HOMOTOPICAL NILPOTENCY

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Introduction

Let X be a topological space with base-point, ΩX its loop space, ΣX its (reduced) suspension. The ordinary multiplication and inversion of loops convert ΩX into an H-space. Eckmann and Hilton [6] have shown that, dually, the identification map resulting by pinching to a point the equatorial $X \subset \Sigma X$ and the reflection of ΣX in X may be used to convert the suspension into an H'-space, the dual of an H-space. Just as in group theory, for every $n \geqq 1$ a commutator map of weight n is available in any H-space; accordingly, we define the nilpotency class of an H-space as the least integer $n \geqq 0$ (if any) with the property that the commutator map of weight $n + 1$ is nullhomotopic. The concepts of a commutator map and of nilpotency class may readily be dualized to H'-spaces: for every $n \geqq 1$ there results a co-commutator map of weight n and the co-nilpotency class of an H'-space is the least integer $n \geqq 0$ (if any) with the property that the co-commutator map of weight $n + 1$ is nullhomotopic. We now revert to the topological space X and introduce two integers, which may be finite or not: the nilpotency class $\operatorname{nil} \Omega X$ and the co-nilpotency class $\operatorname{conil} \Sigma X$. They are uniquely determined by the based homotopy type of X.

The paper is divided into six parts. The first contains basic definitions concerning H- and H'-spaces, commutator and co-commutator maps, nilpotency and co-nilpotency classes. In the second part, we present results relating the nilpotency and co-nilpotency classes of H- and H'-spaces to the nilpotency class of certain groups of homotopy classes of maps; some of these results provide further motivation for our concept of co-nilpotency of an H'-space.

Given a base-points-preserving map $f: X \to Y$, the nilpotency class $\operatorname{nil} \Omega f$ is the least integer $n \geqq 0$ for which the composition

$$\Omega X \times \cdots \times \Omega X \xrightarrow{\varphi_{n+1}} \Omega X \xrightarrow{\Omega f} \Omega Y$$

is nullhomotopic; here, φ_{n+1} is the commutator map of weight $n + 1$, and Ωf is induced by f in the obvious way. In the third section we prove that if $\eta: Q \to Y$ is the inclusion map of the fibre Q in the total space Y, then $\operatorname{nil} \Omega Q \leqq 1 + \operatorname{nil} \Omega \eta$. Dually, if $\eta: X \to P$ is the projection of X onto the "cofibre" P, i.e., η is the identification map resulting by pinching to a point a subset which is smoothly imbedded in X, then $\operatorname{conil} \Sigma P \leqq 1 + \operatorname{conil} \Sigma \eta$; the definition we give of $\operatorname{conil} \Sigma \eta$ stands in evident duality to that of $\operatorname{nil} \Omega \eta$. In particular, $\operatorname{nil} \Omega Q \leqq 1 + \operatorname{nil} \Omega Y$ and $\operatorname{conil} \Sigma P \leqq 1 + \operatorname{conil} \Sigma X$. The first

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theorem was suggested to us by a result of E. H. Spanier and J. H. C. Whitehead [19] according to which, if η is nullhomotopic, then Q is a generalized H-space, i.e., a space having a continuous multiplication with two-sided homotopy unit element. The loop space ΩQ of such a space is well known to be homotopy-commutative, that is, $\operatorname{nil} \Omega Q \leqq 1$. However, there are polyhedra with a homotopy-commutative loop space which fail to be generalized H-spaces; such an example is presented at the end of the third section.

The fourth section gives lower and upper bounds for $\operatorname{nil} \Omega X$ in terms of the usual homotopy invariants of X. Evidently, the nilpotency class of the group $\pi_1(X)$ is $\leqq \operatorname{nil} \Omega X$; the inequality becomes an equality if X is a connected aspherical CW-complex. We then prove that $\operatorname{nil} \Omega X \geqq$ W-long X, the latter invariant representing the maximum length of nonvanishing multiple Whitehead products in X. Next, it is shown that, if X is a 1-connected CW-complex, an upper bound for $\operatorname{nil} \Omega X - 1$ is provided by the number of nontrivial Postnikov invariants of X; in particular, $\operatorname{nil} \Omega X$ does not exceed the number of nonvanishing homotopy groups of the 1-connected CW-complex X.

The fifth section is entirely devoted to proving that, if X is 0-connected, $\operatorname{conil} \Sigma X \geqq \smile$-long X. The latter invariant represents the largest number of singular cohomology classes of positive dimension with nonvanishing cup product in X; the coefficients are taken in an arbitrary commutative field. It should be noted that, within the framework of the Eckmann-Hilton duality theory, W-long and $\smile$-long are dual invariants.

The final section is mainly concerned with some estimations of the nilpotency class of function spaces. If (G, e) is an H-space, then the function space $(G, e)^{(X,a)}$ also is an H-space, and the sets $\pi(X; G)$, $\pi(X, a; G, e)$, of free, respectively based, homotopy classes of maps $X \to G$ are groups. We introduce the weak category, w cat X, a based homotopy type invariant recently discovered by Hilton (see also [1]), and prove under very mild assumptions that

$$\operatorname{conil} \Sigma X \leqq \sup \operatorname{nil} \pi(X, a; G, e) \leqq \sup \operatorname{nil} (G, e)^{(X,a)} \leqq \operatorname{w cat} X - 1.$$

The result of the fifth section adds significance to the lower bound obtained. The upper bound improves a result due to G. W. Whitehead [20] according to which, for any 0-connected H-space (G, e), one has

$$\operatorname{nil} \pi(X; G) \leqq \operatorname{cat} X - 1,$$

where $\operatorname{cat} X$ is the Lusternik-Schnirelmann category of X. For, if G is 0-connected, the groups $\pi(X; G)$ and $\pi(X, a; G, e)$ are isomorphic and, if X is a 0-connected normal space, $\operatorname{w cat} X \leqq \operatorname{cat} X$; as shown by the examples at the end of the paper, the strict inequality may frequently occur. Notice that comparison of the first and last member in the string of inequalities above yields an upper bound for the co-nilpotency class of a suspension. Finally, a further result involving the weak category is provided by the inequality

$$\text{W-long }(Y, b)^{(X,a)} \leqq \text{w cat } X - 1,$$

which will be proved for a large class of spaces X (see [7] for related results).

In conclusion, the authors wish to express their hearty thanks to P. J. Hilton for his interest and many valuable suggestions.

1. Nilpotency and co-nilpotency

Let X_i and Y_i be arbitrary topological spaces with base-points $a_i \in X_i$, $b_i \in Y_i$; let $f_i : (X_i, a_i) \to (Y_i, b_i)$ be continuous maps. The map

$$f_1 \times \cdots \times f_n : (X_1 \times \cdots \times X_n, (a_1, \cdots, a_n)) \to (Y_1 \times \cdots \times Y_n, (b_1, \cdots, b_n))$$

sends the point $(x_1, \cdots, x_n)$ into $(f_1(x_1), \cdots, f_n(x_n))$. We shall frequently need the "wedge", i.e., the subspace

$$X_1 \vee \cdots \vee X_n = \textstyle\bigcup_{i=1}^n a_1 \times \cdots \times a_{i-1} \times X_i \times a_{i+1} \times \cdots \times a_n \subset \prod_{i=1}^n X_i = X_1 \times \cdots \times X_n,$$

and write

$$j : X_1 \vee \cdots \vee X_n \to X_1 \times \cdots \times X_n,$$

$$f_1 \vee \cdots \vee f_n : (X_1 \vee \cdots \vee X_n, (a_1, \cdots, a_n)) \to (Y_1 \vee \cdots \vee Y_n, (b_1, \cdots, b_n)),$$

for the inclusion map and the map defined by $f_1 \times \cdots \times f_n$. If $(X_i, a_i) = (X, a)$, $(Y_i, b_i) = (Y, b)$, $f_i = f$, we write $X^n = X_1 \times \cdots \times X_n$, $f^n = f_1 \times \cdots \times f_n$, ${}^nX = X_1 \vee \cdots \vee X_n$, and ${}^nf = f_1 \vee \cdots \vee f_n$. The diagonal map $\triangle : X \to X^n$ is defined by $\triangle(x) = (x, \cdots, x)$ and the folding map $\nabla : {}^nX \to X$ by $\nabla(a, \cdots, a, x, a, \cdots, a) = x$. The composition of $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f : X \to Z$. The identity map of all spaces involved will consistently be denoted by θ. We consider H-spaces and H'-spaces in the sense of Eckmann and Hilton:

1.1. Definition. *A system (Y, b, μ, ν) consisting of a topological space Y with base-point b and two base-points-preserving maps*

$$\mu : Y \times Y \to Y, \qquad \nu : Y \to Y,$$

is an H-space if

(i) *the composition $Y \vee Y \xrightarrow{j} Y \times Y \xrightarrow{\mu} Y$ is homotopic rel. base-point to the folding map;*

(ii) *the compositions*

$$Y \xrightarrow{\triangle} Y \times Y \xrightarrow{\theta \times \nu} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \xrightarrow{\triangle} Y \times Y \xrightarrow{\nu \times \theta} Y \times Y \xrightarrow{\mu} Y$$

are both nullhomotopic rel. base-point;

(iii) *the compositions*

$$Y \times Y \times Y \xrightarrow{\theta \times \mu} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \times Y \times Y \xrightarrow{\mu \times \theta} Y \times Y \xrightarrow{\mu} Y$$

are homotopic rel. base-point.

We often abbreviate $\mu(y_1, y_2)$ to $y_1 y_2$ and $\nu(y)$ to y^{-1}.

1.2. DEFINITION. *A system (X, a, σ, τ) consisting of a topological space X with base-point a and two base-points-preserving maps*

$$\sigma: X \to X \vee X, \qquad \tau: X \to X,$$

is an H'-space if

(i) *the composition $X \times X \xleftarrow{j} X \vee X \xleftarrow{\sigma} X$ is homotopic rel. base-point to the diagonal map;*

(ii) *the compositions*

$$X \xleftarrow{\nabla} X \vee X \xleftarrow{\theta \vee \tau} X \vee X \xleftarrow{\sigma} X$$

and

$$X \xleftarrow{\nabla} X \vee X \xleftarrow{\tau \vee \theta} X \vee X \xleftarrow{\sigma} X$$

are both nullhomotopic rel. base-point;

(iii) *the compositions*

$$X \vee X \vee X \xleftarrow{\theta \vee \sigma} X \vee X \xleftarrow{\sigma} X$$

and

$$X \vee X \vee X \xleftarrow{\sigma \vee \theta} X \vee X \xleftarrow{\sigma} X$$

are homotopic rel. base-point.

We now introduce commutator and co-commutator maps.

1.3. DEFINITION. *Let (Y, b, μ, ν) be an H-space. The basic commutator map φ is the composition*

$$Y^2 \xrightarrow{\Delta} Y^2 \times Y^2 \xrightarrow{\theta^2 \times \nu^2} Y^2 \times Y^2 \xrightarrow{\mu \times \mu} Y \times Y \xrightarrow{\mu} Y.$$

The commutator map φ_1 of weight 1 is the identity map of Y; the commutator map φ_{n+1} of weight $n+1$ is the composition

$$Y^{n+1} = Y^n \times Y \xrightarrow{\varphi_n \times \theta} Y \times Y \xrightarrow{\varphi} Y$$

in which φ_n is the commutator map of weight $n \geqq 1$.

1.4. DEFINITION. *Let (X, a, σ, τ) be an H'-space. The basic co-commutator map ψ is the composition*

$$ {}^2X \xleftarrow{\nabla} {}^2X \vee {}^2X \xleftarrow{{}^2\theta \vee {}^2\tau} {}^2X \vee {}^2X \xleftarrow{\sigma \vee \sigma} X \vee X \xleftarrow{\sigma} X. $$

The co-commutator map ψ_1 *of weight* 1 *is the identity map of* X; *the co-commutator map* ψ_{n+1} *of weight* $n+1$ *is the composition*

$$ {}^{n+1}X = {}^nX \vee X \xleftarrow{\psi_n \vee \theta} X \vee X \xleftarrow{\psi} X $$

in which ψ_n *is the co-commutator map of weight* $n \geqq 1$.

It is well known [20; 2.4] that

1.5. Lemma. *In an H-space* (Y, b, μ, ν) *the composition*

$$ Y \vee Y \xrightarrow{j} Y \times Y \xrightarrow{\varphi} Y $$

is nullhomotopic rel. base-point.

