THE INITIAL VALUE PROBLEM FOR MAXWELL'S EQUATIONS FOR TWO MEDIA SEPARATED BY A PLANE[1]

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY
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Let the column vectors

$$E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \qquad H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

describe an electromagnetic field. Denoting the space coordinates by x_1, x_2, x_3 and the time by t we put

$$\xi_i = \partial/\partial x_i, \qquad \tau = \partial/\partial t.$$

The "curl" operator is then represented by the matrix

$$C(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

while the "divergence" operator corresponds to the row vector

$$\xi = (\xi_1, \xi_2, \xi_3).$$

Maxwell's equations for a homogeneous, isotropic, nonconducting medium in the absence of charges then take the form

$$\varepsilon\tau E = C(\xi)H, \qquad \mu\tau H = -C(\xi)E, \qquad \xi E = \xi H = 0.$$

We consider now the case of two media separated by the plane $x_1 = 0$. The field in the medium $x_1 < 0$, where the electric capacities shall have values ε, μ, we denote by E, H. We require that

$$\text{(1a)} \quad \varepsilon\tau E = C(\xi)H, \qquad \mu\tau H = -C(\xi)E, \qquad \xi E = \xi H = 0 \quad \text{for } x_1 \leqq 0.$$

The field in the other medium, where the capacities shall have values ε', μ', we denote by E', H'. For our purposes it is convenient to use in the second field a new name x_1' for the first space coordinate x_1. Putting

$$\xi_1' = \partial/\partial x_1', \qquad \xi' = (\xi_1', \xi_2, \xi_3)$$

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we have the equations

$$\varepsilon'\tau E' = C(\xi')H', \qquad \mu'\tau H' = -C(\xi')E', \qquad \xi'E' = \xi'H' = 0 \qquad \text{for } x_1' \geqq 0. \tag{1b}$$

In order to formulate the transition conditions on the interface we introduce the constants

$$p = \varepsilon/\varepsilon', \qquad q = \mu/\mu',$$

and the matrices

$$P = \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is then required that

$$E = P^{-1}E', \qquad H = Q^{-1}H' \qquad \text{for } x_1 = x_1' = 0. \tag{1c}$$

Finally the initial values of the field are prescribed:

$$E = E^0, \qquad H = H^0 \qquad \text{for } x_1 \leqq 0, \quad t = 0 \tag{1d}$$

$$E' = E'^0, \qquad H' = H'^0 \qquad \text{for } x_1' \geqq 0, \quad t = 0. \tag{1e}$$

The problem to be solved consists in determining a solution of (1a, b, c, d, e) for given initial data E^0, H^0, E'^0, H'^0. To insure consistency of the data we assume that E^0, H^0, E'^0, H'^0 are of class C_∞ in their respective half-spaces, satisfy

$$\xi E^0 = \xi H^0 = \xi' E'^0 = \xi' H'^0 = 0, \tag{2}$$

and vanish identically near the boundary of those half-spaces.

The transient field corresponding to a pulse in one of the media has been determined by H. Poritsky [7]. With varying degrees of generality this problem has also been discussed by Sommerfeld [8], Gerjuoy [1], van der Pol [10], Weyl [9], and Garnir [2]. The method used in the present paper is based on a principle used previously by the author to solve the corresponding acoustical problem. (See [3].) It furnishes the solution in terms of finite quadratures extended over the initial values without intervention of any Fourier and Laplace transforms. In fact the problem is reduced to that of solving the boundary-initial value problem for the ordinary wave equation in a half-space, which has a well known simple explicit solution. (See Freda [6].) The resulting solution of the two-media problem obtained here lends itself to the determination of the domains of dependence, which are found to agree with those suggested by geometrical optics.

The principle

Given a field E', H' for $x_1' \geqq 0$ and all t satisfying Maxwell's equations (1b), there cannot exist more than one "complementary" field E, H for $x_1 \leqq 0$

satisfying Maxwell's equations (1a) and the transition conditions (1c). Indeed (1a), (1c) constitute a Cauchy problem (though an improperly posed one), whose solution is determined uniquely by a well known theorem of Holmgren [5] (see also John [4]). There does not always exist a complementary field. Take, for example, a time-independent field

$$E' = \operatorname{grad} \phi(x_1', x_2, x_3), \qquad H' = 0,$$

where ϕ is harmonic for $x_1' \geqq 0$. Then there will not exist a complementary field, unless ϕ can be continued as a harmonic function into the whole $x_1' x_2 x_3$-space. We call a field E', H' *compatible*, if a complementary field exists. We have then the following principle:

If

$$pq = \varepsilon\mu/\varepsilon'\mu' > 1 \tag{3}$$

(*i.e., the right-hand medium has the larger propagation speed*), *and if* $E'(x_1', x_2, x_3, t)$, $H'(x_1', x_2, x_3)$ *form a compatible field, then the complementary field* E, H *can be extended into the whole* $x_1 x_2 x_3$*-space. Moreover with* E', H' *also the translated vectors* $E'(x_1' + y, x_2, x_3, t)$, $H'(x_1' + y, x_2, x_3, t)$ *form a compatible field for any positive constant* y.

