SYMMETRY TYPES OF PERIODIC SEQUENCES

BY

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1. Introduction

This paper gives a short treatment of the problem appearing in Fine [2], which is as follows. Consider periodic sequences $a = (\cdots, a_{-1}, a_0, a_1, \cdots)$ with period n and with a_j limited to the q values $1, 2, \cdots, q$. If two sequences are taken to be equivalent when they can be made alike either by a shift in origin or by a permutation of the element values $1, 2, \cdots, q$, or by both, how many distinct (inequivalent) sequences, or symmetry types of sequences are there?

An example given by Fine is repeated here for concreteness. For $n = 3$, $q = 2$ there are two types, namely (111) and (112); (111) and (222) are equivalent by the permutation (12), and the six remaining sequences (112), (121), (211), (221), (212), (122), are equivalent either by this permutation or a shift in origin.

Section 4 is devoted specifically to Fine's problem. Depending on the intended application, a group G of symmetry transformations (possibly different from Fine's) may be allowed. If only translations $(a_i \to a_{i+s})$ are allowed, G is a cyclic group C_n. This case appears in [5] in connection with counting necklaces made from n beads of q different kinds (translations merely rotate the necklace). It also arises in problems of coding and genetics [3]. The special case $n = 12$, $q = 2$ occurs in finding the number of distinct musical chords (of 0, 1, $\cdots$, or 12 notes) when inversions and transpositions to other keys are equivalences. Turning over the plane of necklace $(a_i \to a_{-i})$ produces a new "mirror image" necklace. If this symmetry is permitted as well as the translations, then G is a dihedral group D_n. Permutations of the element values $1, 2, \cdots, q$ form a symmetric group S_q. Thus, in Fine's problem, G is a product group $C_n \times S_q$. This problem has some applications to switching theory. For example, consider a switching network to control q lights, one at a time, in a periodic cycle; here a_i is the name of the light which changes its state at the i^{th} step. In counting the number of distinct sequences possible, translations merely start the cycle at a different point and permutations of $1, \cdots, q$ merely give the lights new names. If sequences which operate the lights in reverse order are also considered equivalent, then G becomes $D_n \times S_q$. More details on the music and switching applications appear in Section 6.

Our treatment of $C_n \times S_q$ is related to a special case of one of the theorems in de Bruijn [1]. By its use it is also easy to treat the case $D_n \times S_q$.

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2. Pólya's lemma

As Fine has noted, a sequence of period n also has period kn, so it is useful to distinguish sequences of *primitive period* n, that is of period n but no smaller period. If $F_q^*(n)$ is the number of period n, $F_q(n)$ the number of primitive period n, then (Fine's equation (1))

$$F_q^*(n) = \sum_{d|n} F_q(d); \tag{1}$$

this has the inverse

$$F_q(n) = \sum_{d|n} \mu(n/d) F_q^*(d), \tag{1a}$$

where $\mu(n)$ is the Möbius function: $\mu(1) = 1$, $\mu(n) = (-1)^r$ if n is a product of r distinct primes, $\mu(n) = 0$ otherwise.

When only translation symmetries are allowed, the number of types of period n is

$$F_q^*(n) = (1/n) \sum_{d|n} \varphi(d) q^{n/d} \qquad (G = C_n), \tag{2}$$

(see [5, p. 162]) with $\varphi(d)$ the Euler totient function. The number of types with primitive period n is

$$F_q(n) = \sum_{d|n} \mu(d) q^{n/d} \qquad (G = C_n).$$

This follows from (2) by using (1a) or may be derived by a simple direct argument [3]. A proof of (2) will serve to introduce a lemma used by Pólya [4] in his proof of what de Bruijn calls his "fundamental theorem in enumerative combinatorial analysis," namely,

LEMMA. *If G is a finite group, of order g, of transformations operating on a finite set of objects, and if two objects are equivalent when one is transformed into the other by a transformation of G, then the number of inequivalent objects is*

$$T = g^{-1} \sum_t I(t),$$

where $I(t)$ is the number of objects left invariant by transformation t of G, and the sum is over all g members of G.