The dual is

1.6. Lemma. *In an H'-space* (X, a, σ, τ) *the composition*

$$ X \times X \xleftarrow{j} X \vee X \xleftarrow{\psi} X $$

is nullhomotopic rel. base-point.

Proof. In the diagram

$$
\begin{tikzcd}
{}^2X \arrow[d, "j"] & {}^2X \vee {}^2X \arrow[l, "\nabla"'] & {}^2X \vee {}^2X \arrow[l, "{}^2\theta \vee {}^2\tau"'] \arrow[d, "j \vee j"'] & & \\
X^2 & X^2 \vee X^2 \arrow[l, "\nabla"'] & X^2 \vee X^2 \arrow[l, "\theta^2 \vee \tau^2"'] & X \vee X \arrow[l, "\triangle \vee \triangle"'] \arrow[ul, "\sigma \vee \sigma"'] \arrow[dl, "\theta \vee \tau"] & X \arrow[l, "\sigma"'] \\
X \arrow[u, "\triangle"] & & X \vee X \arrow[ll, "\nabla"'] & &
\end{tikzcd}
$$

the triangle is homotopy-commutative rel. base-point according to 1.2 (i); the two other parts are strictly commutative. Therefore, $j \circ \psi$ is homotopic rel. base-point to $\triangle \circ \nabla \circ (\theta \vee \tau) \circ \sigma$. Finally, by 1.2 (ii), the composition $\nabla \circ (\theta \vee \tau) \circ \sigma$ is nullhomotopic rel. base-point.

We now introduce the nilpotency class of an H-space and the co-nilpotency class of an H'-space.

1.7. Definition. *The nilpotency class* nil (Y, b, μ, ν) *of an H-space is the least integer* $n \geqq 0$ *for which the map* φ_{n+1} *is nullhomotopic rel. base-point; if no such integer exists, we put* nil $(Y, b, \mu, \nu) = \infty$.

Thus, nil $(Y, b, \mu, \nu) = 0$ if and only if Y is contractible rel. b, and, as is easily seen, nil $(Y, b, \mu, \nu) \leqq 1$ if and only if (Y, b, μ, ν) is homotopy-commutative.

1.8. DEFINITION. *The co-nilpotency class* conil (X, a, σ, τ) *of an H'-space is the least integer $n \geqq 0$ for which the map ψ_{n+1} is nullhomotopic rel. base-point; if no such integer exists, we put* conil $(X, a, \sigma, \tau) = \infty$.

We shall need homomorphisms in a strict sense of H-spaces and H'-spaces.

1.9. DEFINITION. *Let (B, b_0, μ, ν) and (Y, y_0, μ, ν) be H-spaces. A function $f:B \to Y$ is an H-homomorphism if: f is continuous, $f(b_0) = y_0$, $f \circ \mu(b_1, b_2) = \mu \circ f^2(b_1, b_2)$ and $f \circ \nu(b) = \nu \circ f(b)$ for all $b_1, b_2, b \in B$.*

1.10. DEFINITION. *Let (A, a_0, σ, τ) and (X, x_0, σ, τ) be H'-spaces. A function $g:X \to A$ is an H'-homomorphism if: g is continuous, $g(x_0) = a_0$, $\sigma \circ g(x) = {}^2g \circ \sigma(x)$ and $\tau \circ g(x) = g \circ \tau(x)$ for all $x \in X$.*

The definitions of nilpotency and co-nilpotency classes may be extended to homomorphisms.

1.11. DEFINITION. *The nilpotency class* nil f *of an H-homomorphism $f:B \to Y$ is the least integer $n \geqq 0$ for which the map $f \circ \varphi_{n+1}:B^{n+1} \to Y$ is nullhomotopic rel. base-point; if no such integer exists, we put* nil $f = \infty$.

1.12. DEFINITION. *The co-nilpotency class* conil g *of an H'-homomorphism $g:X \to A$ is the least integer $n \geqq 0$ for which the map $\psi_{n+1} \circ g:X \to {}^{n+1}A$ is nullhomotopic rel. base-point; if no such integer exists, we put* conil $g = \infty$.

The following propositions are easy to prove:

1.13. *If (Y, b, μ, ν) is an H-space and $\theta = \mathrm{id}:Y \to Y$, then*

$$\text{nil } (Y, b, \mu, \nu) = \text{nil } \theta.$$

1.14. *If $f:B \to Q$ and $g:Q \to Y$ are H-homomorphisms, then*

$$\text{nil } g \circ f \leqq \min \{\text{nil } f, \text{nil } g\}.$$

1.15. *If $f:(B, b, \mu, \nu) \to (Y, y, \mu, \nu)$ is an H-homomorphism, then*

$$\text{nil } f \leqq \min \{\text{nil } (B, b, \mu, \nu), \text{nil } (Y, y, \mu, \nu)\}.$$

1.16. *If $f_t:(B, b) \to (Y, y)$ is a homotopy and if f_0 and f_1 are H-homomorphisms, then* nil $f_0 =$ nil f_1.

1.17. *If $f:(B, b) \to (Y, y)$ and $g:(Y, y) \to (B, b)$ are H-homomorphisms and if $g \circ f$ is homotopic rel. b (as a map) to the identity map of B, then*

$$\text{nil } (B, b, \mu, \nu) \leqq \text{nil } (Y, y, \mu, \nu).$$

The duals are automatic.

Finally, let π be an abstract group. The classical definitions of commutator maps γ_n of weight $n \geqq 1$ and of the nilpotency class nil π may be obtained from those given above upon considering π as an H-space with discrete topology. We have nil $\pi = 0$ if and only if π is trivial, and nil $\pi = 1$ if and only if π is a nontrivial Abelian group. We shall also need the concept of nilpotency class of a homomorphism, which is defined in strict analogy to 1.11.

The set $\pi_0(Y)$ of all path-components of an H-space (Y, b, μ, ν) is known to be a group. The following result is easy to prove

1.18. LEMMA. *If the path component of b in Y is contractible rel. b, then* nil $\pi_0(Y) =$ nil (Y, b, μ, ν).

2. Function spaces

Let X and Y be arbitrary topological spaces. We write Y^X for the space of all continuous maps $X \to Y$ taken with the usual compact-open topology; this is defined by selecting as a subbase the collection of all sets

$$(C, V) = \{f \in Y^X \mid f(C) \subset V\},$$

where C is a compact subset of X and V an open subset of Y. If $A \subset X$ and $B \subset Y$, we write $(Y, B)^{(X,A)}$ for the subspace of Y^X consisting of those maps which send A into B. For any two continuous maps

$$\alpha: (Z, C) \to (X, A), \qquad \beta: (Y, B) \to (W, D),$$

the map

$$\beta^\alpha: (Y, B)^{(X,A)} \to (W, D)^{(Z,C)}$$

given by $\beta^\alpha(f) = \beta \circ f \circ \alpha$ is continuous.

The set of all path-components of $(Y, B)^{(X,A)}$ will be denoted by $\pi(X, A; Y, B)$; if $A = B = \emptyset$, we simply write $\pi(X; Y)$. The path-component of f in $(Y, B)^{(X,A)}$ will be denoted by $\{f\}$; we have $\{f_0\} = \{f_1\}$ if and only if there is a homotopy $h_t: (X, A) \to (Y, B)$ such that $h_0 = f_0$, $h_1 = f_1$. Composition with either of the previous maps α and β induces functions

$$\alpha^*: \pi(X, A; Y, B) \to \pi(Z, C; Y, B),$$
$$\beta_*: \pi(X, A; Y, B) \to \pi(X, A; W, D).$$

2.1. LEMMA. *Let X, Y, Z be arbitrary topological spaces. Composition with any homotopy $h: X \times I \to Y$ induces homotopies $s: Z^Y \times I \to Z^X$ and $d: X^Z \times I \to Y^Z$.*

Proof. Let θ denote the identity map of Z.

First, consider the sequence

$$Z^Y \xrightarrow{\theta^h} Z^{X \times I} \to (Z^X)^I$$

in which the second arrow sends any map f into the map g defined by $g(t)(x) = f(x, t)$. This arrow is continuous since I is Hausdorff (see the proof of Theorem 1 in [14]). The first arrow also is continuous so that the composition is continuous. Since I is locally compact Hausdorff, the resulting map s is continuous.

Next, notice that the map d equals the composition

$$X^Z \times I \rightarrow (X \times I)^Z \xrightarrow{h^\theta} Y^Z$$

in which the first arrow sends any pair (f, t) into the map g defined by $g(z) = (f(z), t)$. If $g \in (C, W)$ with $C \subset Z$ compact and $W \subset X \times I$ open, then there exist open subsets $U \subset X$ and $J \subset I$ such that

$$g(C) = f(C) \times t \subset U \times J \subset W.$$

Therefore, $(C, U) \times J$ is a neighborhood of (f, t) which, clearly, is sent into (C, W) so that the first arrow is continuous. The second arrow also is continuous so that the entire composition is continuous.

For future purpose we state the easily proved

2.2. Proposition. *If X_i are Hausdorff spaces and Y_i are arbitrary spaces, then the function*

$$P: Y_1^{X_1} \times \cdots \times Y_n^{X_n} \rightarrow (Y_1 \times \cdots \times Y_n)^{(X_1 \times \cdots \times X_n)}$$

defined by $P(f_1, \cdots, f_n) = f_1 \times \cdots \times f_n$ is continuous.

From 2.2 and 2.1 we obtain the well-known

2.3. Corollary. *If (X, a) is a Hausdorff space with diagonal map $\triangle: X \rightarrow X \times X$ and if (Y, b, μ, ν) is an H-space, setting*

$$f_1 f_2 = \mu \circ (f_1 \times f_2) \circ \triangle \quad \textit{and} \quad (f)^{-1} = \nu \circ f$$

converts Y^X and $(Y, b)^{(X, a)}$ into H-spaces with the constant map $X \rightarrow b$ as base-point.

2.4. *Remark.* Evidently, under the assumptions of 2.3, $\pi(X; Y)$ and $\pi(X, a; Y, b)$ are groups; however, this holds even if X fails to be a Hausdorff space.

In the dual case we only state [6] the

2.5. Proposition. *If (Y, b) is an arbitrary space with folding map $\nabla: Y \vee Y \rightarrow Y$ and if (X, a, σ, τ) is an H′-space, setting*

$$\{f_1\}\{f_2\} = \{\nabla \circ (f_1 \vee f_2) \circ \sigma\} \quad \textit{and} \quad \{f\}^{-1} = \{f \circ \tau\}$$

converts $\pi(X, a; Y, b)$ into a group with the class of the constant map as unit element.

In order to avoid complicated formulae, we shall frequently use the abbreviations

$$^m a = (a, \cdots, a) \in {}^m X \quad \text{and} \quad b^m = (b, \cdots, b) \in Y^m$$

for any $m \geqq 1$ and any based topological spaces (X, a) and (Y, b).

2.6. THEOREM. *For any H'-space (X, a, σ, τ) one has*

$$\operatorname{conil}(X, a, \sigma, \tau) = \sup \operatorname{nil} \pi(X, a; {}^mX, {}^ma) = \sup \operatorname{nil} \pi(X, a; Y, b),$$

where m ranges over all integers $\geqq 1$ and Y over all based topological spaces.

Proof. The n-fold commutator of any elements $\{f_i\} \in \pi(X, a; Y, b)$ is easily seen to equal the based homotopy class of the composition

$$X \xrightarrow{\psi_n} {}^nX \xrightarrow{f_1 \vee \cdots \vee f_n} {}^nY \xrightarrow{\nabla} Y.$$

If $\operatorname{conil} X = n - 1$, then ψ_n is nullhomotopic rel. a, and the same holds for the above composition, so that $\operatorname{nil} \pi(X, a; Y, b) \leqq n - 1$ for any Y. Suppose now that $\operatorname{nil} \pi(X, a; {}^mX, {}^ma) \leqq n - 1$ for any $m \geqq 1$, and let $j_i: X \to {}^nX$ denote the map which imbeds X as the i^{th} summand in ${}^nX$, $1 \leqq i \leqq n$. With $Y = {}^nX$ and $f_i = j_i$, the above composition is nullhomotopic rel. a, and, since

$$\nabla \circ (j_1 \vee \cdots \vee j_n) = \mathrm{id}: {}^nX \to {}^nX,$$

so is ψ_n; therefore, $\operatorname{conil} X \leqq n - 1$, and 2.6 is proved.