We shall see how this principle leads to the construction of the solution of the initial value problem (1a, b, c, d, e) and will itself be verified by that construction. Assume we had a solution of that problem. The principle asserts that there exist functions $e(x_1, y, x_2, x_3, t)$, $h(x_1, y, x_2, x_3, t)$ for $y \geqq 0$ satisfying

$$\varepsilon\tau e = C(\xi)h, \qquad \mu\tau h = -C(\xi)e, \qquad \xi e = \xi h = 0 \tag{4}$$

for which

$$e(0, y, x_2, x_3, t) = P^{-1}E'(y, x_2, x_3, t),$$

$$h(0, y, x_2, x_3, t) = Q^{-1}H'(y, x_2, x_3, t).$$

We write again x_1' for y, and consider e, h as functions of the independent variables x_1, x_1', x_2, x_3, t, which satisfy (4) for $x_1' \geqq 0$ and also

$$e = P^{-1}E', \qquad h = Q^{-1}H' \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0, \tag{5}$$

$$e = E, \qquad h = H \qquad \text{for } x_1' = 0, \quad x_1 \leqq 0. \tag{5a}$$

Equations (4) imply that

$$Le = Lh = 0 \qquad \text{for } x_1 \leqq 0, \quad x_1' \geqq 0, \tag{6}$$

where L is the scalar wave operator

$$L = \tau^2 - (1/\varepsilon\mu)(\xi_1^2 + \xi_2^2 + \xi_3^2).$$

From (5), (4), (1b) we get for $x_1 = 0, x_1' \geqq 0$

$$0 = \varepsilon\tau(e - P^{-1}E') = C(\xi)h - pP^{-1}C(\xi')H' = (C(\xi) - pP^{-1}C(\xi')Q)h,$$

$$0 = \xi h - \xi' H' = (\xi - \xi' Q)h.$$

If we introduce the vectors

$$\text{(7a)} \qquad u = (C(\xi) - pP^{-1}C(\xi')Q)h, \qquad v = (C(\xi) - qQ^{-1}C(\xi')P)e$$

and the scalars

$$\text{(7b)} \qquad \phi = (\xi - \xi' Q)h, \qquad \psi = (\xi - \xi' P)e,$$

we have

$$\text{(7c)} \qquad u = v = \phi = \psi = 0 \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0.$$

Writing relations (7a, b) componentwise we observe that u_1 and v_1 vanish identically, and that (7a, b) are equivalent to the relations

$$\text{(8a)} \qquad \xi_1 h_1 = q\xi_1' h_1 + \phi,$$

$$\text{(8b)} \qquad \xi_1 h_2 = p\xi_1' h_2 - (pq - 1)\xi_2 h_1 + u_3,$$

$$\text{(8c)} \qquad \xi_1 h_3 = p\xi_1' h_3 - (pq - 1)\xi_3 h_1 - u_2,$$

$$\text{(8d)} \qquad \xi_1 e_1 = p\xi_1' e_1 + \psi,$$

$$\text{(8e)} \qquad \xi_1 e_2 = q\xi_1' e_2 - (pq - 1)\xi_2 e_1 + v_3,$$

$$\text{(8f)} \qquad \xi_1 e_3 = q\xi_1' e_3 - (pq - 1)\xi_3 e_1 - v_2.$$

It is clear that also

$$\text{(9)} \qquad Lu = Lv = L\phi = L\psi = 0 \qquad \text{for } x_1' \geqq 0.$$

Since u, v, ϕ, ψ vanish on $x_1 = 0$ and the operator L is even in ξ_1, it follows that u, v, ϕ, ψ must be *odd* functions of x_1. We form now the operator

$$L' = \tau^2 - (1/\varepsilon'\mu')(\xi_1'^2 + \xi_2^2 + \xi_3^2).$$

By (1b) and (5)

$$L'e = L'h = 0 \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0.$$

It follows from relations (8a, b, c, d, e, f) and (7c) that then also

$$\xi_1 L'e = \xi_1 L'h = 0 \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0.$$

$L'e$ and $L'h$ are then as functions of x_1, x_2, x_3, t solutions of the wave equation

$$L(L'e) = 0, \qquad L(L'h) = 0 \qquad \text{for } x_1 \leqq 0$$

with vanishing Cauchy data on the plane $x_1 = 0$, and hence vanish identically:

$$\text{(10)} \qquad L'e = L'h = 0 \qquad \text{for } x_1' \geqq 0.$$

We finally introduce the operator

$$(11)\qquad \Lambda = \varepsilon\mu(L' - L) = \xi_1^2 - pq\xi_1'^2 - (pq - 1)(\xi_2^2 + \xi_3^2).$$

By (6), (10), e and h satisfy the *time-independent* differential equations

$$(12)\qquad \Lambda e = \Lambda h = 0 \qquad \text{for } x_1' \geqq 0.$$

The important point is that Λ is a hyperbolic operator by assumption (3) with x_1 as time-like direction.

