). Then the compact Hausdorff universal minimal set associated with T is strongly effective.*

Proof. By Lemma 9, there is a universal almost periodic minimal set X associated with T. Since T is maximally almost periodic, it is not hard to see that X is strongly effective, and so is the universal minimal set associated with T.

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