In the present instance, the group G is the cyclic group of order n, represented by R^s, $s = 1, 2, \cdots, n$, where R is the permutation $(1, 2, \cdots, n)$ in cyclic form. If a sequence a of period n is invariant under R^s, then for all j

$$a_j = a_{j+s} = a_{j+2s} = \cdots .$$

Since a is of period n, the indices $j + ks$, $k = 0, 1, \cdots$, are integers mod n; the number of these which are distinct is $n/(n, s)$ where (n, s) is the greatest common divisor of n and s. The (n, s) numbers $a_1, \cdots, a_{(n,s)}$ may be chosen in $q^{(n,s)}$ ways. So, by the lemma

$$T = F_q^*(n) = (1/n) \sum_{s=1}^{n} q^{(n,s)} = (1/n) \sum_{d|n} \varphi(d) q^{n/d},$$

the latter by classifying the R^s by their cycle structure.

If we now consider both kinds of equivalence, those arising from permutations of the element values as well as those arising from shifts in the origin, the group G, as noted by Fine [2], is the direct product of the cyclic group C_n and the symmetric group S_q. If π is an arbitrary element of the latter, any element of G may be written in the form $R^s\pi$, with R as above.

The lemma will again give the number $F_q^*(n)$ of types of sequences of period n when we find the number $I(R^s\pi)$ of sequences which a typical element of $C_n \times S_q$ leaves invariant. If $R^s\pi$ leaves the sequence a invariant, then for all j

$$a_j = \pi a_{j+s} = \pi^2 a_{j+2s} = \cdots .$$

Let $d = n/(n, s)$. The sequence a is again specified completely by $a_1, \cdots, a_{n/d}$. Now however, some of $1, 2, \cdots, q$ may be forbidden as choices for the a_j. Since

$$a_j = \pi^d a_{j+sd} = \pi^d a_j,$$

the value of a_j must belong to a cycle of π which has length dividing d. If π has k_i cycles of length i $(i = 1, 2, \cdots, q)$, then each of the n/d elements a_j may be one of only

$$m(d) = \textstyle\sum_{c|d} ck_c \tag{3}$$

possibilities. Thus, $I(R^s\pi) = \{m(d)\}^{n/d}$ which, together with the lemma, is a solution.

Again, the lemma gives a formula for $F_q^*(n)$ which simplifies when terms $R^s\pi$ with like cycle structure are combined. For a given divisor d of n, there are $\varphi(d)$ translations R^s which have cycle structure $d^{n/d}$ (i.e., n/d cycles of length d). For a given partition $k_1 + 2k_2 + \cdots + qk_q = q$, the number of permutations π which have k_i cycles of length i $(i = 1, \cdots, q)$ is

$$N(k_1, \cdots, k_q) = q!/(k_1! \cdots k_q!\, 2^{k_2} \cdots q^{k_q}).$$

Thus, combining terms,

$$F_q^*(n) = (1/q!\,n)\textstyle\sum_{d,k} \varphi(d)N(k_1, \cdots, k_q)(m(d))^{n/d} \quad (G = C_n \times S_q), \tag{4}$$

where the sum is over all divisors d of n and all partitions

$$k_1 + 2k_2 + \cdots + qk_q = q.$$

3. A theorem

We now show a connection between (2) and (3) and a theorem of de Bruijn. For this purpose we introduce the cycle indexes (see [5, Chapter 6]) $C_n(x_1, \cdots, x_n)$ and $S_q(x_1, \cdots, x_q)$ of the groups C_n and S_q:

$$C_n(x_1, \cdots, x_n) = (1/n)\textstyle\sum_{d|n} \varphi(d)x_d^{n/d}, \tag{5}$$

$$S_q(x_1, \cdots, x_q) = (1/q!)\textstyle\sum (q!/k_1! \cdots k_q!)x_1^{k_1}(x_2/2)^{k_2} \cdots (x_q/q)^{k_q}. \tag{6}$$

In the sums the coefficient of $x_1^a x_2^b x_3^c \cdots$ is the number of group elements which produce a cycles of length 1, b of length 2, c of length 3, etc.