The construction of the solution of our problem starts now with the determination of the initial values e^0, h^0 of e, h for $t = 0$. One first obtains e^0, h^0 for $x_1' \geqq 0$, $x_1 \leqq 0$ as the solution of the boundary-initial value problem

$$(13a)\qquad , \qquad \Lambda e^0 = \Lambda h^0 = 0 \qquad \text{for } x_1 \leqq 0, \quad x_1' \geqq 0,$$

$$(13b)\qquad e^0 = E^0, \qquad h^0 = H^0 \qquad \text{for } x_1 \leqq 0, \quad x_1' = 0,$$

$$e^0 = P^{-1}E'^0, \qquad h^0 = Q^{-1}H'^0,$$

$$(13c)\qquad 0 = (C(\xi) - pP^{-1}C(\xi')Q)h^0 = (C(\xi) - qQ^{-1}C(\xi')P)e^0$$
$$= (\xi - \xi'Q)h^0 = (\xi - \xi'P)e^0 \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0.$$

(By (8a, b, c, d, e, f) relations (13c) amount to prescribing the Cauchy data of e, h on $x_1 = 0$, $x_1' \geqq 0$.) With the e^0, h^0 determined we form the functions

$$(13d)\qquad u^0 = (C(\xi) - pP^{-1}C(\xi')Q)h^0, \qquad v^0 = (C(\xi) - qQ^{-1}C(\xi')P)e^0,$$

$$(13e)\qquad \phi^0 = (\xi - \xi'Q)h^0, \qquad \psi^0 = (\xi - \xi'P)e^0$$

for $x_1 \leqq 0$, $x_1' \geqq 0$. We extend these functions into the region $x_1 \geqq 0$, $x_1' \geqq 0$ as odd functions of x_1. We then determine e^0, h^0 in the region $x_1 \geqq 0$, $x_1' \geqq 0$ from the relations (13d, e) using the extended functions u^0, v^0, ϕ^0, ψ^0, and requiring that $e^0 = P^{-1}E'^0$, $h^0 = Q^{-1}H'^0$ on $x_1 = 0$, $x_1' \geqq 0$. In this way we obtain e^0, h^0 for $x_1' \geqq 0$ and all x_1, x_2, x_3. The vectors e, h can then be obtained for $x_1' \geqq 0$ and all x_1, x_2, x_3, t as solutions of the pure initial value problem

$$(13f)\qquad Le = Lh = 0,$$

$$(13g)\qquad e = e^0, \qquad h = h^0, \qquad \varepsilon\tau e = C(\xi)h^0, \qquad \mu\tau h = -C(\xi)e^0 \quad \text{for } t = 0.$$

Finally the desired field vectors E, H, E', H' are given by

$$(13h)\qquad E = e(x_1, 0, x_2, x_3, t), \qquad H = h(x_1, 0, x_2, x_3, t),$$

$$(13i)\qquad E' = Pe(0, x_1', x_2, x_3, t), \qquad H' = Qh(0, x_1', x_2, x_3, t) \quad \text{for } x_1' \geqq 0.$$

Verification of the solution

The first step in the construction consists in the solution of the Cauchy problem (13a, b, c) for e^0, h^0 in the quarter-space $x_1 \leqq 0$, $x_1' \geqq 0$. Under the assumptions made that the data are in C_∞ and vanish near the intersection of the planes $x_1 = 0$, $x_1' = 0$, a unique solution e^0, h^0 of class C_∞ will exist.

The solution can be written conveniently by associating with the differential operator Λ the metric represented by the quadratic form

$$(14)\qquad R(y) = R(y_1, y_1', y_2, y_3) = y_1^2 - \frac{1}{pq} y_1'^2 - \frac{1}{pq - 1}(y_2^2 + y_3^2).$$

The solution e^0 consists of a contribution of the initial values of e^0, $\xi_1 e^0$ for $x_1 = 0$, $x_1' \geqq 0$ and of a contribution of the boundary values of e^0 for $x_1' = 0$, $x_1 \leqq 0$. The contribution of the initial values is identical with the solution of the pure initial value problem obtained by extending the initial values of e^0, $\xi_1 e^0$ for $x_1 = 0$ to all x_1', x_2, x_3 as odd functions of x_1'. The contribution to e^0 at a point $x = (x_1, x_1', x_2, x_3)$ of the initial data is then given by the formula

$$(15a)\qquad \frac{1}{4\pi}\frac{\partial}{\partial x_1}\frac{1}{x_1}\iint_{\substack{R(y-x)=0\\ y_1=0}} e^0(y)\, d\omega + \frac{1}{4\pi}\frac{1}{x_1}\iint_{\substack{R(y-x)=0\\ y_1=0}} \xi_1 e^0(y)\, d\omega.$$

The contribution of the boundary data is given by

$$(15b)\qquad -\frac{pq}{2\pi}\frac{\partial}{\partial x_1'}\frac{1}{x_1'}\iint_{\substack{R(y-x)=0\\ y_1'=0\\ x_1<y_1<0}} e^0(y)\, d\omega.$$

Here in each case $d\omega$ represents the surface element induced by the metric R, i.e., in (15a)

$$d\omega = \left| \frac{\sqrt{pq}\, x_1\, dy_2\, dy_3}{(pq - 1)(x_1' - y_1')} \right|,$$

while in (15b)

$$d\omega = \left| \frac{x_1'\, dy_2\, dy_3}{\sqrt{pq}(pq - 1)(x_1 - y_1)} \right|.$$

Defining next u^0, v^0, ϕ^0, ψ^0 by (13d, e) we have that those expressions are in C_∞ for $x_1 \leqq 0$, $x_1' \geqq 0$, satisfy

$$(16a)\qquad \Lambda u^0 = \Lambda v^0 = \Lambda\phi^0 = \Lambda\psi^0 = 0,$$

and vanish for $x_1 = 0$, $x_1' \geqq 0$. By continuing them as odd functions of x_1 into $x_1 \geqq 0$, $x_1' \geqq 0$, they stay solutions of (16a), and the extended functions are of class C_∞ for $x_1' \geqq 0$.

