ON THE INTEGRAL REPRESENTATION OF THE RATE OF TRANSMISSION OF A STATIONARY CHANNEL

BY

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1. Introduction

For finite-memory channels as defined in [1] the equality of stationary and ergodic capacities has been proved by I. P. Tsaregradsky [2] and L. Breiman [3]. For channels with infinite memory we give a new definition of the rate of transmission which coincides with the usual definition for finite-memory channels. By utilising the representation of a general stationary measure as a direct integral of ergodic measures due to Kryloff and Bogoliouboff [4], we obtain a representation for the rate of transmission of any stationary input in terms of the rates for ergodic inputs. This representation leads to two important results: It shows that for any stationary channel the ergodic capacity is equal to the stationary capacity, and that the ergodic capacity is attained whenever the stationary capacity is attained.

2. Basic properties of stationary and ergodic inputs

In this section we shall consider stationary measures on the Borel field generated by cylinder sets of the product space

$$A^I = \prod_{i=-\infty}^{+\infty} A_i, \qquad A_i = A \quad \text{for all } i,$$

where the product is taken over all integers and A is a finite set consisting of a elements. Then under the product topology we can assume A^I to be a compact metric space. If T is the shift transformation of A^I into itself, then, under the group of automorphisms $T^n : n = \cdots, -1, 0, 1, \cdots$, A^I becomes a compact dynamical system. Hereafter we shall follow the notation and terminology of Oxtoby [5]. If $f(p)$ is a real-valued function on A^I, let

$$M(f, p, k) = f_k(p) = (1/k) \sum_{i=1}^{k} f(T^i p) \quad (k = 1, 2 \cdots) \tag{2.1}$$

and

$$M(f, p) = f^*(p) = \lim_{k\to\infty} M(f, p, k) \tag{2.2}$$

in case this limit exists. A Borel subset E of A^I is said to have invariant measure one if $\mu(E) = 1$ for every invariant probability measure μ. Let Q be the set of points p for which $M(f, p)$ exists for every $f \in C(A^I)$ where $C(A^I)$ is the space of continuous functions on A^I. It follows easily from Riesz's representation theorem that corresponding to any point $p \in Q$ there exists a unique invariant probability measure μ_p such that

$$M(f, p) = \int f \, d\mu_p .$$

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Let $R \subset Q$ be the set of those points for which μ_p is ergodic. R is called the set of regular points. Now we quote from Oxtoby's paper the results of Kryloff and Bogoliouboff for reference.

THEOREM 2.1 *The set R of regular points is Borel measurable, and of invariant measure one.*

THEOREM 2.2. *For any ergodic measure μ, the set of regular points p such that $\mu_p = \mu$ is of μ-measure one.*

THEOREM 2.3. *For any bounded Borel measurable function f on A^I, $\int f\, d\mu_p$ is a Borel measurable function of p on R, and*

$$\int f\, d\mu = \int_R \left[\int f\, d\mu_p\right] d\mu(p)$$

for every finite invariant Borel measure μ.

THEOREM 2.4. *For any Borel set $E \subset A^I$, $\mu_p(E)$ is Borel measurable on R, and*

$$\mu(E) = \int_R \mu_p(E)\, d\mu(p)$$

for every finite invariant Borel measure μ.

We denote by $[x_{i_1} \cdots x_{i_k}]$ the cylinder set of points x in A^I where $i_1{}^{\text{th}}, \cdots, i_k{}^{\text{th}}$ coordinates are $x_{i_1}, \cdots, x_{i_k}$ respectively. Let F_A^- be the Borel field generated by cylinder sets $[x_{i_1} \cdots x_{i_k}]$ where $i_1, \cdots, i_k$ vary over negative integers only. Let Z_α denote the cylinder set of points with zeroth coordinate equal to α. Corresponding to any finite measure μ we consider the following conditional probability function $g_\mu(x, \alpha)$ given by

$$\mu(E \cap Z_\alpha) = \int_E g_\mu(x, \alpha)\, d\mu(x) \tag{2.3}$$

for any Borel set E in F_A^-. We shall now prove the following theorem concerning $g_\mu(x, \alpha)$.