2.7. THEOREM. *For any H-space (Y, b, μ, ν) one has*

$$\operatorname{nil}(Y, b, \mu, \nu) = \sup \operatorname{nil} \pi(Y^m, b^m; Y, b) = \sup \operatorname{nil} \pi(X, a; Y, b),$$

where m ranges over all integers $\geqq 1$ and X over all based topological spaces.

Proof. Replace the composition in the proof of 2.6 by

$$Y \xleftarrow{\varphi_n} Y^n \xleftarrow{f_1 \times \cdots \times f_n} X^n \xleftarrow{\Delta} X,$$

replace the maps j_i by the projections $p_i: Y^n \to Y$, and notice that

$$(p_1 \times \cdots \times p_n) \circ \Delta = \mathrm{id}: Y^n \to Y^n.$$

In the sequel, extensive use will be made of the following well-known [6] examples:

2.8. The *loop functor* Ω associates to every space (X, a) its compact-open topologized loop space $\Omega(X, a)$ with the constant loop Ωa as base-point. We often abbreviate $(\Omega(X, a), \Omega a)$ to $\Omega(X, a)$ or simply to ΩX. The maps

$$\mu: \Omega X \times \Omega X \to \Omega X \quad \text{and} \quad \nu: \Omega X \to \Omega X,$$

defined by

$$\begin{aligned} \mu(\omega_1, \omega_2)(s) &= \omega_1(2s) && \text{for } 0 \leqq s \leqq \tfrac{1}{2}, \\ &= \omega_2(2s - 1) && \text{for } \tfrac{1}{2} \leqq s \leqq 1, \\ \nu(\omega)(s) &= \omega(1 - s) && \text{for } 0 \leqq s \leqq 1, \end{aligned}$$

provide an H-space structure in ΩX. For any continuous map

$$f: (X, a) \to (Y, b),$$

the map

$$\Omega f\colon (\Omega(X, a), \Omega a) \to (\Omega(Y, b), \Omega b),$$

defined by $\Omega f(\omega)(s) = f \circ \omega(s)$, is an H-homomorphism as in 1.9.

2.9. The *suspension functor* Σ associates to every space (X, a) its suspension $\Sigma(X, a)$ which results from the Cartesian product $X \times I$ by pinching the subset $X \times 0 \cup X \times 1 \cup a \times I$ to a point Σa which serves as base-point in $\Sigma(X, a)$. We often abbreviate $(\Sigma(X, a), \Sigma a)$ to $\Sigma(X, a)$ or simply to ΣX; the image in ΣX of $(x, s) \in X \times I$ will be denoted by $\langle x, s\rangle$. The maps

$$\sigma\colon \Sigma X \to \Sigma X \vee \Sigma X \quad \text{and} \quad \tau\colon \Sigma X \to \Sigma X,$$

defined by

$$\begin{aligned} \sigma\langle x, s\rangle &= (\langle x, 2s\rangle, \Sigma a) && \text{for } 0 \leqq s \leqq \tfrac{1}{2}, \\ &= (\Sigma a, \langle x, 2s - 1\rangle) && \text{for } \tfrac{1}{2} \leqq s \leqq 1, \\ \tau\langle x, s\rangle &= \langle x, 1 - s\rangle && \text{for } 0 \leqq s \leqq 1, \end{aligned}$$

provide an H'-space structure in ΣX. For any continuous map

$$f\colon (X, a) \to (Y, b),$$

the map

$$\Sigma f\colon (\Sigma(X, a), \Sigma a) \to (\Sigma(Y, b), \Sigma b),$$

defined by $\Sigma f\langle x, s\rangle = \langle f(x), s\rangle$, is an H'-homomorphism as in 1.10.

2.10. With any based space (X, a) we associate the integers

$$\operatorname{nil} \Omega(X, a) = \operatorname{nil} (\Omega(X, a), \Omega a, \mu, \nu)$$

and

$$\operatorname{conil} \Sigma(X, a) = \operatorname{conil} (\Sigma(X, a), \Sigma a, \sigma, \tau)$$

which are defined according to 2.8, 1.7, and 2.9, 1.8.

2.11. *If (X, a) and (Y, b) have the same based homotopy type, then*

$$\operatorname{nil} \Omega(X, a) = \operatorname{nil} \Omega(Y, b) \quad \textit{and} \quad \operatorname{conil} \Sigma(X, a) = \operatorname{conil} \Sigma(Y, b).$$

If X is a connected CW-complex, then for any $a, b \in X$, (X, a) and (X, b) have the same based homotopy type [16; p. 333]. Therefore,

2.12. *If X is a connected CW-complex, then* $\operatorname{nil} \Omega(X, a)$ *and* $\operatorname{conil} \Sigma(X, a)$ *do not depend on the base-point $a \in X$ and will be abbreviated to* $\operatorname{nil} \Omega X$ *and* $\operatorname{conil} \Sigma X$.

2.13. Let (X, a) and (Y, b) be arbitrary topological spaces. According to 2.4 and 2.5, $\pi(X, a; \Omega Y, \Omega b)$ and $\pi(\Sigma X, \Sigma a; Y, b)$ have group structures; they are related [6] by a natural isomorphism

$$\phi\colon \pi(\Sigma X, \Sigma a; Y, b) \approx \pi(X, a; \Omega Y, \Omega b)$$

which is defined by

$$\phi\{f\} = \{g\} \qquad \text{with} \qquad g(x)(s) = f\langle x, s\rangle.$$

Since ${}^m(\Sigma X)$ and $(\Omega Y)^m$ may obviously be identified with $\Sigma({}^m X)$ and $\Omega(Y^m)$, from 2.6 and 2.7 we now obtain

2.14. COROLLARY. *For any space (X, a) one has*

$$\operatorname{conil} \Sigma(X, a) = \sup \operatorname{nil} \pi(X, a; \Omega\Sigma({}^m X), \Omega\Sigma({}^m a)) = \sup \operatorname{nil} \pi(X, a; \Omega Y, \Omega b),$$

where m ranges over all integers $\geqq 1$ and Y over all based topological spaces.

2.15. COROLLARY. *For any space (Y, b) one has*

$$\operatorname{nil} \Omega(Y, b) = \sup \operatorname{nil} \pi(\Sigma\Omega(Y^m), \Sigma\Omega(b^m); Y, b) = \sup \operatorname{nil} \pi(\Sigma X, \Sigma a; Y, b),$$

where m ranges over all integers $\geqq 1$ and X over all based topological spaces.

2.16. For any (Z, c), the natural imbedding $e: Z \to \Omega\Sigma Z$ is the continuous and univalent map defined by $e(z)(s) = \langle z, s\rangle$. The natural projection $p: \Sigma\Omega Z \to Z$ is the continuous map defined by $p\langle \omega, s\rangle = \omega(s)$; p is onto if and only if Z is 0-connected.

The following result will play a fundamental role later.

2.17. PROPOSITION. *Let (X, a) be an arbitrary space and $n \geqq 1$ an arbitrary integer. The co-commutator map*

$$\psi_n: \Sigma(X, a) \to {}^n\Sigma(X, a)$$

of weight n in the H'-space ΣX is nullhomotopic rel. base-point if and only if so is the composition

$$X \xrightarrow{\triangle} X^n \xrightarrow{j_1 \times \cdots \times j_n} ({}^n X)^n \xrightarrow{e^n} (\Omega\Sigma({}^n X))^n \xrightarrow{\varphi_n} \Omega\Sigma({}^n X)$$

in which $\triangle$ is the diagonal map, $j_i: X \to {}^n X$ imbeds X as the i^{th} summand in ${}^n X$, e is the natural imbedding of ${}^n X$ in $\Omega\Sigma({}^n X)$, and φ_n is the commutator map of weight n in this H-space.

Proof. Let $f_i = \Sigma j_i: \Sigma X \to \Sigma({}^n X)$. The natural isomorphism

$$\phi: \pi(\Sigma X, \Sigma a; \Sigma({}^n X), \Sigma({}^n a)) \approx \pi(X, a; \Omega\Sigma({}^n X), \Omega\Sigma({}^n a))$$

sends $\{f_i\}$ into $\{g_i\}$, where $g_i(x)(s) = f_i\langle x, s\rangle$. Clearly, the n-fold commutators

$$\xi = [\{f_1\}, \cdots, \{f_n\}] \quad \text{and} \quad \eta = \phi(\xi) = [\{g_1\}, \cdots, \{g_n\}]$$

are simultaneously trivial or not. As is easily seen, ξ is represented by the composition

$$\Sigma X \xrightarrow{\psi_n} {}^n\Sigma X \xrightarrow{f_1 \vee \cdots \vee f_n} {}^n\Sigma({}^n X) \xrightarrow{\nabla} \Sigma({}^n X),$$

in which ∇ is the folding map. Since the composition of the last two arrows is a homeomorphism, ξ is trivial if and only if ψ_n is nullhomotopic rel. base-point. Also, η is represented by the composition

$$X \xrightarrow{\triangle} X^n \xrightarrow{g_1 \times \cdots \times g_n} (\Omega\Sigma({}^nX))^n \xrightarrow{\varphi_n} \Omega\Sigma({}^nX),$$

and, as is easily seen, $g_i = e \circ j_i$. Therefore, η is trivial if and only if the composition in the statement is nullhomotopic rel. a.

Dually, we have

2.18. PROPOSITION. *Let (Y, b) be an arbitrary space and $n \geqq 1$ an arbitrary integer. The commutator map*

$$\varphi_n: (\Omega(Y, b))^n \to \Omega(Y, b)$$

of weight n in the H-space ΩY is nullhomotopic rel. base-point if and only if so is the composition

$$Y \xleftarrow{\nabla} {}^nY \xleftarrow{p_1 \vee \cdots \vee p_n} {}^n(Y^n) \xleftarrow{{}^np} {}^n(\Sigma\Omega(Y^n)) \xleftarrow{\psi_n} \Sigma\Omega(Y^n)$$

in which ∇ is the folding map, $p_i: Y^n \to Y$ projects Y^n on its i^{th} factor, p is the natural projection of $\Sigma\Omega(Y^n)$ on Y^n, and ψ_n is the co-commutator map of weight n in this H'-space.

3. Fibrations and cofibrations

We consider fibrations as given by the

3.1. DEFINITION. *A sequence $(Q, q_0) \xrightarrow{\eta} (Y, y_0) \xrightarrow{\beta} (B, b_0)$ of spaces and maps is a fibration if*

(i) *η defines a homeomorphism of the fibre Q onto the subspace $\beta^{-1}(b_0)$ of Y, and if*

(ii) *for any space (E, e_0), any homotopy $h_t: (E, e_0) \to (B, b_0)$ and any map $k: (E, e_0) \to (Y, y_0)$ satisfying $\beta \circ k = h_0$, there is a homotopy $H_t: (E, e_0) \to (Y, y_0)$ such that $H_0 = k$ and $\beta \circ H_t = h_t$.*

We do not require that β be onto. A familiar example is provided by the sequence $\Omega B \xrightarrow{\eta} \mathcal{E}B \xrightarrow{\beta} B$ in which $\mathcal{E}B$ is the space of all paths λ in B emanating from the base-point, η is the inclusion map of the loop space, and $\beta(\lambda) = \lambda(1)$.

Fibrations are dual to cofibrations which are given [6] by the

3.2. DEFINITION. *A sequence $(P, p_0) \xleftarrow{\eta} (X, x_0) \xleftarrow{\alpha} (A, a_0)$ of spaces and maps is a cofibration if*

(i) *η induces a homeomorphism of the cofibre P onto the identification space obtained by pinching the subset $\alpha(A)$ of X to a point, and if*

(ii) *for any space (E, e_0), any homotopy $h_t: (A, a_0) \to (E, e_0)$ and any map $k: (X, x_0) \to (E, e_0)$ satisfying $k \circ \alpha = h_0$, there is a homotopy $H_t: (X, x_0) \to (E, e_0)$ such that $H_0 = k$ and $H_t \circ \alpha = h_t$.*

We do not require that α be univalent.

3.3. THEOREM. *If* $(Q, q_0) \xrightarrow{\eta} (Y, y_0) \xrightarrow{\beta} (B, b_0)$ *is a fibration, then* $\operatorname{nil} \Omega(Q, q_0) \leqq 1 + \operatorname{nil} \Omega\eta$.

