ON POSITIVE HARMONIC POLYNOMIALS[1]

To Hans Rademacher, in friendship

BY

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1. Introduction

1. In 1915, Georg Pick [2] made the following observation. Let $u(x, y, z)$ belong to the class K of all harmonic functions which are not identically zero, moreover regular and nonnegative in the unit sphere $x^2 + y^2 + z^2 < 1$. We expand u in a series of spherical harmonics:

$$(1.1) \qquad u(x, y, z) = \sum_{n=0}^{\infty} H_n(x, y, z),$$

where H_n is a homogeneous harmonic polynomial of degree n. We have then for $x^2 + y^2 + z^2 \leq 1$,

$$(1.2) \qquad | H_1/H_0 | \leq 3.$$

The constant 3 is the best possible one. Of course, $H_0 = u(0, 0, 0) > 0$.

Inequality (1.2) can be proved and generalized easily with the aid of the well-known representation (cf. [1, p. 444]):

$$(1.3) \qquad H_n(x, y, z) = \frac{2n + 1}{4\pi} r^n \iint u(x', y', z') P_n(\cos \gamma) \, d\sigma'.$$

Here we use the polar equivalent r, θ, ϕ of the cartesian coordinates x, y, z. The integration refers to x', y', z' or 1, θ', ϕ', and γ denotes the spherical distance of the points 1, θ, ϕ and 1, θ', ϕ' on the unit sphere; $d\sigma'$ is the surface element at 1, θ', ϕ', and P_n denotes Legendre's polynomial. We find from (1.3)

$$(1.4) \qquad | H_n/H_0 | \leq 2n + 1.$$

The bound is the best possible and is attained for the function

$$(1.5) \qquad u(x, y, z) = \frac{1 - r^2}{(1 - 2r \cos \gamma + r^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) r^n.$$

2. In 1927, I discussed [3] the following refinement of Pick's inequality. Let N be a nonnegative integer; we denote by K_N the class of finite (terminating) harmonic polynomials

$$(1.6) \qquad u(x, y, z) = \sum_{n=0}^{N} H_n(x, y, z),$$

which are again not identically zero and nonnegative in the unit sphere. The

refinement of (1.2) is then:

(1.7) $|H_1/H_0| \leqq 3\rho_N$,

where ρ_N is the largest zero of the Legendre polynomial $P_{q+1}(t)$ if $N = 2q$ is
even, and the largest zero of the polynomial $P_{q+1}(t) + P_{q+2}(t)$ if $N = 2q + 1$
is odd. In both cases, the bound $3\rho_N$ is sharp, i.e., attained for a suitable
member of the class K_N at a suitable point x, y, z of the unit sphere. The
exact range of the quotient H_1/H_0 as u is arbitrary in K_N and x, y, z is arbitrary
in the unit sphere, is the closed interval $[-3\rho_N, 3\rho_N]$.

3. The purpose of the present note is to deal with the corresponding prob-
lem for the quotient H_2/H_0 referring again to the class K_N. Thus we aim at
a refinement of the inequality (1.4) in the special case $n = 2$. This case can
be treated by methods essentially similar to those used for $n = 1$. The nature
of the result is however slightly different since for $n = 2$ the range of the quo-
tient in question is a nonsymmetrical interval around the origin. The reason
for this discrepancy is that the range of Legendre's polynomial $P_n(x)$ in
$-1 \leqq x \leqq 1$ is the symmetrical interval $[-1, +1]$ if n is odd, and a certain
nonsymmetrical interval $[-\mu_n, +1]$, $0 < \mu_n < 1$, if n is even.