THEOREM 2.5. *If μ, μ_1, and μ_2 are invariant measures in A^I, $\mu = a\mu_1 + (1 - a)\mu_2$ $(0 \leqq a \leqq 1)$, and μ_1 and μ_2 are orthogonal, then*

$$g_\mu(x, \alpha) = g_{\mu_1}(x, \alpha) \qquad \text{a.e. } x\ (\mu_1).$$

Proof. Since μ_1 and μ_2 are invariant and orthogonal, the critical sets in which their masses are concentrated can be taken to be invariant and hence in F_A^-. It is then immediate from the definition of conditional probabilities that

$$g_\mu(x, \alpha) = g_{\mu_1}(x, \alpha) \qquad \text{a.e. } x\ (\mu_1).$$

THEOREM 2.6. *If μ is an invariant probability measure, then*

$$g_\mu(x, \alpha) = g_{\mu_p}(x, \alpha) \qquad \text{a.e. } x\ (\mu_p)$$

for almost all p (μ).

Proof. For any invariant measure μ, we have from (2.3) and Theorem 2.3,

$$(2.4) \qquad \mu(E \cap Z_\alpha) = \int_E g_\mu(x, \alpha)\, d\mu(x) = \int_R \left[\int_E g_\mu(x, \alpha)\, d\mu_p(x) \right] d\mu(p).$$

From Theorems 2.4 and (2.3) we have

$$(2.5) \quad \mu(E \cap Z_\alpha) = \int \mu_p(E \cap Z_\alpha)\, d\mu(p) = \int_R \left[\int_E g_{\mu_p}(x, \alpha)\, d\mu_p(x) \right] d\mu(p),$$

where R is the set of regular points and E is any set in F_A^-.

For any invariant set A for which $\mu(A)$ is neither zero nor one we can write

$$\mu = a\mu_1 + (1 - a)\mu_2,$$

where $a = \mu(A)$, $\mu_1(E) = \mu(E \cap A)/\mu(A)$, and $\mu_2(E) = \mu(E \cap A')/\mu(A')$ for any Borel set E. Then μ_1 and μ_2 are invariant and orthogonal. Hence, by Theorem 2.6,

$$g_\mu(x, \alpha) = g_{\mu_1}(x, \alpha) \qquad \text{a.e. } x\ (\mu_1)$$

Substituting μ_1 for μ in (2.4) and (2.5), equating the two expressions, and making use of Theorem 2.3, we obtain

$$(2.6) \quad \int_{A \cap R} \left[\int_E g_\mu(x, \alpha)\, d\mu_p(x) \right] d\mu(p) = \int_{A \cap R} \left[\int_E g_{\mu_p}(x, \alpha)\, d\mu_p(x) \right] d\mu(p)$$

for any invariant set A and any set E in F_A^-. Since the functions of p within the square brackets in (2.6) are invariant and thus measurable with respect to the σ-field of invariant Borel sets, we have, for all cylinder sets $E \in F_A^-$ and almost all p (μ),

$$\int_E g_\mu(x, \alpha)\, d\mu_p(x) = \int_E g_{\mu_p}(x, \alpha)\, d\mu_p(x).$$

The required result now follows from the uniqueness of the Radon-Nikodým derivative.

3. Properties of the rate per letter of an information source and transmission function of a channel

As is well known, the rate per letter of a stationary information source $[A^I, \mu]$ is defined as the limit

$$(3.1) \qquad \mathcal{H}(\mu, A) = \lim_{n\to\infty} -(1/n) \sum_{[x_1\cdots x_n]} \mu[x_1 \cdots x_n] \log \mu[x_1 \cdots x_n].$$

For any point $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$, let

$$h_\mu(x) = g_\mu(x, x_0).$$

Then by McMillan's theorem [1], $-\log h_\mu(x)$ is integrable with respect to μ, and

$$\mathcal{H}(\mu, A) = -\int \log h_\mu(x)\, d\mu.$$

Now we shall prove the following representation theorem.