The functions e^0, h^0 are obtained for $x_1 \geqq 0$, $x_1' \geqq 0$ by means of equations (13d, e), using initial conditions $e^0 = P^{-1}E'^0$, $h^0 = Q^{-1}H'^0$ for $x_1 = 0$, $x_1' \geqq 0$. Writing (13d) in the equivalent form (8a, b, c, d, e, f) with superscripts 0 added, we see that e_1^0, e_2^0, e_3^0, h_1^0, h_2^0, h_3^0 can be obtained successively by quadratures along the lines

$$p\, dx_1 + dx_1' = dx_2 = dx_3 = 0 \quad \text{respectively} \quad q\, dx_1 + dx_1' = dx_2 = dx_3 = 0.$$

Since p and q are positive, these lines issuing from points with $x_1 = 0, x_1' \geqq 0$ cover the whole quarter-space $x_1 \geqq 0, x_1' \geqq 0$. The resulting extended functions e^0, h^0 are again in C_∞ for all $x_1' \geqq 0$ and all x_1, x_2, x_3. Moreover they again satisfy (13a), since Λe^0 and Λh^0 vanish for $x_1 = 0$, and in addition satisfy first order equations obtained from (13d, e) by applying the Λ-operator, with the inhomogeneous parts vanishing by (16a).

We now make use of the identity

$$\begin{aligned}(C(\xi) - pP^{-1}C(\xi')Q)C(\xi) &+ pP^{-1}C(\xi')Q(C(\xi) - qQ^{-1}C(\xi')P)\\ &= C^2(\xi) - pqP^{-1}C^2(\xi')P\\ &= (\xi^T - pqP^{-1}\xi'^T)\xi + pqP^{-1}\xi'^T(\xi - \xi'P) - \Lambda I,\end{aligned}$$

where I denotes the unit-matrix, and ξ^T, ξ'^T are the transposed vectors to ξ and ξ', i.e., the gradient operators. Applying this identity to e^0, it becomes

$$\begin{aligned}(C(\xi) - pP^{-1}C(\xi')Q)C(\xi)e^0 &+ pP^{-1}C(\xi')Qv^0\\ &= (\xi^T - pqP^{-1}\xi'^T)\xi e^0 + pqP^{-1}\xi'^T\psi^0.\end{aligned} \tag{16b}$$

The first row of this identity reads

$$-\xi_3 v_2^0 + \xi_2 v_3^0 = (\xi_1 - q\xi_1')\xi e^0 + q\xi_1'\psi^0. \tag{16c}$$

By (13c), (2) we have for $x_1 = 0, x_1' \geqq 0$

$$\xi e^0 = \xi' P e^0 + \psi^0 = \xi' E'^0 = 0.$$

Then also by (16c)

$$\xi_1 \xi e^0 = q\xi_1' \xi e^0 = 0 \qquad \text{for } x_1 = 0, \quad x_1' \geqq 0,$$

since v^0 and ψ^0 vanish there. Moreover by (13b), (2)

$$\xi e^0 = \xi E^0 = 0 \qquad \text{for } x_1' = 0, \quad x_1 \leqq 0.$$

Then ξe^0 is a solution of the differential equation

$$\Lambda \xi e^0 = 0 \qquad \text{for } x_1 \leqq 0, \quad x_1' \geqq 0$$

with vanishing boundary and initial data. It follows that ξe^0 vanishes in the quarter-space $x_1 \leqq 0, x_1' \geqq 0$. Then also $(\xi_1 - q\xi_1')\xi e^0$ vanishes there. By (16c) this expression is an odd function of x_1, since v^0 and ψ^0 are odd. Hence also

$$(\xi_1 - q\xi_1')\xi e^0 = 0 \qquad \text{for } x_1 \geqq 0, \quad x_1' \geqq 0.$$

Since ξe^0 vanishes on $x_1 = 0, x_1' \geqq 0$, this implies that $\xi e^0 = 0$ also for $x_1 \geqq 0$, $x_1' \geqq 0$. Similarly for ξh^0. We have then

$$\xi e^0 = \xi h^0 = 0 \qquad \text{for } x_1' \geqq 0. \tag{16d}$$

Identity (16b) then yields that

$$(C(\xi) - pP^{-1}C(\xi')Q)C(\xi)e^0$$

and similarly

$$(C(\xi) - qQ^{-1}C(\xi')P)C(\xi)h^0$$

are odd in x_1 for $x_1' \geqq 0$.