We shall prove the following

THEOREM. *The range of the quotient H_2/H_0, as u runs over the class K_N and
(x, y, z) runs in the unit sphere, is the interval*

(1.8) $5 \cdot (3t'^2 - 1)/2 \leqq H_2/H_0 \leqq 5 \cdot (3t''^2 - 1)/2,$

*where t' and t'' denote the smallest and largest positive zero of $P_{[N/2]+2}(t)$,
$0 < t' < t'' < 1$, respectively, provided $[N/2]$ is even. The meaning of t'' is still
the same if $[N/2]$ is odd, but t' will be in the latter case the smallest positive zero
of the derivative $P'_{[N/2]+2}(t)$.*

The general case, i.e., the corresponding refinement of (1.4) for the class
K_N for an arbitrary n, $n \geqq 3$, seems to be rather difficult. There is, however,
no difficulty in determining the exact range of the quotient H_n/H_0 for arbitrary
n if $u \subset K$. This is the interval $[-\mu_n(2n + 1), 2n + 1]$ where $-\mu_n$ is the
minimum of Legendre's polynomial $P_n(t)$ for $-1 < t < 1$. This minimum
is in fact $= -1$ if n is odd, and it is the first relative minimum of $P_n(t)$ from
the right end point of the interval $-1 < t < 1$ if n is even (cf. [4, Theorem
7.3.1, p. 162]). The quantity μ_n is decreasing with increasing n, n even, [5] and
converges for $n \to \infty$ to minus the absolute minimum of the Bessel function
$J_0(x)$ which is 0.4028.

4. As in the proof of (1.7), we shall make use of an important theorem of
F. Lukács on polynomials $f(t)$ which are nonnegative in the real interval
$a \leqq t \leqq b$. If the degree of $f(t)$ is n, we have the following representation:

(1.9) $f(t) = [A(t)]^2 + (t - a)(b - t)[B(t)]^2$ if n is even, $n = 2q$,

and

(1.10) $f(t) = (t - a)[C(t)]^2 + (b - t)[D(t)]^2$ if n is odd, $n = 2q + 1$.

Here A, B, C, D are suitable polynomials of t whose degrees do not exceed q, $q - 1$, q, q, respectively.

For the proof of this theorem, cf. for instance [4, p. 4].

2. Symmetrization. Reduction of the problem

1.   For the proof of the theorem formulated above, we observe first that in view of the extremum principle it is sufficient to consider only the points on the unit sphere $x^2 + y^2 + z^2 = 1$.   Also, any rotation of the unit sphere leaves the class K_N unchanged; thus it is sufficient to establish the range of the values of H_2/H_0 for $x = y = 0$, $z = 1$, i.e., for $r = 1$, $\theta = 0$.   Now for every n

(2.1) $H_n(0, 0, 1) = \dfrac{2n + 1}{4\pi} \iint u(x', y', z') P_n (\cos \theta') \sin \theta' \, d\theta' \, d\phi',$

where $(x', y', z') = (1, \theta', \phi')$.   Using the standard representation of the surface harmonics (cf. [1, p. 444]), we see that

(2.2) $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} u(x', y', z') \, d\phi'$

is a zonal harmonic polynomial of the form

(2.3) $f(\cos \theta') = \sum_{n=0}^{N} a_n P_n(\cos \theta')$

where the coefficients a_n are real.   Obviously the associated solid harmonic $\sum_{n=0}^{N} a_n r^n P_n(\cos \theta')$ belongs also to K_N, and the values $H_N(0, 0, 1)$ do not change when we pass from the original function $u(x, y, z)$ to the "symmetrized" zonal function (2.2).

Thus without impairing the generality we are led to the following reduced

PROBLEM.   *Let* $f(t) = \sum_{n=0}^{N} a_n P_n(t)$ *denote an arbitrary polynomial of degree* N *which is not identically zero and moreover nonnegative in the interval* $-1 \leqq t \leqq 1$.   *We have to determine the range of the ratio* a_2/a_0.

2.   The expression

(2.4) $(f(t) + f(-t))/2 = g(t^2) = g(\tau)$

represents a polynomial in $t^2 = \tau$ of degree $[N/2]$, and

$$a_0 = \frac{1}{2} \int_{-1}^{+1} f(t) \, dt = \frac{1}{2} \int_0^1 g(\tau) \tau^{-1/2} \, d\tau,$$

$$a_2 = \frac{5}{2} \int_{-1}^{+1} f(t) P_2(t) \, dt = \frac{5}{2} \int_0^1 g(\tau) \frac{3\tau - 1}{2} \tau^{-1/2} \, d\tau,$$

so that

(2.5) $\dfrac{a_2}{a_0} = 5 \displaystyle\int_0^1 g(\tau)\, \dfrac{3\tau - 1}{2}\, \tau^{-1/2}\, d\tau \quad : \quad \displaystyle\int_0^1 g(\tau)\tau^{-1/2}\, d\tau.$

Here $g(\tau)$ is an arbitrary polynomial of degree $[N/2]$ which is not identically zero and moreover nonnegative in the interval $0 \leqq \tau \leqq 1$.