Now, (2) and (3) can be written (in the style of de Bruijn)

(2a) $$F_q^*(n) = C_n(\delta_1, \cdots, \delta_n)y_1^q \qquad (G = C_n),$$

(3a) $$F_q^*(n) = C_n(\delta_1, \cdots, \delta_n)S_q(y_1, \cdots, y_q) \qquad (G = C_n \times S_q),$$

where

$$y_j = \exp j(z_j + z_{2j} + \cdots), \qquad \delta_i = \partial/\partial z_i,$$

and the evaluation is at $z_1 = z_2 = \cdots = 0$.

To verify (2a) and (3a), observe that

(7) $$y_1^{k_1}y_2^{k_2} \cdots \exp\left(\textstyle\sum_j z_j \sum_{c|j} ck_c\right) = \exp\left(\textstyle\sum_j z_j m(j)\right),$$

and

(8) $$\delta_j^e(y_1^{k_1}y_2^{k_2} \cdots) = m(j)^e y_1^{k_1}y_2^{k_2} \cdots.$$

Since y_1^q is the cycle index for a group of degree q consisting only of the identity element, both (2a) and (3a) are instances of the following theorem proved by de Bruijn [1]. (It is a special case both of his Theorem 1 and of his Theorem 2, when weight functions appearing are ignored.) In the terminology of [5, Chapter 6], it reads as follows:

THEOREM (de Bruijn). *If n objects are chosen independently from a store of q different objects, if equivalence for objects is specified by a group J_q with cycle index $J_q(x_1, \cdots, x_q)$, and equivalence for order of choice by a group H_n with cycle index $H_n(x_1, \cdots, x_n)$, then the number of inequivalent choices is given by*

(9) $$P_{n,q} = H_n(\partial/\partial z_1, \cdots, \partial/\partial z_n)J_q(y_1, \cdots, y_q)$$

evaluated at $z_1 = z_2 = \cdots = 0$; $y_k = \exp[k(z_k + z_{2k} + \cdots)]$.

This theorem may be proved by the same argument that derived (2a) and (3a). The group G is now $H_n \times J_q$ with elements $\chi\pi$ ($\chi \in H_n$, $\pi \in J_q$). To find $I(\chi\pi)$, let χ have e_d cycles of length d ($d = 1, \cdots, n$), and let π have k_i cycles of length i ($i = 1, \cdots, q$). A sequence a, left invariant by $\chi\pi$, is determined by prescribing values for one element a_j in each of the

$$e_1 + \cdots + e_n$$

cycles of χ. For an element in a cycle of length d, the number of allowed choices from $1, 2, \cdots, q$ is again $m(d)$, given by (3). Thus

$$I(\chi\pi) = m(1)^{e_1}m(2)^{e_2} \cdots m(n)^{e_n}.$$

By using the lemma and combining terms of like cycle structure,

$$F_q^*(n) = (1/hj)\textstyle\sum_{e,k} \varphi(e_1, \cdots, e_n)N(k_1, \cdots, k_q) \prod_d \{m(d)\}^{e_d} \qquad (G = H_n \times J_q),$$

where h and j are the orders of H_n and J_q, $\varphi(e_1, \cdots, e_n)$ is the number of permutations χ of cycle structure $1^{e_1}, \cdots, n^{e_n}$, $N(k_1, \cdots, k_q)$ is again the number of permutations π of cycle structure $1^{k_1}, \cdots, q^{k_q}$, and the sum is over all partitions $e_1 + 2e_2 + \cdots + ne_n = n$, $k_1 + \cdots + qk_q = q$. The expression $P_{n,q}$ of (9) is a shorthand for this as may be verified using (7) and (8).