Proof. Let $\pi_n(D, R) = \pi(\Sigma_n D, \Sigma_n d; R, r)$, where (D, d) and (R, r) are arbitrary spaces and Σ_n denotes n-fold suspension ($n \geqq 1$). As a generalization of the familiar homotopy sequence of a fibration, for any space (X, a) there is [11; p. 24] an exact sequence of groups and homomorphisms

$$\cdots \xrightarrow{\beta_*} \pi_2(X, B) \xrightarrow{\partial} \pi_1(X, Q) \xrightarrow{\eta_*} \pi_1(X, Y) \xrightarrow{\beta_*} \pi_1(X, B)$$

with the property [11; p. 22] that $\partial\pi_2(X, B)$ lies in the center of $\pi_1(X, Q)$. As is easily seen, this implies that

$$\operatorname{nil} \pi_1(X, Q) \leqq 1 + \operatorname{nil} \eta_* .$$

Since the isomorphism ϕ in 2.13 is natural, we have the commutative diagram

$$\begin{array}{ccc} \pi(\Sigma X, \Sigma a; Q, q_0) & \xrightarrow{\eta_*} & \pi(\Sigma X, \Sigma a; Y, y_0) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \pi(X, a; \Omega Q, \Omega q_0) & \xrightarrow{(\Omega\eta)_*} & \pi(X, a; \Omega Y, \Omega y_0) \end{array}$$

so that $\operatorname{nil} \eta_* = \operatorname{nil} (\Omega\eta)_*$. Evidently, $\operatorname{nil} (\Omega\eta)_* \leqq \operatorname{nil} \Omega\eta$. Finally, by 2.15,

$$\operatorname{nil} \Omega(Q, q_0) = \sup \operatorname{nil} \pi_1(X, Q)$$

with X ranging over all based topological spaces, and 3.3 is proved.

Dually, we have

3.4. THEOREM. *If* $(P, p_0) \xleftarrow{\eta} (X, x_0) \xleftarrow{\alpha} (A, a_0)$ *is a cofibration, then* $\operatorname{conil} \Sigma(P, p_0) \leqq 1 + \operatorname{conil} \Sigma\eta$.

Proof. With the above notations, for any space (Y, b) we now have [11; p. 25] the exact sequence

$$\cdots \xrightarrow{(\Sigma_2 \alpha)^*} \pi_2(A, Y) \xrightarrow{\partial} \pi_1(P, Y) \xrightarrow{(\Sigma\eta)^*} \pi_1(X, Y) \xrightarrow{(\Sigma\alpha)^*} \pi_1(A, Y)$$

with $\partial\pi_2(A, Y)$ lying in the center of $\pi_1(P, Y)$. Therefore,

$$\operatorname{nil} \pi_1(P, Y) \leqq 1 + \operatorname{nil} (\Sigma\eta)^*.$$

One has $\operatorname{nil} (\Sigma\eta)^* \leqq \operatorname{conil} \Sigma\eta$, and 2.6 yields the desired result.

Application of 1.15 and of its dual now yields

3.5. COROLLARY. *If* $(Q, q_0) \rightarrow (Y, y_0) \rightarrow (B, b_0)$ *is a fibration, then* $\operatorname{nil} \Omega(Q, q_0) \leqq 1 + \operatorname{nil} \Omega(Y, y_0)$.

3.6. Corollary. *If* $(P, p_0) \leftarrow (X, x_0) \leftarrow (A, a_0)$ *is a cofibration, then* conil $\Sigma(P, p_0) \leqq 1 +$ conil $\Sigma(X, x_0)$.

3.7. Now let $\mathfrak{F}:(Q, q_0) \xrightarrow{\eta} (Y, y_0) \xrightarrow{\beta} (B, b_0)$ be a fibration and $f:(C, c_0) \to (B, b_0)$ a continuous map. Let

$$Z = \{(c, y) \mid f(c) = \beta(y)\} \subset C \times Y \quad \text{and} \quad z_0 = (c_0, y_0).$$

The sequence $(Q, q_0) \xrightarrow{\zeta} (Z, z_0) \xrightarrow{\gamma} (C, c_0)$, in which

$$\zeta(q) = (c_0, \eta(q)) \quad \text{and} \quad \gamma(c, y) = c,$$

is the well-known fibration induced by $\mathfrak{F}$ via f.

We shall need the following consequence of 3.5:

3.8. Corollary. *Let* $f:(C, c_0) \to (B, b_0)$ *be a continuous map and* $(\Omega B, \Omega b_0) \to (Z, z_0) \to (C, c_0)$ *the fibration induced by* $\Omega B \to \mathcal{E}B \to B$ *via* f. *Then* nil $\Omega(Z, z_0) \leqq 1 +$ nil $\Omega(C, c_0)$.

Proof. Introduce the space

$$Y = \{(c, \lambda) \mid f(c) = \lambda(1)\} \subset C \times B^I, \qquad y_0 = (c_0, b_0^I).$$

One has $(Z, z_0) \subset (Y, y_0)$ and the sequence

$$(Z, z_0) \xrightarrow{\eta} (Y, y_0) \xrightarrow{\beta} (B, b_0),$$

in which $\beta(c, \lambda) = \lambda(0)$ and η is the inclusion map, is a fibration. The result now follows from 3.5 upon noticing that (Y, y_0) has the based homotopy type of (C, c_0).

3.9 *Remark.* Condition 3.1 (ii) is more restrictive than Serre's classical definition of a fibre space in which it is required that the covering homotopy theorem hold only for maps of polyhedra. Nevertheless, there are two important cases in which 3.1 (ii) is fulfilled: the first is that of a fibre space obtained upon transforming by means of spaces of paths any map into a fibre map; the second is that of a locally trivial fibre space in which both the base and the total space are metrizable (the proof given in [4] may readily be adjusted so as to provide homotopies keeping base-points fixed).

3.10. We conclude by giving an example of a space Y which fails to be a generalized H-space although its loop space is homotopy-commutative.

The space Y results by adding cells to the complex projective plane M so as to kill its homotopy groups in dimensions $\geqq 6$. Let S^5 denote the 5-sphere, and recall that $\pi_q(M) \approx \pi_q(S^5)$ if $q \geqq 3$. Since $\pi_q(S^5)$ is a finite group for $q \geqq 6$, it follows from [18] that Y has the rational cohomology groups of M. Application of the Hopf [2] theorem then implies that Y cannot have a continuous multiplication with two-sided homotopy unit element.

We now prove that nil $\Omega Y \leqq 1$. Notice first that, without altering the homotopy type of Y, we may assume that there is a fibration

$$(Q,\, q_0) \xrightarrow{\ \eta\ } (Y, y_0) \xrightarrow{\ \beta\ } (B, b_0)$$

with fibre Q of type $(Z,\, 5)$ and base B of type $(Z,\, 2)$; here and below, Z stands for the integers. Introduce the diagram

$$\begin{tikzcd}
\Omega Q \arrow[r, "\Omega\eta"] & \Omega Y \arrow[r, "\Omega\beta"] & \Omega B \\
\Omega Y \mathbin{\overline{\wedge}} \Omega Y \arrow[u, "\xi"] \arrow[ur, "\phi"] & \Omega Y \times \Omega Y \arrow[l, "j"'] \arrow[u, "\varphi"] \arrow[r, "\Omega\beta \times \Omega\beta"] & \Omega B \times \Omega B \arrow[u, "\varphi"]
\end{tikzcd}$$

in which φ denotes the basic commutator map and j imbeds $\Omega Y \times \Omega Y$ in the space obtained by attaching to the latter the reduced cone over the subset $\Omega Y \vee \Omega Y$ (see [16; p. 329]). It follows from 1.5 that φ may be extended to $\Omega Y \mathbin{\overline{\wedge}} \Omega Y$ yielding a map ϕ for which the lower triangle in the left square is commutative. Since $\Omega\beta$ is an H-homomorphism, the square on the right commutes. Evidently, B is an H-space so that the map φ on the right is nullhomotopic rel. base-point. By commutativity, so is also the composition $\Omega\beta \circ \phi \circ j$, and, by [16; Satz 14], it follows that $\Omega\beta \circ \phi$ already is nullhomotopic rel. base-point. Therefore, 3.1 yields a map ξ for which the upper triangle in the left is homotopy-commutative rel. base-point.

We have $\pi_1(\Omega Y) \approx Z$ and $\pi_q(\Omega Y) = 0$ for $q = 2, 3$, so that

$$H_q(\Omega Y; Z) \approx H_q(Z, 1; Z) = 0 \qquad\qquad \text{for} \quad q = 2, 3,$$

and the Künneth formula now yields $H_q(\Omega Y \mathbin{\overline{\wedge}} \Omega Y; Z) = 0$ for $q = 3, 4$. As a result,

$$H^4(\Omega Y \mathbin{\overline{\wedge}} \Omega Y; Z) = 0.$$

Since ΩQ is of type $(Z,\, 4)$ and, as is implied by [12; Theorems 3 and 2], $\Omega Y \mathbin{\overline{\wedge}} \Omega Y$ has the based homotopy type of a CW-complex, the map ξ is nullhomotopic rel. base-point. It follows now easily that the map φ in the center also is nullhomotopic rel. base-point so that nil $\Omega Y \leqq 1$ as asserted.

Let us finally mention that it is easy to prove that any "M_n-Raum", in the sense of [5], has a homotopy commutative loop space; however, we ignore whether such a space necessarily is an H-space.

4. The nilpotency class of a loop space

For any topological spaces $(Y,\, b)$ and $(Z,\, c)$, let $Y \wedge Z$ and $p: Y \times Z \to Y \wedge Z$ denote the identification space and identification map resulting by pinching the subset $Y \vee Z$ of $Y \times Z$ to a point. We write $y \wedge z$ for $p(y, z)$ and take $b \wedge c$ as base-point in $Y \wedge Z$.

Let $(K_i\,,\, a_i)$, $i \geqq 1$, be pairs, each consisting of a countable CW-complex and one of its vertices. Let

$$(W_1, d_1) = (K_1, a_1) \quad \text{and} \quad p_1 = \theta:(K_1, a_1) \to (W_1, d_1);$$

also, for every $n \geqq 1$, define

$$(W_{n+1}, d_{n+1}) \qquad \text{as} \qquad (W_n \wedge K_{n+1}, d_n \wedge a_{n+1})$$

and

$$p_{n+1}:(K_1 \times \cdots \times K_{n+1}, (a_1, \cdots, a_{n+1})) \to (W_{n+1}, d_{n+1})$$

as the composition

$$(K_1 \times \cdots \times K_n) \times K_{n+1} \xrightarrow{p_n \times \theta} W_n \times K_{n+1} \xrightarrow{p} W_{n+1},$$

where θ stands for the identity map of any space. We write $x_1 \wedge \cdots \wedge x_n$ for $p_n(x_1, \cdots, x_n)$ so that $d_n = a_1 \wedge \cdots \wedge a_n$. For any $n \geqq 1$, W_n is a countable CW-complex having d_n as a vertex [16; p. 339].

Let (G, e) be a fixed H-space with commutator maps φ and φ_n.

4.1. Lemma. *For any* $n \geqq 1$,

$$p_n^*:\pi(W_n, d_n; G, e) \to \pi(K_1 \times \cdots \times K_n, (a_1, \cdots, a_n); G, e)$$

is a monomorphism.

Proof. According to [16; p. 300], a map $f:(Y, b) \to (Z, c)$ is called monomorphic if, whatever be the space (V, v), a map $g:(Z, c) \to (V, v)$ is null-homotopic rel. base-point whenever so is $g \circ f$.

Now, as is easily seen, p_{n+1} equals the composition

$$(K_1 \times \cdots \times K_n) \times K_{n+1} \xrightarrow{p} (K_1 \times \cdots \times K_n) \wedge K_{n+1}$$
$$\xrightarrow{p_n \wedge \theta} (K_1 \wedge \cdots \wedge K_n) \wedge K_{n+1},$$

in which $p_n \wedge \theta((x_1, \cdots, x_n) \wedge x_{n+1}) = (x_1 \wedge \cdots \wedge x_n) \wedge x_{n+1}$ for all $x_i \in K_i$. According to [16; Sätze 14 and 16], p is monomorphic; according to [16; Satz 22] so is also $p_n \wedge \theta$ provided p_n is monomorphic, and 4.1 follows by induction upon noticing that, anyhow, p_n^* is a homomorphism.

