A PROBLEM ABOUT PRIME NUMBERS AND THE RANDOM WALK I

BY

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Consider the set Q of 3-dimensional lattice points (l_1, l_2, l_3) with $l_1 \geqq 2$ prime, $l_2 = l_3 = 0$. K. Itô and H. P. McKean, Jr. [1, p. 131] posed the problem of computing the probability γ that the standard 3-dimensional random walk hits Q an infinite number of times.

Given a string B of m ($\geqq 2$) consecutive integers $\subset [2^{n-1}, 2^n)$, A. Selberg's sieve estimate [2, p. 290] provides the upper bound $\pi(B) < c_1 m/\lg m$ to the number of primes in B, and this can be used to prove that $\gamma = 1$.

Wiener's test (see [1, p. 128]) indicates that it is enough to check

$$\sum_{n \geqq 1} 1/ne(n) = +\infty, \qquad e(n) = \max_{\substack{2^{n-1} \leqq a < 2^n \\ a \text{ prime}}} \int_{2^{n-1}}^{2^n} \frac{\pi(db)}{|b - a|},$$

where $|b - a|$ is defined to be 1 in case $a = b$. Now, using Selberg's estimate, it is clear that, for $2^{n-1} \leqq a < 2^n$ and $n \uparrow \infty$,

$$\int_{2^{n-1}}^{2^n} \frac{\pi(db)}{|b - a|} = 1 + \int_{2^{n-1}}^{a-1} \frac{\pi(db)}{a - b} + \int_{a+1}^{2^n} \frac{\pi(db)}{b - a}$$

$$= 1 + \frac{\pi[2^{n-1}, a)}{a - 2^{n-1}} + \int_{2^{n-1}}^{a-1} \frac{\pi[b, a)\, db}{(b - a)^2} + \frac{\pi(a, 2^n]}{2^n - a} + \int_{a+1}^{2^n} \frac{\pi(a, b]\, db}{(b - a)^2}$$

$$< c_2 + c_3 \int_e^{2^n} \frac{db}{b \lg b} < c_4 \lg n,$$

i.e., $e(n) < c_4 \lg n$, and this is good enough.

P. Erdös has proved (see the following note) that the number of points of Q with $l_1 \leqq n$ that the sample path visits is $\sim c_5 \times \lg_2 n$ $(n \uparrow \infty)$.

I learned of Selberg's estimate through the kindness of N. C. Ankeny.

REFERENCES

1. K. ITÔ AND H. P. McKEAN, JR., *Potentials and the random walk*, Illinois J. Math., vol. 4 (1960), pp. 119–132.
2. A. SELBERG, *The general sieve-method and its place in prime number theory*, Proceedings of the International Congress of Mathematicians, 1950, vol. I, pp. 286–292.

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