4. Application of the theorem ($G = C_n \times S_q$)

Equation (3a) is a particularly useful form because of the following generating function [5, p. 68] for the cycle index of S_q:

$$\sum_{q=0}^{\infty} x^q S_q(y_1, \cdots, y_q) = \exp [xy_1 + (x^2/2)y_2 + \cdots + (x^n/n)y_n + \cdots]. \tag{10}$$

Then if

$$P_n(x) = \sum_{q=0}^{\infty} x^q P_{n,q}$$

is the generating function for the numbers $P_{n,q}$, by (3a),

$$P_n(x) = (1/n) \sum_{d|n} \varphi(d)(\partial/\partial z_d)^e \exp (xy_1 + (x^2/2)y_2 + \cdots), \tag{11}$$

again evaluated at $z_1 = z_2 = \cdots = 0$, and with $de = n$,

$$y_k = \exp [k(z_k + z_{2k} + \cdots)].$$

To compute the derivative in (11), first set $z_d = z$, and $z_i = 0$ for all $i \neq d$, in the function $\exp (xy_1 + (x^2/2)y_2 + \cdots)$. The desired derivative is

$$\left(\frac{\partial}{\partial z}\right)^e \exp \left\{ \sum_{c=1}^{\infty} \frac{x^c}{c} + \sum_{c|d} \frac{x^c}{c} (e^{cz} - 1) \right\} \Bigg|_{z=0} = (1 - x)^{-1} \left(\frac{\partial}{\partial z}\right)^e \exp \sum_{c|d} \frac{x^c}{c} (e^{cz} - 1) \Bigg|_{z=0}.$$

Hence, if polynomials $A_{d,n}(x)$ are defined by the exponential generating function

$$\sum_{n=0}^{\infty} A_{d,n}(x) z^n/n! = \exp \sum_{c|d} (x^c/c)(e^{cz} - 1), \tag{12}$$

equation (11) is evaluated by

$$(1 - x)P_n(x) = (1/n) \sum_{d|n} \varphi(d) A_{d,e}(x), \qquad de = n, \tag{13}$$

which completely determines the differences $P_{n,q} - P_{n,q-1}$, the variables with the simplest structure. Note that $P_{n,1} = 1$. The difference

$$Q_{n,q} = P_{n,q} - P_{n,q-1}$$

itself is combinatorially significant as the number of types of periodic sequences in which each of $1, 2, \cdots, q$ actually appears as some a_j.

Turn now to the polynomials $A_{d,n}(x)$. First, by expansion of (12),

$$A_{d,0}(x) = 1, \qquad A_{d,1}(x) = \sum_{c|d} x^c.$$

Next, for $d = 1$

$$\sum_{n=0}^{\infty} A_{1,n}(x)z^n/n! = \exp x(e^z - 1),$$

so that [5, p. 76] $A_{1,n}(x) = a_n(x)$, the enumerator of permutations by number of ordered cycles. Note that

$$a_n(x) = \sum_{k=0}^{n} S(n, k)x^k,$$

with $S(n, k) = \Delta^k 0/k!$, the Stirling number of the second kind. Thus, the polynomials $A_{d,n}(x)$ are a generalization of Stirling number polynomials $a_n(x)$.

For numerical results, the following recurrence relation, obtained by differentiation of (12), is convenient (the prime denotes a derivative)

$$A_{d,n+1}(x) = A_{d,1}(x)A_{d,n}(x) + xA'_{d,n}(x). \tag{14}$$

The first few values (omitting arguments) are as follows

$$A_{1,1} = x, \qquad A_{1,2} = x + x^2,$$
$$A_{2,1} = x + x^2, \qquad A_{2,2} = x + 3x^2 + 2x^3 + x^4,$$
$$A_{3,1} = x + x^3, \qquad A_{3,2} = x + x^2 + 3x^3 + 2x^4 + x^6.$$