4.2. Lemma. *For any two countable CW-complexes* (Y, b) *and* (Z, c), *in which* b *and* c *are vertices, there is a function*

$$\Gamma = \Gamma(Y, Z):\pi(Y, b; G, e) \times \pi(Z, c; G, e) \to \pi(Y \wedge Z, b \wedge c; G, e)$$

such that the diagram

$$\begin{array}{ccc} Y \times Z & \xrightarrow{f \times g} & G \times G \\ \downarrow p & & \downarrow \varphi \\ Y \wedge Z & \xrightarrow{h} & G \end{array}$$

is homotopy-commutative rel. base-point for any base-points-preserving maps f, g, *and* $h \in \Gamma(\{f\}, \{g\})$.

Proof. One has $f \times g(Y \vee Z) \subset G \vee G$ so that, by 1.5, the map $\varphi \circ (f \times g) \mid Y \vee Z$ is nullhomotopic rel. base-point. The CW-pair $(Y \times Z, Y \vee Z)$ has the homotopy extension property, and there results a homotopy $k_t : Y \times Z \to G$ such that

$$k_0 = \varphi \circ (f \times g) \quad \text{and} \quad k_1(Y \vee Z) = k_t(b, c) = e.$$

Since p is an identification, there is a map $h : (Y \wedge Z, b \wedge c) \to (G, e)$ such that $h \circ p = k_1$; also, $\{h\}$ is uniquely determined by the pair $(\{f\}, \{g\})$ since, by 4.1, p^* is a monomorphism. The required function Γ is now defined by setting $\Gamma(\{f\}, \{g\}) = \{h\}$.

Reverting to the countable CW-complexes (K_i, a_i), we define a sequence of functions

$$\Gamma_n : \prod_{i=1}^{n} \pi(K_i, a_i; G, e) \to \pi(W_n, d_n; G, e)$$

as follows: Γ_1 is the identity map and Γ_{n+1} equals the composition

$$\prod_{i=1}^{n+1} \pi(K_i; G) \xrightarrow{\Gamma_n \times \theta} \pi(W_n; G) \times \pi(K_{n+1}; G) \xrightarrow{\Gamma} \pi(W_{n+1}; G),$$

in which θ is the appropriate identity map and $\Gamma = \Gamma(W_n, K_{n+1})$ is given by 4.2; base-points have been discarded to simplify notation.

An immediate induction argument yields

4.3. Lemma. *Whatever be $n \geqq 1$, the diagram*

$$\begin{array}{ccc} K_1 \times \cdots \times K_n & \xrightarrow{f_1 \times \cdots \times f_n} & G \times \cdots \times G \\ \downarrow p_n & & \downarrow \varphi_n \\ W_n & \xrightarrow{\qquad h \qquad} & G \end{array}$$

is homotopy-commutative rel. base-point for any base-points-preserving maps $f_1, \cdots, f_n$, and $h \in \Gamma_n(\{f_1\}, \cdots, \{f_n\})$.

Now let (X, x_0) be an arbitrary space. Evidently,

4.4. Theorem. nil $\pi_1(X, x_0) \leqq$ nil $\Omega(X, x_0)$.

We shall give an extension of this result involving Whitehead products, generally denoted by

$$[\alpha_1, \alpha_2] \in \pi_{q_1+q_2-1}(X, x_0) \qquad \text{if} \qquad \alpha_i \in \pi_{q_i}(X, x_0), \quad q_i \geqq 1.$$

We define $(n + 1)$-fold Whitehead products

$$[\alpha_1, \cdots, \alpha_{n+1}] \qquad \text{as} \qquad [[\alpha_1, \cdots, \alpha_n], \alpha_{n+1}]$$

agreeing that, for $n = 0$, $[\alpha] = \alpha$.

4.5. Definition. W-long (X, x_0) *is the least integer $n \geqq 0$ such that $[\alpha_1, \cdots, \alpha_{n+1}] = 0$ for all $\alpha_i \in \pi_{q_i}(X, x_0)$, $q_i \geqq 1$; if no such integer exists, we put* W-long $(X, x_0) = \infty$.

Next, consider the natural isomorphism

$$T:\pi_{r+1}(X, x_0) \approx \pi_r(\Omega(X, x_0), \Omega x_0) \qquad (r \geqq 0)$$

arising from the fibration of the space of paths in X emanating from x_0. Since $S^j \wedge S^k = S^{j+k}$ (S^m = m-sphere), a well-known result by Samelson [17] may be stated as

$$T[\alpha_1, \alpha_2] = \varepsilon\Gamma(T\alpha_1, T\alpha_2) \qquad (\varepsilon = \pm 1),$$

where $\Gamma = \Gamma(S^{q_1-1}, S^{q_2-1})$ is given by 4.2 with $(G, e) = (\Omega X, \Omega x_0)$. Although not explicitly stated in [17], Samelson's result is also valid if $q_1 = 1$ or $q_2 = 1$ (see [13; Proposition 1]); the actual value of ε is irrelevant for the sequel. The inductive definition of the functions Γ_n, with the K_i replaced by spheres of suitable dimensions, now yields

$$T[\alpha_1, \cdots, \alpha_n] = \varepsilon\Gamma_n(T\alpha_1, \cdots, T\alpha_n). \tag{1}$$

If nil $\Omega(X, x_0) = n - 1$, then the commutator map φ_n in $\Omega(X, x_0)$ is nullhomotopic rel. base-point so that, by 4.3 and 4.1,

$$\Gamma_n(\gamma_1, \cdots, \gamma_n) = 0 \qquad \text{for all} \quad \gamma_1, \cdots, \gamma_n.$$

Since T is an isomorphism, (1) finally implies

4.6. THEOREM. W-long $(X, x_0) \leqq$ nil $\Omega(X, x_0)$.

4.7. *Remark.* The sequences (K_i, a_i) and (W_n, d_n) may also be used to obtain results similar to 4.6 concerning more general homotopy products

$$\textstyle\prod_{i=1}^{2} \pi(\Sigma A_i; X) \to \prod_{i=1}^{2} \pi(A_i; \Omega X)$$
$$\to \pi(A_1 \wedge A_2; \Omega X) \to \pi(B; \Omega X) \to \pi(\Sigma B; X)$$

where, as Hilton suggested, B and $A_1 \wedge A_2$ are related by a fixed map $\phi: B \to A_1 \wedge A_2$; base-points have been discarded to simplify notation.

Now let $P(E)$ denote the singular polytope of an arbitrary space E, and let $p_E: P(E) \to E$ denote the canonical map inducing homotopy isomorphisms [9]. The following result, which will be used below, may be of independent interest (compare with [8; 2.4 and 4.7]).

4.8. PROPOSITION. *If X is a 0-connected space then, for any $x_0 \in X$,* nil $\Omega P(X) \leqq$ nil $\Omega(X, x_0)$.

Proof. Let $P = P(X)$. Since p_X is onto, we may select a base-point $a \in P$ with $p_X(a) = x_0$, and, since P is a connected CW-complex, 2.12 yields nil ΩP = nil $\Omega(P, a)$. Suppose nil $\Omega(X, x_0) = n - 1$. Then the commutator map φ_n in ΩX is nullhomotopic rel. base-point so that its values all lie in the path-component $(\Omega X)_0$ of the constant loop in ΩX. Therefore, φ_n defines a map f as indicated in the diagram

$$\begin{tikzcd}
\Omega P \times \cdots \times \Omega P \arrow[rr, "\Omega p_X \times \cdots \times \Omega p_X"] \arrow[d, "g"] & & \Omega X \times \cdots \times \Omega X \arrow[d, "f"] \\
(\Omega P)_0 \arrow[rr, "(\Omega p_X)_0"] \arrow[dr, "h"'] & & (\Omega X)_0 \\
& P((\Omega X)_0) \arrow[ur, "p_{(\Omega X)_0}"'] &
\end{tikzcd}$$

Next, p_X induces an isomorphism of fundamental groups, and since it is an H-homomorphism, Ωp_X commutes with the commutator maps in ΩP and ΩX. Therefore, the commutator map φ_n in ΩP also has its values in the path-component $(\Omega P)_0$ of the constant loop in ΩP and, thus, defines a map g as indicated in the diagram. With $(\Omega p_X)_0$ defined by Ωp_X in the obvious way, the square commutes. By [12], ΩP, hence also $(\Omega P)_0$, has the based homotopy type of a CW-complex so that there is a map h yielding homotopical commutativity rel. base-point in the triangle. Let $h(\Omega a)$ serve as base-point in $P((\Omega X)_0)$. Since $(\Omega p_X)_0$ and $p_{(\Omega X)_0}$ both induce homotopy isomorphisms, so does also h; since its domain and range have the based homotopy type of connected CW-complexes, it follows that

(2) h *is a based homotopy equivalence.*

Since f is nullhomotopic rel. base-point, homotopical commutativity implies that so is also the composition $p_{(\Omega X)_0} \circ h \circ g$. By [12; Theorem 2], $\Omega P \times \cdots \times \Omega P$ has the based homotopy type of a CW-complex so that the map $h \circ g$ already is nullhomotopic rel. base-point, and (2) finally implies the desired result: g, hence also φ_n in ΩP, is nullhomotopic rel. base-point.

We now proceed to find upper bounds for nil $\Omega(X, x_0)$.

First, let (X, x_0) be a connected aspherical CW-complex. Then $\pi_q(\Omega(X, x_0), \Omega x_0) = 0$ for all $q \geqq 1$; also, by [12], $\Omega(X, x_0)$ has the based homotopy type of a CW-complex. Therefore, the path-component of the constant loop in $\Omega(X, x_0)$ is contractible rel. Ωx_0, and 1.18 yields

4.9. THEOREM. *If X is a connected aspherical CW-complex, then* nil $\pi_1(X) =$ nil ΩX.

For further reference we state the easily proved

4.10. LEMMA. nil $\Omega(X \times Y, (a, b)) = \max\{\text{nil } \Omega(X, a), \text{nil } \Omega(Y, b)\}$.

We now prove the

4.11. THEOREM. *Let X be a 1-connected, i.e., a connected and simply connected, CW-complex. Suppose the invariants k^{n+2} of a Postnikov system for X are trivial for all but r values of n. Then* nil $\Omega X \leqq r + 1$.

Proof. Select a base-point $a \in X$ and let $\pi_n = \pi_n(X, a)$. Let (X_n, p_{n+1}, f_n) be a Postnikov [15] system for X consisting of spaces (X_n, a_n), each having the based homotopy type of a CW-complex, and base-points-preserving maps p_{n+1}, f_n, such that

(i) $p_{n+1}: X_{n+1} \to X_n$ is a fibre map in the sense of Serre with fibre F_{n+1} of type $K(\pi_{n+1}, n+1)$ and characteristic class k^{n+2};

(ii) $f_n: X \to X_n$ induces homotopy isomorphisms in dimensions $\leqq n$;

(iii) $f_n = p_{n+1} \circ f_{n+1}$;

(iv) $X_0 = a_0$.

Let (Y_n, b_n) be of type $K(\pi_{n+1}, n+2)$, and introduce a map $g: (X_n, a_n) \to (Y_n, b_n)$ such that $g^*(\iota) = k^{n+2}$, where ι is the fundamental class in $H^{n+2}(Y_n; \pi_{n+1})$; if $k^{n+2} = 0$, we take for g the constant map. Let further

$$(\Omega Y_n, \Omega b_n) \to (Z_{n+1}, c_{n+1}) \xrightarrow{\beta} (X_n, a_n)$$

be the fibration induced by $\Omega Y_n \to \mathcal{E} Y_n \to Y_n$ via g. Since X_n is 1-connected and $F_{n+1} \in K(\pi_{n+1}, n+1)$, there is a map

$$h_{n+1}: (X_{n+1}, a_{n+1}) \to (Z_{n+1}, c_{n+1})$$

satisfying $\beta \circ h_{n+1} = p_{n+1}$ and inducing homotopy isomorphisms in all dimensions (see for instance [11; Theorem 7.1, p. 43]). As a result, X_{n+1} has the based homotopy type of the singular polytope of Z_{n+1}. Therefore, consecutive application of 4.8 and 3.8 yields

$$\text{(3)} \qquad \operatorname{nil} \Omega(X_{n+1}, a_{n+1}) \leqq \operatorname{nil} \Omega(Z_{n+1}, c_{n+1}) \leqq 1 + \operatorname{nil} \Omega(X_n, a_n)$$

for arbitrary k^{n+2}. If $k^{n+2} = 0$, then $g(X_n) = b_n$ so that

$$\text{(4)} \qquad Z_{n+1} = X_n \times \Omega Y_n \quad \text{with projection} \quad \rho_n: Z_{n+1} \to \Omega Y_n,$$

and consecutive application of 4.8 and 4.10 now yields

$$\text{(5)} \qquad \operatorname{nil} \Omega(X_{n+1}, a_{n+1}) \leqq \operatorname{nil} \Omega(Z_{n+1}, c_{n+1}) \leqq \max\{\operatorname{nil} \Omega(X_n, a_n), 1\}.$$

Let q be such that $k^{n+2} = 0$ for $n \geqq q$. An easy computation using (3) and (5) yields

$$\text{(6)} \qquad \operatorname{nil} \Omega(X_q, a_q) \leqq r + 1.$$

Notice next that (4) certainly holds if $n \geqq q$. We may therefore define a base-points-preserving map

$$\phi: X \to X_q \times \textstyle\prod_{n \geqq q} \Omega Y_n$$

by setting

$$\phi(x) = (f_q(x), (\rho_q \circ h_{q+1} \circ f_{q+1}(x), \cdots, \rho_n \circ h_{n+1} \circ f_{n+1}(x), \cdots)).$$

The Cartesian product $\prod \Omega Y_n$ obviously is an H-space so that, according to 4.10,

$$\text{(7)} \qquad \operatorname{nil} \Omega(X_q \times \prod \Omega Y_n) \leqq \max \{\operatorname{nil} \Omega X_q , 1\}.$$

As is easily seen, ϕ induces homotopy isomorphisms in all dimensions. There results a based homotopy equivalence of X and the singular polytope of $X_q \times \prod \Omega Y_n$, and the desired result finally follows from 4.8, (7), and (6).