3. Now we apply the representation described in §1.4:

(2.6) $\quad \begin{aligned} g(\tau) &= [A(\tau)]^2 + \tau(1 - \tau)[B(\tau)]^2 \quad \text{if } [N/2] \text{ is even, } [N/2] = 2q, \\ g(\tau) &= \tau[C(\tau)]^2 + (1 - \tau)[D(\tau)]^2 \quad \text{if } [N/2] \text{ is odd, } [N/2] = 2q + 1 \cdot \end{aligned}$

The degrees of the polynomials A, B, C, D are q, $q - 1$, q, q, respectively. We seek the range of the quotients:

$$\int_0^1 [A(\tau)]^2 \frac{3\tau - 1}{2}\, \tau^{-1/2}\, d\tau \quad : \quad \int_0^1 [A(\tau)]^2 \tau^{-1/2}\, d\tau,$$

(2.7)
$$\int_0^1 \tau(1 - \tau)[B(\tau)]^2 \frac{3\tau - 1}{2}\, \tau^{-1/2}\, d\tau \quad : \quad \int_0^1 \tau(1 - \tau)[B(\tau)]^2 \tau^{-1/2}\, d\tau,$$

$$\int_0^1 \tau[C(\tau)]^2 \frac{3\tau - 1}{2}\, \tau^{-1/2}\, d\tau \quad : \quad \int_0^1 \tau[C(\tau)]^2 \tau^{-1/2}\, d\tau,$$

$$\int_0^1 (1 - \tau)[D(\tau)]^2 \frac{3\tau - 1}{2}\, \tau^{-1/2}\, d\tau \quad : \quad \int_0^1 (1 - \tau)[D(\tau)]^2 \tau^{-1/2}\, d\tau.$$

In §3 we shall prove that the second range is contained in the first one. Since in (2.5) and (2.6) we can choose $B(\tau) \equiv 0$, the range of (2.5) for $[N/2]$ even, will be identical with the range of the first quotient in (2.7). A similar reasoning applies in the case when $[N/2]$ is odd.

3. Mechanical quadrature

1. To the integrals occurring in (2.7) we apply the formula of Gauss-Jacobi [4, p. 47]. Let $w(\tau)$ be an arbitrary weight function in the interval $0 \leqq \tau \leqq 1$, $p_m(\tau)$ the associated orthogonal polynomial of degree m, τ_v the zeros of $p_m(\tau)$, λ_v the corresponding Christoffel numbers, $1 \leqq v \leqq m$, $0 < \tau_v < 1$, $\lambda_v > 0$. We have then

$$\int_0^1 [S(\tau)]^2 \frac{3\tau - 1}{2}\, w(\tau)\, d\tau \quad : \quad \int_0^1 [S(\tau)]^2 w(\tau)\, d\tau$$

(3.1)
$$= \sum_{v=1}^m \lambda_v [S(\tau_v)]^2 \frac{3\tau_v - 1}{2} \quad : \quad \sum_{v=1}^m \lambda_v [S(\tau_v)]^2,$$

where $S(\tau)$ is any polynomial of degree s; these formulas hold provided $2s + 1 \leqq 2m - 1$, $s \leqq m - 1$. If s is given, we choose $m = s + 1$.

From the representation (3.1), we conclude that the range of the ratio in (3.1), when $S(\tau)$ runs over all polynomials not identically zero and of degree

s, is the closed interval

(3.2) $[(3\tau' - 1)/2, (3\tau'' - 1)/2]$,

where τ' and τ'' denote the smallest and largest zero of $p_m(\tau) = p_{s+1}(\tau)$, respectively. Indeed, by prescribing the values of the polynomial $S(\tau)$ at $m = s + 1$ points, it will be uniquely determined. If the "data" are not all zero, $S(\tau)$ will be not identically zero. We use the fact that $(3\tau - 1)/2$ is monotonic for $0 < \tau < 1$.