TABLE I

Number of Types $F_2^*(n)$ with Period n

n	Symmetry Group G			
	C_n	D_n	$C_n \times S_2$	$D_n \times S_2$
1	2	2	1	1
2	3	3	2	2
3	4	4	2	2
4	6	6	4	4
5	8	8	4	4
6	14	13	8	8
7	20	18	10	9
8	36	30	20	18
9	60	46	30	23
10	108	78	56	44
11	188	126	94	63
12	352	224	180	122
13	632	380	316	190
14	1,173	687	596	362
15	2,192	1,224	1,096	612
16	4,116	2,250	2,068	1,162
17	7,712	4,112	3,856	2,056
18	14,602	7,685	7,316	3,912
19	27,596	14,310	13,798	7,155
20	52,488	27,012	26,272	13,648

The first few values of $Q_n(x) = (1 - x)P_n(x)$, again omitting arguments, are as follows:

$$Q_1 = x, \qquad Q_4 = x + 3x^2 + 2x^3 + x^4,$$
$$Q_2 = x + x^2, \qquad Q_5 = x + 3x^2 + 5x^3 + 2x^4 + x^5,$$
$$Q_3 = x + x^2 + x^3, \qquad Q_6 = x + 7x^2 + 18x^3 + 13x^4 + 3x^5 + x^6.$$

Note that for p a prime

$$Q_p(x) = p^{-1}[a_p(x) + (p - 1)(x + x^p)].$$

Since $Q_n(x)$ is a polynomial with integral coefficients, this entails

$$a_p(x) \equiv x + x^p \pmod p,$$

a congruence known otherwise [5, p. 81].

5. Addition of mirror inversion

Adding the equivalence of mirror inversion is accomplished by replacing the cyclic group by the dihedral group D_n. The cycle index $D_n(x_1, \cdots, x_n)$ of D_n [3, p. 150] is given by

TABLE II

Number of Types $F_2(n)$ with Primitive Period n

n	Symmetry Group G			
	C_n	D_n	$C_n \times S_2$	$D_n \times S_2$
1	2	2	1	1
2	1	1	1	1
3	2	2	1	1
4	3	3	2	2
5	6	6	3	3
6	9	8	5	5
7	18	16	9	8
8	30	24	16	14
9	56	42	28	21
10	99	69	51	39
11	186	124	93	62
12	335	208	170	112
13	630	378	315	189
14	1,152	668	585	352
15	2,182	1,214	1,091	607
16	4,080	2,220	2,048	1,144
17	7,710	4,110	3,855	2,055
18	14,532	7,630	7,280	3,883
19	27,594	14,308	13,797	7,154
20	52,377	26,931	26,214	13,602

$$2D_n(x_1, \cdots, x_n) = C_n(x_1, \cdots, x_n) + (x_1^2 x_2^{m-1} + x_2^m)/2, \quad n = 2m,$$

$$2D_n(x_1, \cdots, x_n) = C_n(x_1, \cdots, x_n) + x_1 x_2^m, \quad n = 2m + 1.$$

Hence if $R_{n,q}$ is the number $F_q^*(n)$ of inequivalent sequences for $G = D_n \times S_q$ and

$$R_n(x) = \sum_{q=0}^{\infty} x^q R_{n,q},$$

it follows from the theorem and some simple calculations that

$$(15) \quad \begin{aligned} 2R_{2n}(x) &= P_{2n}(x) + (1 - x)^{-1} A_{2,n}(x) \\ 2R_{2n+1}(x) &= P_{2n+1}(x) + x(1 - x)^{-1} \sum_{j=0}^{n} \tbinom{n}{j} A_{2,j}(x). \end{aligned}$$

6. Some calculations

The results of Section 5 gave $F_q^*(n)$ when $G = C_n \times S_q$ for $n \leqq 6$ and all values of q. Table I extends these results to $n \leqq 20$ for binary sequences ($q = 2$) only. Numbers of types with primitive period n appear in Table II. Fine gave numerical results for $G = C_n \times S_q$, $1 \leqq n \leqq 10$, which agree with ours.