THEOREM 3.1. *There exists a function $h(p)$ defined over R such that for every invariant probability measure μ,*

$$\mathcal{H}(\mu, A) = \int_R h(p)\, d\mu(p).$$

Proof. Define

$$h(p) = -\int \log g_{\mu_p}(x)\, d\mu_p(x) \tag{3.2}$$

for any regular point p. By Theorem 2.6,

$$h_\mu(x) = h_{\mu_p}(x) \qquad \text{a.e. } x\ (\mu_p)$$

for almost all p (μ). Since $-\int \log h_{\mu_p}\, d\mu_p$ is finite for almost all p (μ), by Theorem 2.4 and Fubini's theorem we have,

$$\mathcal{H}(\mu, A) = -\int \log h_\mu(x)\, d\mu = -\int_R \left[\int \log h_\mu(x)\, d\mu_p(x)\right] d\mu$$

$$= -\int_R \left[\int \log h_{\mu_p}(x)\, d\mu_p(x)\right] d\mu = \int_R h(p)\, d\mu(p).$$

This completes the proof.

Remark. It has been pointed out earlier by Breiman that the rate per letter of an information source is linear in the convex set of stationary probability measures. The above theorem shows that it is not only linear but given by an integral.

Next, we consider an arbitrary stationary channel $[A, \nu_x, B]$, where ν_x is a measure in B^I for every fixed x in A^I possessing the usual stationarity properties, viz., $\nu_x(F) = \nu_{Tx}(TF)$ for any Borel set F in B^I, T being the usual shift operator. For each fixed F, the function $\nu_x(F)$ is assumed to be measurable.

Let $[y_{i_1}\, y_{i_2} \cdots y_{i_k}]$ denote the cylinder set of all points y in B^I whose i_1th, i_2th, $\cdots$, i_kth coordinates are $y_{i_1}, y_{i_2}, \cdots, y_{i_k}$ respectively. Write

$$\mathcal{H}_n(x) = -(1/n) \sum\nolimits_{[y_1 \cdots y_n]} \nu_x[y_1 \cdots y_n] \log \nu_x[y_1 \cdots y_n], \tag{3.3}$$

where the summation is over all cylinders of the type $[y_1 \cdots y_n]$. For any stationary probability measure μ in F_A, let

$$\mathcal{H}(\mu, B \mid A) = \lim_{n\to\infty} \int_{A^I} \mathcal{H}_n(x)\, d\mu(x). \tag{3.4}$$

The existence of the above limit is a well-known result in information theory.

THEOREM 3.2. *There exists a function $H(x)$ such that*

$$\mathcal{H}(\mu, B \mid A) = \int_{A^I} H(x)\, d\mu(x).$$

$H(x)$ is given by

$$H(x) = -\int \log \nu_x[y_0 \mid y_{-1} y_{-2} \cdots]\, d\nu_x(y), \tag{3.5}$$

and

$$\nu_x[y_0 \mid y_{-1} y_{-2} \cdots] = \lim_{n\to\infty} \frac{\nu_x[y_{-(n-1)} \cdots y_0]}{\nu_x[y_{-(n-1)} \cdots y_{-1}]}. \tag{3.6}$$

Proof. The existence of the limit (3.6) for almost all y (ν_x) is a well-known result in the theory of martingales. Further, proceeding in the same way as in the proof of Lemmas 7.3 to 7.7 in pages 67 to 70 of [6] we have

$$\lim_{n\to\infty} \int | g_n(x, y) - g(x, y) |\, d\nu_x(y) = 0, \tag{3.7}$$

where

$$g_n(x, y) = -\log \left\{ \frac{\nu_x[y_{-(n-1)} \cdots y_0]}{\nu_x[y_{-(n-1)} \cdots y_{-1}]} \right\} \tag{3.8}$$

and

$$g(x, y) = -\log \nu_x[y_0 \mid y_{-1} y_{-2} \cdots]. \tag{3.9}$$

Thus,

$$\lim_{n\to\infty} -\int g_n(x, y)\, d\nu_x(y) = -\int g(x, y)\, d\nu_x(y). \tag{3.10}$$