The functions e, h of x_1, x_1', x_2, x_3, t are defined for $x_1' \geqq 0$ as solutions of the pure initial value problem (13f, g). They can be obtained in terms of e^0, h^0 by an explicit formula of the type (15a), and are again of class C_∞. We have for $t = 0$, $x_1' \geqq 0$

$$\varepsilon\tau e - C(\xi)h = C(\xi)h^0 - C(\xi)h^0 = 0,$$

and

$$\begin{aligned}\mu\tau(\varepsilon\tau e - C(\xi)h) &= \varepsilon\mu\tau^2 e - C(\xi)\mu\tau h \\ &= \varepsilon\mu(Le)_{t=0} + \xi\xi^T e^0 + C^2(\xi)e^0 \\ &= \xi^T\xi e^0 = 0\end{aligned}$$

as a consequence of (16d). Thus $\varepsilon\tau e - C(\xi)h$ is a solution of the hyperbolic equation

$$L(\varepsilon\tau e - C(\xi)h)) = 0$$

with vanishing initial data, and hence vanishes identically. Similarly for $\mu\tau h + C(\xi)e$. Consequently e and h satisfy Maxwell's equations

$$\varepsilon\tau e = C(\xi)h, \qquad \mu\tau h = -C(\xi)e \qquad \text{for } x_1' \geqq 0. \tag{16e}$$

These equations imply that ξe and ξh are independent of t, and thus

$$\xi e = \xi h = 0 \qquad \text{for } x_1' \geqq 0, \tag{16f}$$

since (16f) holds for $t = 0$ by (16d).

We define u, v for $x_1' \geqq 0$ by (7a). All these functions are annihilated by the operator L. The initial values for u at $t = 0$ are

$$u = u^0,$$

$$\tau u = (C(\xi) - pP^{-1}C(\xi')Q)\tau h = -(1/\mu)(C(\xi) - pP^{-1}C(\xi')Q)C(\xi)e^0.$$

Since these initial values have been shown to be odd functions of x_1, it follows that u is an odd function of x_1 for all t. Similarly v is seen to be odd in x_1.

The field E', H' is defined by (13i). It is clear that E', H' are in C_∞ for $x_1' \geqq 0$. We have by (16e)

$$\begin{aligned}\varepsilon'\tau E' - C(\xi')H' &= (\varepsilon'\tau Pe - C(\xi')Qh)_{x_1=0} \\ &= (p^{-1}P(C(\xi) - pP^{-1}C(\xi')Q)h)_{x_1=0} \\ &= (p^{-1}Pu)_{x_1=0} = 0\end{aligned}$$

since the odd function u vanishes for $x_1 = 0$. We see then that E', H' satisfy Maxwell's equations

$$\varepsilon'\tau E' = C(\xi')H', \qquad \mu'\tau H' = -C(\xi')E' \qquad \text{for } x_1' \geqq 0. \tag{16g}$$

The remaining equations

$$\xi' E' = \xi' H' = 0 \tag{16h}$$

follow from (16g), if we observe that E', H' have for $t = 0$ the initial values

$$E' = (Pe)_{t=0, x_1=0} = (Pe^0)_{x_1=0} = E'^0 \qquad H' = H'^0, \tag{16i}$$

which satisfy (16h).

Finally the field E, H is defined by (13h) for all arguments x_1, x_2, x_3, t. It satisfies Maxwell's equations as the special case $x_1' = 0$ of (16e, f). For $t = 0$, $x_1' \leqq 0$ we have the initial values

$$E = (e)_{x_1'=0, t=0} = (e^0)_{x_1'=0} = E^0, \qquad H = H^0.$$

The transition conditions

$$E = P^{-1}E', \qquad H = Q^{-1}H' \qquad \text{for } x_1 = x_1' = 0$$

are a direct consequence of the definitions (13h, i) of E, H, E', H'. This completes the solution of the original problem.

Discussion of the solution

We first consider the effect of an initial disturbance concentrated at a single point $(\eta', 0, 0)$ of the medium with the larger speed, i.e., $\eta' > 0$. We first have to determine the domain of influence on e^0, h^0. In the quarter-space $x_1 \leqq 0$, $x_1' \geqq 0$ the vector e^0 is given by formula (15a), since by assumption the boundary values (13b) of e^0 vanish. Since the initial data on $x_1 = 0$ are to be continued as odd functions of x_1', only the point

$$y = (y_1, y_1', y_2, y_3) = (0, \eta', 0, 0)$$

and the symmetric point

$$\bar{y} = (y_1, -y_1', y_2, y_3) = (0, -\eta', 0, 0)$$

make a contribution. It follows that e^0, h^0 have their support in $x_1 \leqq 0$, $x_1' \geqq 0$ in the union of the cones

$$R(x - y) = x_1^2 - \frac{1}{pq}(x_1' - \eta')^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) = 0,$$

$$R(x - \bar{y}) = x_1^2 - \frac{1}{pq}(x_1' + \eta')^2 - \frac{1}{pq - 1}(x_3^2 + x_3^2) = 0.$$

The expressions u^0, v^0, ϕ^0, ψ^0 then have their support on the same cones. They are continued into $x_1 \geqq 0$, $x_1' \geqq 0$ as odd functions of x_1 and hence have their support in that quarter-space again on the same cones. It follows that u^0, v^0, ϕ^0, ψ^0 vanish in the set

$$x_1^2 - \frac{1}{pq}(x_1' - \eta')^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) < 0, \qquad x_1' \geqq 0, \quad x_1 \geqq 0. \tag{17}$$

The vectors e^0, h^0 are continued into $x_1 \geqq 0$, $x_1' \geqq 0$ by equations (13d, e). We shall show that they also vanish in the set (17).