In (2.7), we have

$$w(\tau) = \tau^{-1/2}, \; \tau^{1/2}(1 - \tau), \; \tau^{1/2}, \; \tau^{-1/2}(1 - \tau),$$

(3.3) $s = q, q - 1, q, q,$

$$m = q + 1, q, q + 1, q + 1,$$

respectively.

2. Transforming the interval $[0, 1]$ into the interval $[-1, 1]$, we obtain the following Jacobi polynomials corresponding to the cases (3.3) (notation as in [4, Chapter IV]):

$$P_{q+1}^{(0,-1/2)}(2\tau - 1), \quad P_q^{(1,1/2)}(2\tau - 1), \quad P_{q+1}^{(0,1/2)}(2\tau - 1), \quad P_{q+1}^{(1,-1/2)}(2\tau - 1).$$

Substituting $\tau = t^2$ and taking [4, (4.1.5), p. 59] into account, we find the following polynomials:

(3.4) $P_{2q+2}(t), \quad t^{-1}P_{2q+1}^{(1,1)}(t), \quad t^{-1}P_{2q+3}(t), \quad P_{2q+2}^{(1,1)}(t),$

where P_n is Legendre's polynomial and $P_n^{(1,1)}(t)$ denotes the Jacobi polynomial associated with the weight function $1 - t^2$ in $[-1, 1]$. We remark that $P_n^{(1,1)}(t)$ is, except for a constant factor, identical with $P'_{n+1}(t)$ (derivative of $P_{n+1}(t)$; cf. [4, (4.21.7), p. 63]).

3. Let $[N/2] = 2q$. Since the second function in (3.4) is, except for the factor t^{-1}, the derivative of the first one, the largest zero of the first function will surpass that of the second function; a similar conclusion holds for the smallest positive zeros of these functions.

Let $[N/2] = 2q + 1$. The same as before holds for the largest zeros. But the smallest positive zero of $P'_{2q+3}(t)$ is less than the smallest positive zero of $P_{2q+3}(t)$.

This establishes the assertion of the theorem announced in §1.

4. As an example, we treat the special cases $N = 1$ and $N = 2$ or 3. In the first case $a_2 = 0$, and indeed $P_2(t)$ has only one positive zero which makes the two quantities occurring in (1.8) vanish. In the second case ($N = 2$ or 3), $[N/2] = 1$ is odd. The polynomial $P_3(t) = \frac{5}{2}t^3 - \frac{3}{2}t$ has one positive zero t'', $t''^2 = \frac{3}{5}$, and $P'_3(t) = \frac{15}{2}t^2 - \frac{3}{2}$ has one positive zero t', $t'^2 = \frac{1}{5}$. Thus

the range (1.8) will be

$$-1 \leqq a_2/a_0 \leqq 2.$$

This can be seen directly from the representation

$$(u_0 + u_1 t)^2 + (1 - t^2)u_2^2 = a_0 P_0(t) + a_1 P_1(t) + a_2 P_2(t).$$

We have

$$a_0 = u_0^2 + \tfrac{1}{3}u_1^2 + \tfrac{2}{3}u_2^2 ,$$

$$a_1 = 2u_0 u_1 ,$$

$$a_2 = \tfrac{2}{3}u_1^2 - \tfrac{2}{3}u_2^2 ,$$

and

$$(\tfrac{2}{3}u_1^2 - \tfrac{2}{3}u_2^2)/(u_0^2 + \tfrac{1}{3}u_1^2 + \tfrac{2}{3}u_2^2)$$

has indeed the minimum -1 and the maximum 2.

For $N = 4$ we obtain by an easy but more elaborate calculation the following range:

$$-1.633 = (5 - 3\sqrt{30})/7 \leqq a_2/a_0 \leqq (5 + 3\sqrt{30})/7 = 3.061.$$

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