Musical chords are related to a number $F_2^*(12)$ as follows. Number the notes of the scale in order $\cdots, -1, 0, 1, 2, \cdots$, say with 0 at middle C. A chord specifies a sequence a with $a_j = 1$ if note j is in the chord, and with $a_j = 0$ otherwise. In naming chords (G major, $C\sharp$ minor, etc.) inversion is considered an equivalence; thus, we restrict attention to sequences a of period 12 (one octave). The 12 possible transpositions of chords into other keys form a cyclic group C_{12}. If these are allowed as symmetries, then all 12 major chords will count as just one chord type, all minor chords will be another, etc. The number of chord types is $F_2^*(12)$ for the group C_{12}, namely 352. From Table II, only 335 of these types have primitive period 12. Among the 17 chords with shorter periods are found: silence (period 1); all notes played at once (period 1); 6 notes separated by whole-tone steps

TABLE III

Number of Even Types with $q = 2$, Period n

n	Symmetry Group G			
	C_n	D_n	$C_n \times S_2$	$D_n \times S_2$
2	2	2	1	1
4	4	4	3	3
6	8	8	4	4
8	20	18	12	12
10	56	44	28	22
12	180	122	94	71
14	596	362	298	181
16	2,068	1,162	1,044	618
18	7,316	3,914	3,658	1,957
20	26,272	13,648	13,164	6,966

(period 2); diminished seventh chords (period 3); augmented chords (period 4).

Section 1 cited a switching application. One must distinguish between the cycles of operations of the lights (which Tables I and II count) and cycles of states of the lights. For example, 1, 2, 2, 3, 1, 2, 2, 3, $\cdots$ represents a cycle in which operations have period 4; however, after 4 operations, lights 1 and 3 have changed state. The corresponding sequence of states of the lights has period 8. Thus, we are also led to counting types of *even sequences* of period n, i.e., sequences in which each of 1, 2, $\cdots$, q appears an even number of times within a period. The theorem of Section 3 is inadequate for this because not all q^n sequences are to be classified. The lemma still applies if objects are restricted to be even sequences. For example, in computing $I(R^s\pi)$, $(n, s) = n/d$ values $a_1, \cdots, a_{(n,s)}$ are to be chosen from $m(d)$ possibilities as in Section 2. However, there is now an additional restriction to make the sequence even. If the value of a_j is chosen from a cycle of π of length c (where c divides d), each of the c values in this cycle appears d/c times among $a_j, a_{j+(n,s)}, a_{j+2(n,s)}, \cdots$, in one period. Thus, cycles with d/c even may be chosen freely, but cycles with d/c odd must each be chosen an even number of times.

There are

$$E = \sum_{d/c \text{ even}} ck_c$$

values in cycles with d/c even. If $d = 2^bD$ with D odd, there are

$$M = \sum_{c|d} k_c$$

cycles with d/c odd. Let their lengths be called $c_1, \cdots, c_M$. Then

$$I(R^s\pi) = 2^{-M}\sum_{x_i=\pm1} (x_1 c_1 + \cdots + x_M c_M + E)^{(n,s)},$$

where the sum extends over all 2^M choices of ± 1 for $x_1, \cdots, x_M$.

Again, the lemma provides a solution in which terms of like cycle structure may be combined. There is no further simplification as in Section 4. Table III lists some numbers of types of even sequences when $q = 2$ and

$$n = 2, 4, \cdots, 20.$$

There are no even sequences for odd n.

References

1. N. G. de Bruijn, *Generalization of Pólya's fundamental theorem in enumerative combinatorial analysis*, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 62 = Indag. Math., vol. 21 (1959), pp. 59–69.
2. N. J. Fine, *Classes of periodic sequences*, Illinois J. Math., vol. 2 (1958), pp. 285–302.
3. S. W. Golomb, Basil Gordon, and L. R. Welch, *Comma-free codes*, Canadian J. Math., vol. 10 (1958), pp. 202–209.
4. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math., vol. 68 (1937), pp. 145–253.
5. John Riordan, *An introduction to combinatorial analysis*, New York, Wiley, 1958.

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