We write the equations (13d, e) in the form (8a, b, c, d, e, f). We have e.g.

$$(\xi_1 - q\xi_1')h_1^0 = \phi^0.$$

Using the identity

$$p(\xi_1 - q\xi_1')^2 = (p-q)\Lambda + q(\xi_1 - p\xi_1')^2 + (p-q)(pq-1)(\xi_2^2 + \xi_3^2)$$

and the fact that $\Lambda h_1^0 = 0$, we obtain for h_1^0 the second order equation

$$Th_1^0 = p(\xi_1 - q\xi_1')\phi^0, \tag{18}$$

where

$$T = q(\xi_1 - p\xi_1')^2 + (p-q)(pq-1)(\xi_2^2 + \xi_3^2). \tag{19}$$

Since T is a degenerate quadratic form, the differential equation (18) connects the values of h_1^0 and ϕ in each hyperplane $px_1 + x_1' = \text{const.}$

Consider now a plane $px_1 + x_1' = \text{const.} = c > 0$. In that plane we can introduce x_1, x_2, x_3 as independent variables. The operator T in those variables takes the form

$$T = q\frac{\partial^2}{\partial x_1^2} + (p-q)(pq-1)\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right). \tag{20}$$

In that plane Th_1^0 and the initial data of h_1^0 vanish for

$$\begin{gathered}(q-p)\left[\frac{1}{q}\left(x_1 + \frac{c-\eta'}{q-p}\right)^2 \right.\\ \left. + \frac{1}{(p-q)(pq-1)}(x_2^2 + x_3^2) - \frac{(c-\eta')^2}{p(p-q)^2}\right] < 0;\\ 0 < x_1 < c/p.\end{gathered} \tag{21}$$

For $p > q$ the operator (20) is elliptic, and hence h_1^0 is analytic in the set (20). Moreover that set is connected and contains a neighborhood of the point $x_1 = x_2 = x_3 = 0$, where the Cauchy data vanish. It follows that h_1^0 vanishes everywhere in the set (20).

For $p < q$ the operator T is hyperbolic. The set (21) in that case is bounded by the plane $x_1 = 0$, a portion of the plane $x_1 = c/p$ and a portion of one sheet of a hyperbolic surface. The asymptotic cone of the hyperbolic surface is a characteristic cone with respect to the operator (20). It follows that vanishing Cauchy data on $x_1 = 0$ imply again vanishing of h_1^0 throughout the set (21). Since every point of the set (17) lies in a suitable plane $px_1 + x_1' = c$, it follows that h_1^0 vanishes throughout the set (17).

The same argument yields that $e_1^0 = 0$ in (17). Then also the remaining components of e^0, h^0 vanish in (17), since each of them is annihilated either by $\xi_1 - p\xi_1'$ or $\xi_1 - q\xi_1'$ in the set (17).

For $x_1' = 0$ we find that the initial values of E, H as solution of the differential equations $LE = LH = 0$ have their support in the set

$$x_1^2 - \frac{1}{pq-1}(x_2^2 + x_3^2) \geqq \frac{1}{pq}\eta'^2, \qquad x_1 > 0. \tag{22}$$

This set consists of the interior of one sheet of a hyperboloid of revolution with focus at the point $(\eta', 0, 0)$, vertex at $(\sqrt{\varepsilon'\mu'/\varepsilon\mu}\,\eta', 0, 0)$, and center at the origin. The field E, H is identical with the field obtained in a material with constants ε, μ extending throughout the whole space and with a suitable initial disturbance distributed over the set (22).

We introduce the propagation speeds in the two media, which are respectively

$$c = 1/\sqrt{\varepsilon\mu}, \qquad c' = 1/\sqrt{\varepsilon'\mu'}.$$

At the time t the support of E, H will be in the ct-neighborhood of the set (22). This neighborhood will be bounded by the outer parallel surface of distance ct to the hyperbolic surface

$$x_1^2 - \frac{1}{pq-1}(x_2^2 + x_3^2) = \frac{1}{pq}\eta'^2, \qquad x_1 > 0. \tag{23}$$

The boundary surface of the ct-neighborhood of (22), i.e., the wave front at the time t, can also be obtained by laying off the distance ct along the outer normal of any point of (23). Only that portion of the wave front lying in $x_1 \leqq 0$ has physical meaning. Let $F' = (\eta', 0, 0)$ be the focus of the hyperboloid, which is also the location of the original disturbance in the second medium. Let $P = (x_1, x_2, x_3)$ be a point of (23), i.e., a point of the apparent wave front at the time $t = 0$. Let Q be the intersection of the plane $x_1 = 0$ with the normal to the hyperboloid at P. One easily verifies that

$$(1/c)\overline{QP} = (1/c')\overline{QF'},$$

and that the sines of the angles the lines QP and QF' form with the x_1-axis are in the ratio c/c'. That means that the points of the wave front at the time t with $x_1 < 0$ can also be obtained by moving from F' with velocity c' along a line leading to a point Q of $x_1 = 0$ and then proceeding with velocity c along the direction obtained from Snell's law of refraction. (See Figure 1.)