4.12. Corollary. *Let X be a 1-connected CW-complex. If $\pi_n(X) = 0$ for all but r values of n, then* nil $\Omega X \leqq r$.

Proof. One has $k^{n+2} \epsilon H^{n+2}(X_n ; \pi_{n+1})$, and if π_{q+1} is the first nonvanishing homotopy group of X, then X_q still is a point so that $k^{q+2} = 0$.

Next, comparison of 4.6 and 4.11 yields

4.13. Corollary. *Let X be a 1-connected CW-complex, If the Postnikov invariants $k^{n+2}(X)$ vanish for all but r values of n, then* W-long $X \leqq r + 1$.

4.14. *Remark.* According to [21] there exists a connected CW-complex X such that $\pi_1(X)$ is cyclic of order 2, $\pi_2(X)$ is cyclic infinite, $\pi_q(X) = 0$ for $q \geqq 3$, and $\pi_1(X)$ operates nontrivially on $\pi_2(X)$. Since

$$[\alpha, \xi] = \xi\alpha - \alpha = -2\alpha$$

for arbitrary $\alpha \epsilon \pi_2(X)$ and nontrivial $\xi \epsilon \pi_1(X)$, X has nonvanishing iterated Whitehead products of arbitrary length so that, by 4.6, nil $\Omega X = \infty$. Therefore, the restriction that X be simply connected cannot be removed from 4.11 and 4.12. However, if X is n-simple for every $n \geqq 2$, the relation nil $\Omega X \leqq r + \max \{\operatorname{nil} \pi_1(X), 1\}$ is a candidate to replace 4.11.

4.15. *Remark.* For every integer $n \geqq 1$ there exists a CW-complex X such that nil $\Omega X = n$. By 4.9, it suffices to take X in class $K(\pi, 1)$, where π is an abstract group with nil $\pi = n$. Simply connected CW-complexes with loop spaces of preassigned nilpotency class are also available. Thus, let X result by adding cells to $S^2 \vee S^2$ so as to kill its homotopy groups in dimensions $\geqq n + 2$. By [10], X will have nonvanishing $(n + 1)$-fold Whitehead products so that, according to 4.6, $n \leqq$ nil ΩX; also, by 4.12, nil $\Omega X \leqq n$. This example was kindly communicated to us by P. J. Hilton who used it in a slightly different situation; also, semisimplicial versions of 4.6 and 4.12 may be found in his papers [11] and [11a].

5. The co-nilpotency class of a suspension

Let K be a fixed commutative coefficient field whose unit element is denoted by 1. We consider singular homology vector spaces over K. For any space Y we introduce the direct sums

$$H_*(Y; K) = \textstyle\sum_{q\geqq 0} H_q(Y; K), \qquad H_+(Y; K) = \textstyle\sum_{q>0} H_q(Y; K).$$

All tensor products will be taken over K. The following natural isomorphisms, of which the second is given by the Künneth formula, will always be regarded as identifications

$$H_0(Y; K) \approx K \qquad \text{if} \quad Y \text{ is 0-connected,}$$

$$H_*(Y_1 \times \cdots \times Y_n, K) \approx H_*(Y_1; K) \otimes \cdots \otimes H_*(Y_n; K).$$

The n^{th} diagonal map $\triangle_n : Y \to Y^n$, with $\triangle_1 =$ identity and $\triangle_2 = \triangle$, induces a homomorphism

$$D_n : H_*(Y; K) \to H_*(Y; K) \otimes \cdots \otimes H_*(Y; K)$$

such that, whatever be the "place" of $D = D_2$,

(8) $$D_{n+1} = (\theta \otimes \cdots \otimes D \otimes \cdots \otimes \theta) \circ D_n ;$$

here θ is the identity map of $H = H_*(Y; K)$ and

$$\theta \otimes \cdots \otimes D \otimes \cdots \otimes \theta : H \otimes \cdots \otimes H \otimes \cdots \otimes H$$
$$\to H \otimes \cdots \otimes H \otimes H \otimes \cdots \otimes H.$$

For any $n \geqq 1$ we have the direct sum decomposition

$$H_*(Y; K) \otimes \cdots \otimes H_*(Y; K) = P_n + Z_n$$

in which, with $q_1 \cdots q_n$ denoting the ordinary product of integers,

$$P_n = \sum \{H_{q_1} \otimes \cdots \otimes H_{q_n} \mid q_i \geqq 0, q_1 \cdots q_n > 0\},$$
$$Z_n = \sum \{H_{q_1} \otimes \cdots \otimes H_{q_n} \mid q_i \geqq 0, q_1 \cdots q_n = 0\}.$$

There result homomorphisms

$$p_n : H_* \to H_* \otimes \cdots \otimes H_* \quad \text{and} \quad z_n : H_* \to H_* \otimes \cdots \otimes H_*$$

such that, for any $n \geqq 1$ and $a \in H_*$

(9) $$p_n(a) \in P_n, \qquad z_n(a) \in Z_n, \qquad D_n(a) = p_n(a) + z_n(a).$$

For any $n \geqq 1$, (8) implies that, whatever be the "place" of p_2

(10) $$p_{n+1} = (\theta \otimes \cdots \otimes p_2 \otimes \cdots \otimes \theta) \circ p_n .$$

Finally, an element $a \in H_*(Y; K)$ will be called "primitive" if it is homogeneous, of positive dimension, and if $p_2(a) = 0$.

Now let (X, x_0) be a 0-connected space. Suppose

(11) *$u \in H_+(X; K)$ is a homogeneous element and $n \geqq 1$ an integer such that $p_n(u) \neq 0$ and $p_{n+1}(u) = 0$.*

Our main purpose is to prove that (11) implies the nontriviality of the homology homomorphism induced by the composition

(12) $$X \xrightarrow{\triangle_n} X^n \xrightarrow{j_1 \times \cdots \times j_n} ({}^nX)^n \xrightarrow{e^n} (\Omega\Sigma({}^nX))^n \xrightarrow{\varphi_n} \Omega\Sigma({}^nX)$$

in which j_i, e, and φ_n are as in 2.17. We first prove the

5.1. LEMMA. *There exist primitive elements $u_{ri} \in H_*(X; K)$ such that $p_n(u) = \sum_r u_{r1} \otimes \cdots \otimes u_{rn} \in H_*(X; K) \otimes \cdots \otimes H_*(X; K)$.*

Proof. Let $H_i = H_+(X; K)$ and $\theta_i = \mathrm{id}: H_* \to H_*$. Since we are dealing with vector spaces, we have a direct sum decomposition

$$H_i = R_i + Q_i \qquad \text{where} \quad Q_i = H_i \cap \operatorname{Ker} p_2 .$$

As a result, for any k with $0 \leqq k \leqq n - 1$, we have the direct sum decomposition $T_{k+1} = S_k + T_k$ involving the n-fold tensor products

$$S_k = H_1 \otimes \cdots \otimes H_k \otimes R_{k+1} \otimes Q_{k+2} \otimes \cdots \otimes Q_n ,$$

$$T_k = H_1 \otimes \cdots \otimes H_k \otimes Q_{k+1} \otimes Q_{k+2} \otimes \cdots \otimes Q_n .$$

We have $p_n(u) \in H_1 \otimes \cdots \otimes H_n = T_n$. Suppose $p_n(u) \in T_{k+1}$ for some k with $0 \leqq k \leqq n - 1$. Then $p_n(u) = b + c$, where $b \in S_k$ and $c \in T_k$. Since $p_2 \mid R_{k+1}$ is monomorphic and $p_2(Q_{k+1}) = 0$, the map

$$\phi_k = \theta_1 \otimes \cdots \otimes \theta_k \otimes p_2 \otimes \theta_{k+2} \otimes \cdots \otimes \theta_n$$

is monomorphic on S_k and vanishes on T_k. By (10) and (11) we have

$$\phi_k(b) = \phi_k(b) + \phi_k(c) = \phi_k \circ p_n(u) = p_{n+1}(u) = 0$$

so that $b = 0$ and $p_n(u) \in T_k$. Thus, we finally obtain

$$p_n(u) \in T_0 = Q_1 \otimes \cdots \otimes Q_n .$$

The desired result now follows upon noticing that any $a \in Q_i$ is a finite sum of homogeneous elements a_q satisfying $\dim a_q = q > 0$ and $p_2(a_q) = 0$.

Our next step consists in deriving certain homological properties of $\Omega = \Omega\Sigma(^nX)$ and of φ_n. Since X is 0-connected, $\Sigma(^nX)$ is 1-connected and Ω is 0-connected; the identifications introduced at the beginning of the section are valid and will be used without further reference. Let $A = H_*(\Omega; K)$. Also, let

$$\mu:\Omega \times \Omega \to \Omega \quad \text{and} \quad \nu:\Omega \to \Omega$$

be the usual multiplication and inversion of loops which convert Ω into an H-space. The homomorphism

$$M:A \otimes A \to A$$

induced by μ converts the vector space A into the (associative) Pontrjagin algebra of Ω; its unit element is $1 \in H_0(\Omega; K)$. We abbreviate $M(a \otimes b)$ to ab. Let

$$N:A \to A$$

be the vector space homomorphism induced by ν.

The Cartesian product $\Omega \times \Omega$ also is an H-space in the obvious way. Its Pontrjagin algebra is naturally isomorphic to the skew tensor product $A \otimes A$ of the graded algebras A and A, in which the product is defined by setting

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{pq}(a_1 b_1 \otimes a_2 b_2)$$

for any homogeneous elements with $\dim b_1 = p$, $\dim a_2 = q$. Since $\triangle : \Omega \to \Omega \times \Omega$ is an H-homomorphism, $D: A \to A \otimes A$ is an algebra homomorphism. Notice that

$$D(a) = a \otimes 1 + 1 \otimes a \quad \text{if} \quad a \in A \quad \text{is primitive.} \tag{13}$$

5.2. LEMMA. *If $a \in A$ is primitive, then $N(a) = -a$.*

Proof. By (13) and since $N(1) = 1$, the composition

$$A \xrightarrow{D} A \otimes A \xrightarrow{\theta \otimes N} A \otimes A \xrightarrow{M} A$$

sends a into $a + N(a)$. By 1.1 (ii) this composition is trivial so that $a + N(a) = 0$.