Let

$$H_n(\mu, B \mid A) = -\frac{1}{n} \int \sum_{[y_1 \cdots y_n]} \nu_x[y_1 \cdots y_n] \log \nu_x[y_1 \cdots y_n]\, d\mu(x). \tag{3.11}$$

Then, by applying the well-known result that

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} (a_n - a_{n-1})/(b_n - b_{n-1})$$

whenever the second limit exists and b_n is monotonic, to the sequence H_n, we have

$$\lim_{n\to\infty} H_n(\mu, B \mid A) = \lim_{n\to\infty} -\int \left\{ \sum \nu_x[y_1 \cdots y_n] \log \frac{\nu_x[y_1 \cdots y_n]}{\nu_x[y_1 \cdots y_{n-1}]} \right\} d\mu(x) \tag{3.12}$$

provided the limit on the right side of (3.12) exists. Further

$$\begin{aligned} &-\int \left[\sum \nu_x[y_1 \cdots y_n] \log \frac{\nu_x[y_1 \cdots y_n]}{\nu_x[y_1 \cdots y_{n-1}]} \right] d\mu(x) \\ &\quad = -\int \left[\sum \nu_{T^{-n}x}[y_{-(n-1)} \cdots y_0] \log \frac{\nu_{T^{-n}x}[y_{-(n-1)} \cdots y_0]}{\nu_{T^{-n}x}[y_{-(n-1)} \cdots y_{-1}]} \right. d\mu(x) \end{aligned} \tag{3.13}$$

where T^{-n} is the shift transformation applied n times in the reverse direction. Changing the variable x to $T^n x$ in (3.13), we obtain

$$\lim_{n\to\infty} H_n(\mu, B \mid A) = \lim_{n\to\infty} - \int_{A^I} \int_{B^I} \log \frac{\nu_x[y_{-(n-1)} \cdots y_0]}{\nu_x[y_{-(n-1)} \cdots y_{-1}]} \, d\nu_x(y) \, d\mu(x),$$

which becomes

$$\mathcal{H}(\mu, B \mid A) = - \iint \log \nu_x[y_0 \mid y_{-1} y_{-2} \cdots] \, d\nu_x(y) \, d\mu(x) = \int H(x) \, d\mu(x)$$

by the bounded convergence theorem and (3.10). This completes the proof of Theorem 3.2.

For any stationary measure μ in F_A, let the measure η be defined by

$$\eta(E) = \int_{A^I} \nu_x(F) \, d\mu(x).$$

η is defined in F_B.

DEFINITION 3.1. For any stationary channel $[A, \nu_x, B]$ the rate of transmission for any stationary input measure μ is defined by the equation

$$\mathcal{R}(\mu) = \mathcal{H}(\eta, B) - \mathcal{H}(\mu, B \mid A).$$

It is easy to see that this definition of rate of transmission coincides with the usual definition for finite-memory channels.

THEOREM 3.3. *For any stationary channel* $[A, \nu_x, B]$

$$\mathcal{R}(\mu) = \int_R \mathcal{R}(\mu_p) \, d\mu(p).$$

Proof. This is an immediate consequence of Theorems 2.3, 3.1, and 3.2 and (3.2).

COROLLARY 1. $\operatorname{Sup}_{(\mu \text{ stationary})} \mathcal{R}(\mu) = \operatorname{Sup}_{(\mu \text{ ergodic})} \mathcal{R}(\mu)$.

COROLLARY 2. *The set of stationary measures at which the capacity can be achieved is a closed convex set whose extreme points are ergodic.*

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