The points on the plane $x_1 = 0$ in which E, H can be different from 0 lie in or on the sphere of radius $c't$ about the point $F' = (\eta', 0, 0)$, i.e., lie in the set

$$x_2^2 + x_3^2 + \eta'^2 \leqq c'^2t^2. \tag{24}$$

The vectors E', H' for $x_1' = 0$ have their support in the same set by virtue of the transition conditions. We can consider E', H' as solutions of the hyperbolic equations

$$L'E' = L'H' = 0 \qquad \text{for } x_1' \geqq 0,$$

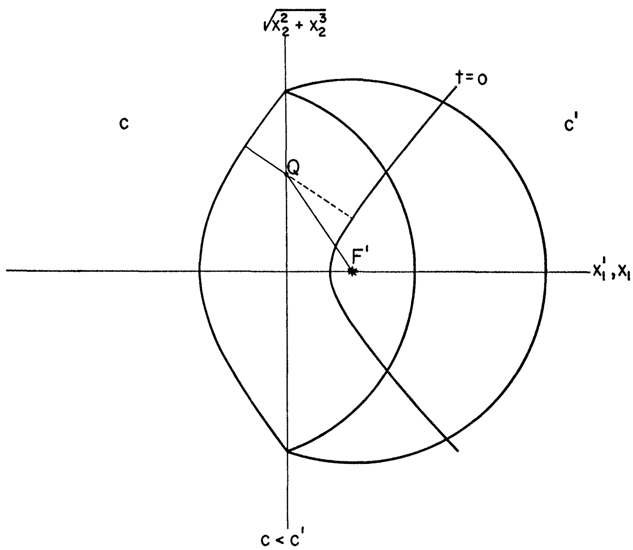

Figure 1

which have their initial values for $t = 0$, $x_1' \geqq 0$ concentrated at F', and have nonvanishing boundary data on $x_1' = 0$ only in the portion (24) of the boundary plane. The contribution of the initial disturbance to the state of E', H' at the time t is confined to the sphere

$$(x_1' - \eta')^2 + x_2^2 + x_3^2 = c'^2 t^2. \tag{25a}$$

One easily verifies that the contribution of the boundary data is confined to the symmetric sphere

$$(x_1' + \eta')^2 + x_2^2 + x_3^2 \leqq c'^2 t^2. \tag{25b}$$

Hence E', H' have their support at the time t in the union of the sets (25a) and (25b).

Next we consider the effect of an initial disturbance concentrated at a point $F = (\eta, 0, 0)$ of the first medium:

$$\eta < 0.$$

The vectors e^0, h^0 are solutions of the boundary-initial value problem (13a, b, c) for $x_1 \leqq 0$, $x_1' \geqq 0$. The initial data vanish, and the boundary data are concentrated at the point $y = (y_1, y_1', y_2, y_3) = (\eta, 0, 0, 0)$. Formula (15b) shows that e^0, h^0 will have their support in the set $R(y - x) = 0$, i.e., on

$$(x_1 - \eta) - \frac{1}{pq} x_1'^2 - \frac{1}{pq - 1} (x_2^2 + x_3^2) = 0, \qquad x_1 \leqq \eta, \quad x_1' \geqq 0. \tag{26}$$

The same holds for u^0, v^0, ϕ^0, ψ^0. Those quantities continued as odd functions of x_1 will then have their support for $x_1 \geqq 0$, $x_1' \geqq 0$ in the symmetric set

$$(x_1 + \eta)^2 - \frac{1}{pq} x_1'^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) = 0, \qquad x_1 \geqq -\eta, \quad x_1' \geqq 0.$$

The vectors e^0, h^0 for $x_1 \geqq 0$, $x_1' \geqq 0$ are obtained by integrating along lines

$$p\, dx_1 + dx_1' = dx_2 = dx_3 = 0$$

or

$$q\, dx_1 + dx_1' = dx_2 = dx_3 = 0.$$

It follows that the support of e^0, h^0 for $x_1 \geqq 0$, $x_1' \geqq 0$ is confined to the set

$$(27) \quad (x_1 + \eta)^2 - \frac{1}{pq} x_1'^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) \geqq 0, \qquad x_1 \geqq -\eta, \quad x_1' \geqq 0.$$

For $x_1' = 0$ the vectors e^0, h^0 reduce to E^0, H^0. It follows that the field E, H can be obtained by solving a pure initial value problem for E, H as solutions of

$$LE = LH = 0$$

with initial values for $t = 0$ confined to the point

$$F = (\eta, 0, 0)$$

and to the conical set

$$(28) \qquad (x_1 + \eta)^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) \geqq 0, \qquad x_1 \geqq -\eta.$$

It follows that at the time t the field E, H has its support in the union of the sphere

$$(x_1 - \eta)^2 + x_2^2 + x_3^2 = c^2 t^2$$

and of the ct-neighborhood of (28). The ct-neighborhood of the set (18) consists of the truncated cone

$$\left(x_1 + \eta + \frac{ct\sqrt{pq}}{\sqrt{pq - 1}}\right)^2 - \frac{1}{pq - 1}(x_2^2 + x_3^2) \geqq 0, \qquad x_1 \geqq -\eta - \frac{ct\sqrt{pq - 1}}{\sqrt{pq}}$$

and the spherical cap

$$(x_1 + \eta)^2 + x_2^2 + x_3^2 \leqq c^2 t^2, \qquad -\eta - ct \leqq x_1 \leqq -\eta - ct\frac{\sqrt{pq - 1}}{\sqrt{pq}}.$$

(See Figure 2.)