Evidently, the basic commutator map φ equals the composition

$$\Omega \times \Omega \xrightarrow{\triangle \times \triangle} \Omega^2 \times \Omega^2 \to \Omega^2 \times \Omega^2 \xrightarrow{\theta^2 \times \nu^2} \Omega^2 \times \Omega^2 \xrightarrow{\mu \times \mu} \Omega \times \Omega \xrightarrow{\mu} \Omega$$

in which the second arrow sends $((\omega_1, \omega_2), (\omega_3, \omega_4))$ into the element $((\omega_1, \omega_3), (\omega_2, \omega_4))$. Therefore, the vector space homomorphism F induced by φ equals the composition

$$\begin{aligned} &A \otimes A \xrightarrow{D \otimes D} B \otimes B \to B \otimes B \\ &\xrightarrow{(\theta \otimes \theta) \otimes (N \otimes N)} B \otimes B \xrightarrow{M \otimes M} A \otimes A \xrightarrow{M} A \end{aligned} \tag{14}$$

in which B stands for $A \otimes A$ and the second arrow is defined by

$$(a_1 \otimes a_2) \otimes (a_3 \otimes a_4) \to (-1)^{pq}(a_1 \otimes a_3) \otimes (a_2 \otimes a_4)$$

for any homogeneous elements with $\dim a_3 = p$, $\dim a_2 = q$. Direct computation using 5.2, (13), and (14) now yields (compare with [17]):

5.3. LEMMA. *If a, $b \in A$ are primitive elements with* $\dim a = p$ *and* $\dim b = q$, *then $F(a \otimes b) = ab - (-1)^{pq}ba$.*

5.4. LEMMA. *If a, $b \in A$ are primitive, then $c = F(a \otimes b)$ also is a primitive element.*

Proof. Direct computation using 5.3, (13), and the fact that D is an algebra homomorphism yields $D(c) = c \otimes 1 + 1 \otimes c$.

It follows from 1.3 that the vector space homomorphism F_{m+1} induced by the commutator map φ_{m+1} of weight $m + 1$ equals the composition

$$A \otimes \cdots \otimes A \otimes A \xrightarrow{F_m \otimes \theta} A \otimes A \xrightarrow{F} A. \tag{15}$$

An induction argument using 5.4 and 5.3 now yields

5.5. LEMMA. *If $a_i \in A$ are primitive elements, then*

$$F_n(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n + \textstyle\sum_s \pm a_{s(1)} \cdots a_{s(n)},$$

where the summation subscript s runs through a set of permutations of $(1, \cdots, n)$ *which is entirely determined by n and does not contain the identity.*

5.6. LEMMA. *If* $a_i \in A$ *are homogeneous elements and if* $\dim a_k = 0$ *for some* k, *then* $F_n(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_n) = 0$.

Proof. Since F_{m+1} equals the composition (15), it obviously suffices to prove that $F(a \otimes b) = 0$ if, say, $\dim b = 0$. We may then also assume that $b = 1$. An easy computation now yields

$$F(a \otimes 1) = M \circ (\theta \otimes N) \circ D(a),$$

and the right side vanishes according to 1.1 (ii).

After this digression we revert to the consideration of (12). Let

$$J_i: H_*(X; K) \to H_*({}^nX; K) \quad \text{and} \quad E: H_*({}^nX; K) \to H_*(\Omega; K)$$

be the homomorphisms induced by the maps j_i and e, where $\Omega = \Omega\Sigma({}^nX)$. We identify $(j_1 \otimes \cdots \otimes j_n)_*$ with $J_1 \otimes \cdots \otimes J_n$ and $(e^n)_*$ with $E \otimes \cdots \otimes E$.

Let $u \in H_+(X; K)$ be given by (11). According to (9), there exist homogeneous elements $v_{ti} \in H_*(X; K)$ such that

$$z_n(u) = \textstyle\sum_t v_{t1} \otimes \cdots \otimes v_{tn}$$

and

$$\dim v_{tk} = 0 \qquad \text{for some} \quad k = k(t).$$

Clearly, $\dim E \circ J_k(v_{tk}) = 0$ so that, according to 5.6,

$$F_n \circ (E \otimes \cdots \otimes E) \circ (J_1 \otimes \cdots \otimes J_n) \circ z_n(u) = 0. \tag{16}$$

Let the primitive u_{ri} correspond to u according to 5.1 and let

$$c_{ri} = J_i(u_{ri}), \qquad a_{ri} = E(c_{ri}). \tag{17}$$

The naturality of the Künneth formula implies that every a_{ri} is primitive so that, by 5.1 and 5.5,

$$\begin{aligned} F_n \circ (E \otimes \cdots \otimes E) \circ (J_1 \otimes \cdots \otimes J_n) \circ p_n(u) \\ = \textstyle\sum_r w_r + \sum_r \sum_s w_{rs}, \end{aligned} \tag{18}$$

where, with the dot representing Pontrjagin multiplication in A,

$$w_r = a_{r1} \cdots a_{rn} \quad \text{and} \quad w_{rs} = \pm a_{rs(1)} \cdots a_{rs(n)}. \tag{19}$$

Now let $C = H_+({}^nX; K)$; introduce the direct sum

$$T(C) = \textstyle\sum_{q=0}^{\infty} C^{(q)}$$

in which

$$C^{(0)} = K \qquad \text{and} \qquad C^{(q)} = C \otimes \cdots \otimes C \quad \text{if} \quad q > 0.$$

The natural isomorphisms $C^{(p)} \otimes C^{(q)} \approx C^{(p+q)}$, where $p, q \geqq 0$, extend linearly to a homomorphism $T(C) \otimes T(C) \to T(C)$ which converts the vector space $T(C)$ into the (associative) tensor algebra of C; its unit element is $1 \in C^{(0)}$. We have the direct sum decomposition

$$C = C_1 + \cdots + C_n + C_{n+1}$$

in which $C_i = J_i(H_+(X; K))$ for $1 \leqq i \leqq n$; the presence of an extra term C_{n+1}, which is a direct summand as we work with vector spaces, is due to a possibly bad behavior of the base-point. As a result, we obtain the direct sum decomposition

$$(20) \quad T(C) = C^{(0)} + \textstyle\sum_{q=1}^{\infty} \sum \{C_{i_1} \otimes \cdots \otimes C_{i_q} \mid 1 \leqq i_k \leqq n+1, 1 \leqq k \leqq q\}.$$

Evidently, every J_i is a monomorphism; working with vector spaces implies that so is also $J_1 \otimes \cdots \otimes J_n$. We have $p_n(u) \neq 0$ and $\dim u_{ri} > 0$. Therefore, 5.1 and (17) yield

$$(21) \quad 0 \neq (J_1 \otimes \cdots \otimes J_n) \circ p_n(u) = \textstyle\sum_r c_{r1} \otimes \cdots \otimes c_{rn} \in C_1 \otimes \cdots \otimes C_n;$$

also, for any permutation s of $(1, \cdots, n)$, by (17) we have

$$(22) \qquad \textstyle\sum_r c_{rs(1)} \otimes \cdots \otimes c_{rs(n)} \in C_{s(1)} \otimes \cdots \otimes C_{s(n)}.$$

If s is not the identity, then the right-hand vector spaces in (22) and (21) are distinct direct summands in (20). Therefore, adding in $T(C)$ yields

$$(23) \quad \textstyle\sum_r c_{r1} \otimes \cdots \otimes c_{rn} + \sum_s \sum_r \pm c_{rs(1)} \otimes \cdots \otimes c_{rs(n)} \neq 0;$$

here, the summation subscripts r and s run precisely as in 5.1 and 5.5 respectively, and the signs are taken as in (19). After interchanging, as we may, summation on r and s in (23), we abbreviate its left member to d.

By the Bott-Samelson theorem ([3]; see also [13]) applied to the 0-connected space ${}^nX$, there is a natural algebra isomorphism

$$\phi: T(C) \approx A \qquad \text{such that} \qquad \phi \mid C^{(1)} = E \mid C.$$

The definition of multiplication in $T(C)$, (17), and (19) now yield

$$\phi(d) = \textstyle\sum_r w_r + \sum_r \sum_s w_{rs}.$$

Since ϕ is an isomorphism, (23) and (18) yield

$$(24) \qquad F_n \circ (E \otimes \cdots \otimes E) \circ (J_1 \otimes \cdots \otimes J_n) \circ p_n(u) \neq 0.$$

By (9), (16), and (24) we finally obtain

$$F_n \circ (E \otimes \cdots \otimes E) \circ (J_1 \otimes \cdots \otimes J_n) \circ D_n(u) \neq 0.$$

Thus, it follows from (11) that the homology homomorphism induced by (12) is nontrivial, and 2.17 now implies that the co-commutator map ψ_n of weight n in the H'-space ΣX fails to be nullhomotopic rel. base-point.

Recall now the

5.7. Definition. $\smile$-long X *is the least integer* $k \geqq 0$ *such that, for any commutative coefficient field, the cup product of any* $k + 1$ *singular cohomology* *classes of positive dimension vanishes; if no such integer exists, we put* $\smile$-long $X = \infty$.

Suppose $\smile$-long $X \geqq m \geqq 1$. Then, the definition of the cup product in terms of the diagonal map provides a commutative coefficient field K, an element $u \in H_+(X; K)$ and an integer $n \geqq m$ satisfying (11). Therefore,

5.8. THEOREM. *If X is a 0-connected space, then, for any base-point $x_0 \in X$,* $\smile$-long $X \leqq$ conil $\Sigma(X, x_0)$.

5.9. An upper bound for conil $\Sigma(X, x_0)$ will be given by Theorem 6.13 in the next section. Meanwhile, we show that for any $n \geqq 1$ a space X satisfying conil $\Sigma X = n$ exists. To this end, let X be the Cartesian product of n copies of a q-sphere, $q \geqq 1$. With Z_2 as coefficient field, there are n singular q-dimensional cohomology classes with nonvanishing cup product in X so that, by 5.8, we have $n \leqq$ conil ΣX. On the other hand, the Lusternik-Schnirelmann category of X equals $n + 1$ so that, by 6.13 and 6.8 in the next section, we have conil $\Sigma X \leqq n$.

6. Weak category and nilpotency of function spaces

We first recall [16; Hilfssatz 14] the following

6.1. DEFINITION. *The base-point $b \in Y$ is nondegenerate if there are a neighborhood $U \ni b$ which is contractible rel. b in Y, and a continuous function $u: Y \to I$ with $u(b) = 1$ and $u(Y - U) = 0$.*

The advantage of this definition lies in the fact that the property of having a nondegenerate base-point is a based homotopy type invariant [16; p. 333]. We list without proofs some properties of nondegenerate base-points.

6.2. *If the base-point $b \in Y$ is nondegenerate, then there exist a neighborhood $V \ni b$, a continuous function $v: Y \to I$, and a homotopy $\rho_t: Y \to Y$ such that $v(b) = 1$, $v(Y - V) = 0$, $\rho_0(y) = y$, and $\rho_1(\bar{V}) = \rho_t(b) = b$.*

6.3. *If the base-point $a \in X$ is nondegenerate and if the map*

$$f: (X, a) \to (Y, b)$$

is freely homotopic to the constant map $X \to b$, then f also is homotopic rel. a to the constant map.

6.4. *If the base-point $a \in X$ is nondegenerate and if Y is 0-connected, then any map $X \to Y$ is freely homotopic to a map $(X, a) \to (Y, b)$.*

6.5. *If the base-point $b \in Y$ is nondegenerate, then the base-point Ωb is nondegenerate in the loop space $\Omega(Y, b)$.*

Now let $n \geqq 1$. For any space (Y, b) let $T(Y, b; n)$ denote the subset of the Cartesian product Y^n which consists of all points $(y_1, \cdots, y_n)$ satisfying $y_i = b$ for some i. Let

$$Y^{(n)} \quad \text{and} \quad p: Y^n \to Y^{(n)}$$

denote the identification space and identification map resulting by pinching to a point the subset $T(Y, b; n)$ of Y^n. This construction is related to that introduced at the beginning of the fourth section, but we shall not need this fact here. We owe to Hilton the following

6.6. DEFINITION. *For any space* (X, a), w cat (X, a) *is the least integer* $n \geqq 1$ *such that the composition* $X \xrightarrow{\triangle} X^n \xrightarrow{p} X^{(n)}$ *is nullhomotopic rel.* a; *if no such integer exists*, w cat $(X, a) = \infty$.

As usual, $\triangle$ stands here for the diagonal map. Clearly

6.7. *If* (X, a) *and* (Y, b) *have the same based homotopy type, then*

$$\text{w cat } (X, a) = \text{w cat } (Y, b).$$

Recall that the Lusternik-Schnirelmann category cat X is the least integer $n \geqq 1$ such that X may be covered by n open subsets which are contractible in X; if no such integer exists, cat $X = \infty$.

6.8. PROPOSITION. *If X is a 0-connected normal space with nondegenerate base-point $a \in X$, then* w cat $(X, a) \leqq$ cat X.