For the purpose of finding the support of the field E', H' we first find that of e, h and then use that

$$E' = (Pe)_{x_1=0}, \qquad H' = (Qh)_{x_1=0}.$$

The vectors e^0, h^0 have their support in the union of the sets (26), (27). For

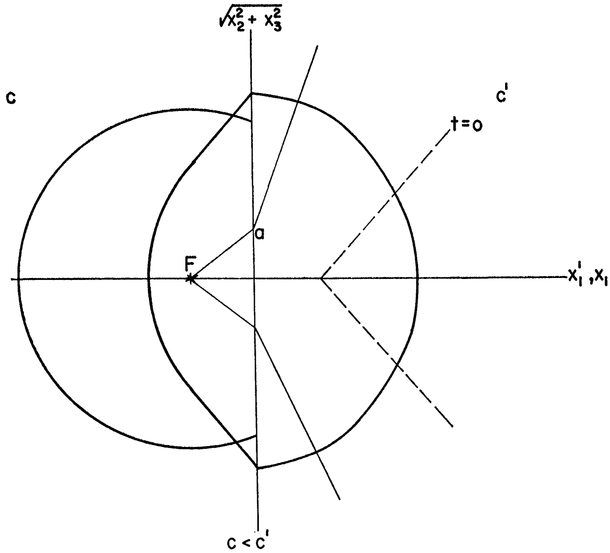

Figure 2

any fixed $x_1' \geqq 0$ the vectors e, h are the solutions of the pure initial value problem (13f, g). It follows that for fixed $x_1' \geqq 0$ and positive t the support of e, h is contained in the ct-neighborhood of the sets (26), (27) in $x_1 x_2 x_3$-space. The plane $x_1 = 0$ in $x_1 x_2 x_3$-space lies outside the sets (26), (27). A point of that plane lies in a ct-neighborhood of the sets (26), (27) if it lies in a ct-neighborhood of a boundary point of one of the sets. Since the boundaries of the sets (26), (27) are symmetric with respect to the plane $x_1 = 0$, we find that it is sufficient to consider the ct-neighborhood of the set (26). We see then that $E'(x_1', x_2, x_3, t)$, $H'(x_1', x_2, x_3, t)$ can be different from 0 for $x_1' \geqq 0$, $t > 0$ only, if there exist y_1, y_2, y_3 with

$$(y_1 - \eta)^2 - \frac{1}{pq} x_1'^2 - \frac{1}{pq-1}(y_2^2 + y_3^2) = 0, \qquad y_1 \leqq \eta, \tag{29}$$

$$y_1^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 \leqq c^2 t^2. \tag{30}$$

We can take here for y_1, y_2, y_3 those quantities satisfying (29), for which the left-hand side of (30) assumes its smallest value. Then

$$y_1, \quad x_2 - y_2, \quad x_3 - y_3$$

are proportional to

$$y_1 - \eta, \quad \frac{1}{pq-1} y_2, \quad \frac{1}{pq-1} y_3.$$

Introducing

$$(z_1, z_2, z_3) = \left(\eta, \frac{pq}{pq-1} y_2, \frac{pq}{pq-1} y_3\right), \tag{31}$$

we also have

$$(32) \qquad (z_1, z_2, z_3) = (\theta y_1, \theta y_2 + (1 - \theta)x_2, \theta y_3 + (1 - \theta)x_3)$$

where $\theta = \eta/y_1$ lies between 0 and 1. By (29), (31)

$$(33) \qquad \begin{aligned} &\sqrt{pq}\,\sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2} \\ &\qquad = \sqrt{(z_1 - \eta + x_1')^2 + z_2^2 + z_3^2}, \end{aligned}$$

while by (32)

$$(34) \qquad \begin{aligned} &\sqrt{y_1^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2} \\ &\qquad = \sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2} \\ &\qquad\qquad + \sqrt{z_1^2 + (z_2 - x_2)^2 + (z_3 - x_3)^2}. \end{aligned}$$

It follows from (33), (34), (30) that

$$c'^{-1}\sqrt{(z_1 - \eta + x_1')^2 + z_2^2 + z_3^2} + c^{-1}\sqrt{z_1^2 + (z_2 - x_2)^2 + (z_3 - x_3)^2} \leqq t.$$

Introducing finally a_1, a_2, a_3 by

$$a_1 = 0, \qquad a_2 = x_2 - z_2, \qquad a_3 = x_3 - z_3$$

we see that in order that E', H' do not vanish at the point (x_1', x_2, x_3) it is necessary that there exist quantities a_2, a_3 such that

$$(35) \qquad c^{-1}\sqrt{\eta^2 + a_2^2 + a_3^2} + c'^{-1}\sqrt{x_1'^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} \leqq t.$$

Relation (35) expresses that there exists a point $(0, a_2, a_3)$ such that moving from $(\eta, 0, 0)$ to $(0, a_2, a_3)$ with speed c and then from $(0, a_2, a_3)$ to (x_1', x_2, x_3) with speed c' will take a total time at most t. That is, the support of E', H' due to an initial disturbance at $(\eta, 0, 0)$ is contained in the set given by the law of refraction of geometrical optics.

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