Proof. Suppose cat $X = n$, and let X be covered by n open subsets U_i, each of which is contractible in X under a homotopy $h_i: U_i \times I \to X$ satisfying $h_i(x, 0) = x$, $h_i(x, 1) = a_i$. Since X is 0-connected, we may assume that $a_i = a$ for all i. The normality of X yields closed subsets A_i of X, open subsets V_i of X, and continuous functions $f_i: X \to I$ such that

$$X = A_1 \cup \cdots \cup A_n, \qquad A_i \subset V_i \subset \bar{V}_i \subset U_i,$$

$$f_i(A_i) = 1, \qquad f_i(X - V_i) = 0.$$

For every i, define a homotopy $k_i: X \times I \to X$ by

$$\begin{aligned} k_i(x, t) &= x && \text{if } x \in X - \bar{V}_i, \\ &= h_i(x, tf_i(x)) && \text{if } x \in U_i. \end{aligned}$$

Let $k(x, t) = (k_1(x, t), \cdots, k_n(x, t)) \in X^n$; then $k(x, 0) = \triangle(x)$ and since every x belongs to some A_i, $k(x, 1) \in T(X, a; n)$. The composition $p \circ \triangle$ is now freely homotopic to the constant map $X \to p(a, \cdots, a)$ under the homotopy $p \circ k$. Since $a \in X$ is nondegenerate, application of 6.3 yields a homotopy rel. a connecting the map $p \circ \triangle$ and the constant map, so that w cat $(X, a) \leqq n$.

6.9. LEMMA. *If (G, e) is an H-space with nondegenerate base-point, then, for any $n \geqq 1$, there is a homotopy $h_t: G^n \to G$ such that $h_0 = \varphi_n$ and*

$$h_1(T(G\ e; n)) = h_t(e, \cdots, e) = e.$$

Proof. Let φ denote the basic commutator map in (G, e). By 1.5, there is a homotopy $\phi_t: G \vee G \to G$ such that

$$\phi_0 = \varphi \mid G \vee G \quad \text{and} \quad \phi_1(G \vee G) = \phi_t(e, e) = e.$$

Let the neighborhood $V \ni e$, the continuous function $v: G \to I$, and the homotopy $\rho_t: G \to G$ be given by 6.2. Select a continuous function $r: I \times I \to I$ such that $r(0, 0) = 0$ and $r(s, 1) = r(1, t) = 1$. Let $d(x, y) = r(v(x), v(y))$, $W = G \times V \cup V \times G$ and $M = G \times G - W$. The map $k_t: G \times G \to G$ defined by

$$\begin{aligned} k_t(x, y) &= \varphi(\rho_{2t}(x), \rho_{2t}(y)) && \text{for } 0 \leqq t \leqq \tfrac{1}{2}, \quad (x, y) \in G \times G, \\ &= \varphi(\rho_1(x), \rho_1(y)) && \text{for } \tfrac{1}{2} \leqq t \leqq 1, \quad (x, y) \in M, \\ &= \phi_{(2t-1)d(x,y)}(\rho_1(x), \rho_1(y)) && \text{for } \tfrac{1}{2} \leqq t \leqq 1, \quad (x, y) \in \bar{W}, \end{aligned}$$

satisfies $k_0 = \varphi$ and $k_1(G \vee G) = k_t(e, e) = e$. Now, if $n = 1$, we put $h_t =$ identity. Assume the homotopy h_t "of weight $n \geqq 1$" has been defined. The homotopy "of weight $n + 1$" is then given by $k_t(h_t(x_1, \cdots, x_n), x_{n+1})$.

Now let $\mathcal{G}$ denote the class of all H-spaces with nondegenerate base-point.

6.10. THEOREM. *If (X, a) is a Hausdorff space with nondegenerate base-point, then* $\sup \{\text{nil } (G, e)^{(X,a)} \mid (G, e) \in \mathcal{G}\} \leqq \text{w cat } (X, a) - 1$.

Proof. Assume that $\text{w cat } (X, a) = n \geqq 1$. Let the neighborhood $V \ni a$ and the homotopy $\rho_t: X \to X$ be given by 6.2. With $X_i = X$ and $V_i = V$, the set

$$W = \textstyle\bigcup_{i=1}^{n} X_1 \times \cdots \times X_{i-1} \times V_i \times X_{i+1} \times \cdots \times X_n$$

is a neighborhood of $T(X, a; n)$ in X^n and the homotopy

$$\eta_t = \rho_t \times \cdots \times \rho_t: (X^n, (a, \cdots, a)) \to (X^n, (a, \cdots, a))$$

satisfies $\eta_0 =$ identity and $\eta_1(W) \subset T(X, a; n)$. Introduce the space $X^{(n)}$ and the identification map $p: X^n \to X^{(n)}$; let $b = p(a, \cdots, a)$.

Let $(G, e) \in \mathcal{G}$. For every map $f: (X^n, T(X, a; n)) \to (G, e)$ there is a unique continuous map $f': (X^{(n)}, b) \to (G, e)$ satisfying $f' \circ p = f$. Setting $F(f) = f'$ defines a continuous map

$$F: (G, e)^{(X^n, W)} \to (G, e)^{(X^{(n)}, b)}; \tag{25}$$

for, if $F(f) \in (C, U)$, where $C \subset X^{(n)}$ is compact and $U \subset G$ is open, then $p^{-1}(C) - W$ is a compact subset of X^n, so that

$$\Lambda = (p^{-1}(C) - W, U) \cap (G, e)^{(X^n, W)}$$

is a neighborhood of f in the domain of F and $F(\Lambda) \subset (C, U)$.

Let $\triangle: X \to X^n$ be the diagonal map. Since $\text{w cat } (X, a) = n$, there is a homotopy $\xi_t: X \to X^{(n)}$ such that $\xi_0 = p \circ \triangle$ and $\xi_1(X) = \xi_t(a) = b$. Let

$$\phi_n: (G, e)^{(X,a)} \times \cdots \times (G, e)^{(X,a)} \to (G, e)^{(X,a)}$$

be the commutator map of weight n in the H-space $(G, e)^{(X,a)}$. Then, for any $f_i \in (G, e)^{(X,a)}$, $\phi_n(f_1, \cdots, f_n)$ equals the composition

$$X \xrightarrow{\triangle} X^n \xrightarrow{f_1 \times \cdots \times f_n} G^n \xrightarrow{\varphi_n} G,$$

where φ_n is the commutator map of weight n in the H-space (G, e). By 6.9, there is a homotopy $h_t : G^n \to G$ such that

$$h_0 = \varphi_n \quad \text{and} \quad h_1(T(G, e; n)) = h_t(e, \cdots, e) = e.$$

Define

$$k_t : (G, e)^{(X,a)} \times \cdots \times (G, e)^{(X,a)} \to (G, e)^{(X,a)}$$

by

$$\begin{aligned} k_t(f_1, \cdots, f_n) &= h_{2t} \circ (f_1 \times \cdots \times f_n) \circ \eta_{2t} \circ \triangle && \text{if } 0 \leqq t \leqq \tfrac{1}{2}, \\ &= F(h_1 \circ (f_1 \times \cdots \times f_n) \circ \eta_1) \circ \xi_{2t-1} && \text{if } \tfrac{1}{2} \leqq t \leqq 1. \end{aligned}$$

That k_t actually is a homotopy, i.e., that it depends continuously on $(f_1, \cdots, f_n, t)$, is easily checked by expanding it as a composition of maps and homotopies of function spaces, each of which is continuous by 2.2, 2.1, and (25). Moreover, $k_0 = \phi_n$, and, with ε denoting the constant map $X \to e$, $k_1(f_1, \cdots, f_n) = k_t(\varepsilon, \cdots, \varepsilon) = \varepsilon$. Thus, nil $(G, e)^{(X,a)} \leqq n - 1$.

6.11. Theorem. *If (X, a) is an arbitrary space with nondegenerate base-point, then* conil $\Sigma(X, a) \leqq \sup \{\text{nil } \pi(X, a; G, e) \mid (G, e) \in \mathcal{G}\}$.

Proof. Since X has a nondegenerate base-point, so does ${}^mX$ for any $m \geqq 1$; it follows from [16; Satz 17 and (33)] that $\Sigma({}^mX)$ also has a nondegenerate base-point, and 6.5 now implies that $\Omega\Sigma({}^mX)$ belongs to $\mathcal{G}$ for any $m \geqq 1$. The result now follows from 2.14.

6.12. Corollary. *If (X, a) is a 0-connected normal Hausdorff space with nondegenerate base-point, then, with (G, e) ranging over $\mathcal{G}$,*

$$\smile\text{-long } X \leqq \sup \text{nil } \pi(X, a; G, e) \leqq \sup \text{nil } (G, e)^{(X,a)} \leqq \text{cat } X - 1.$$

Proof. The first inequality follows from 5.8 and 6.11, the second is obvious, and the third follows from 6.10 and 6.8.

As another consequence, we obtain the promised upper bound for the conilpotency class of a suspension:

6.13. Theorem. *If (X, a) is a Hausdorff space with nondegenerate base-point, then* conil $\Sigma(X, a) \leqq$ w cat $(X, a) - 1$.

Now let $\mathcal{G}_0$ denote the subclass of $\mathcal{G}$ consisting of all 0-connected H-spaces in $\mathcal{G}$. Suppose (X, a) is a Hausdorff space with nondegenerate base-point, and let $(G, e) \in \mathcal{G}_0$. Then, it follows from 6.3 and 6.4 that the inclusion map $(G, e)^{(X,a)} \to G^X$ induces an isomorphism

$$\pi(X, a; G, e) \approx \pi(X; G). \tag{26}$$

Therefore, by 6.10, we have

6.14. PROPOSITION. *If* (X, a) *is a Hausdorff space with nondegenerate base-point, then* $\sup\{\text{nil}\, \pi(X; G) \mid (G, e) \in \mathcal{G}_0\} \leqq \text{w cat}\,(X, a) - 1$.

Furthermore, if X is 0-connected, so is also the H-space $\Omega\Sigma(^m X)$, and, as shown in the proof of 6.11, its base-point is nondegenerate provided so is $a \in X$; by 2.14 and (26) we therefore obtain

6.15. PROPOSITION. *If* (X, a) *is a* 0-*connected space with nondegenerate base-point, then* $\text{conil}\, \Sigma(X, a) \leqq \sup\{\text{nil}\, \pi(X; G) \mid (G, e) \in \mathcal{G}_0\}$.

Our last result concerns Whitehead products in function spaces.

6.16. THEOREM. *Let* (X, a) *and* (Y, b) *have nondegenerate base-points. If* X *is a Hausdorff space, then the relation*

$$\text{W-long}\,((Y, b)^{(X,a)}, b^X) \leqq \text{w cat}\,(X, a) - 1$$

holds in each of the following three cases: (i) X *is locally compact*, (ii) X *is a CW-complex*, (iii) X *satisfies the first countability axiom.*

Proof. By 4.6 we have

$$\text{W-long}\,((Y, b)^{(X,a)}, b^X) \leqq \text{nil}\,\Omega((Y, b)^{(X,a)}, b^X); \tag{27}$$

by 6.10 and 6.5 we have

$$\text{nil}\,((\Omega(Y, b), \Omega b)^{(X,a)}, (\Omega b)^X) \leqq \text{w cat}\,(X, a) - 1. \tag{28}$$

According to [14; Theorem 6], in each of the three cases there is an H-structure-preserving homeomorphism

$$\Omega((Y, b)^{(X,a)}, b^X) \approx (\Omega(Y, b), \Omega b)^{(X,a)},$$

and the right-hand member of (27) now equals the left member of (28).

We conclude by giving two examples related to 6.8 (see [1]).

6.17. Let Y be the complex obtained by removing an open 3-cell from a Poincaré space, i.e., a nonsimply connected closed 3-manifold which is a homology sphere. Since Y is not contractible, by [8; Theorem 1.1] we have $\text{cat}\, X \geqq n + 1$ for $X = Y^n$. Nevertheless, $X^{(2)}$ is simply connected and acyclic, whence contractible, so that $\text{w cat}\, X = 2$.

6.18. Let $Y = S^3 \vee S^3$, and let $i_1, i_2 : S^3 \to Y$ denote the left and right inclusion maps. Let X result by attaching to Y an 8-cell with characteristic map in the class $[i_1, [i_1, i_2]] \in \pi_7(Y)$, where $[f, g]$ denotes the obvious Whitehead product. Then, X satisfies $\pi_1(X) = 0$, $\text{w cat}\, X = 2$, $\text{cat}\, X = 3$.

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