FOURIER SERIES OF AUTOMORPHIC FORMS OF NONNEGATIVE DIMENSION[1]

BY

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I. Introduction

In [2] Rademacher introduced a method for recapturing the functional equation $J(-1/\tau) = J(\tau)$ directly from the Fourier series expansion for $J(\tau)$, the well-known modular invariant. In this paper we extend his method to entire forms of positive even integral dimension for the modular group and for the groups $G(\sqrt{2})$, $G(\sqrt{3})$, $G(2)$, where $G(\sqrt{l})$ is generated by $\tau' = \tau + l^{1/2}$, $\tau' = -1/\tau$ $(l = 2, 3)$, and $G(2)$ is the principal congruence subgroup of level two, of the modular group. This group is generated by $\tau' = \tau + 2$, $\tau' = \tau/(2\tau + 1)$.

We start here with the Fourier series given by Rademacher and Zuckerman [3], Raleigh [4], and Simons [6]. In [3] Fourier expansions are given for entire modular forms (i.e., modular forms regular in the upper half plane) of positive dimension. While every entire modular form of positive dimension has a Fourier series of the type given in [3], the converse is not true. That is, not every function defined by such a Fourier series is a modular form. However, it is reasonable to expect "decent" behavior under modular substitutions for all such Fourier series. We show that this is indeed true in a certain special case described below. Using these results it is a simple matter to construct modular forms of positive even integral dimension by means of their Fourier series.

Similar results will then be obtained for the groups $G(\sqrt{2})$, $G(\sqrt{3})$, and $G(2)$.

The result of Rademacher and Zuckerman [3] is as follows.

THEOREM (1.01). *Let $F(\tau)$ be a modular form of dimension $r > 0$; that is,*

(a) $F((a\tau + b)/(c\tau + d)) = \varepsilon(a, b, c, d)\cdot(-i(c\tau + d))^{-r}\cdot F(\tau)$, *where a, b, c, d are integers with $ad - bc = 1$ and $c > 0$, ε does not depend on τ, $|\varepsilon| = 1$, and we choose $|\arg(-i(c\tau + d))| < \pi/2$;*

(b) $F(\tau + 1) = \varepsilon(1, 1, 0, 1)(-i)^{-r}\cdot F(\tau) = e^{2\pi i\alpha}F(\tau)$, $0 \leqq \alpha < 1$;

(c) *the Fourier expansion of* $\exp(-2\pi i\alpha\tau)\cdot F(\tau)$ *contains only a finite number of terms with negative exponents.*

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Assume, in addition, that $F(\tau)$ *is analytic in the upper half plane. Then the Fourier expansion of* $\exp(-2\pi i\alpha\tau)\cdot F(\tau)$ *is of the form*

$$\exp(-2\pi i\alpha\tau)\cdot F(\tau) = \sum_{m=-\mu}^{\infty} a_m \exp(2\pi i m\tau),$$

$$(1.02)\qquad a_m = 2\pi \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{k=1}^{\infty} k^{-1} A_{k,\nu}(m)\cdot\left(\frac{\nu-\alpha}{m+\alpha}\right)^{(r+1)/2} \cdot I_{r+1}\left\{\frac{4\pi}{k}(\nu-\alpha)^{1/2}(m+\alpha)^{1/2}\right\},\qquad m \geqq 0,$$

where I_n *is the modified Bessel function of the first kind and*

$$A_{k,\nu}(m) = \sum_{\substack{0\leq h<k\\(h,k)=1}} \varepsilon\left(h', -\frac{hh'+1}{k}, k, -h\right)^{-1} \cdot \exp\left[\frac{-2\pi i}{k}((\nu-\alpha)h' + (m+\alpha)h)\right],$$

with $hh' \equiv -1 \pmod{k}$.

The modular group is the set of transformations $\tau' = (a\tau+b)/(c\tau+d)$ such that a, b, c, d are integers and $ad - bc = 1$. It is well known that the full group is generated by $\tau' = \tau + 1$ and $\tau' = -1/\tau$. It is easily verified that in the definition of a modular form, conditions (a) and (b) can be replaced by

$$(1.03)\qquad \begin{aligned}&(a)\quad F(-1/\tau) = \varepsilon(0, -1, 1, 0)\cdot(-i\tau)^{-r}\cdot F(\tau),\\ &(b)\quad F(\tau+1) = \varepsilon(1,1,0,1)(-i)^{-r}F(\tau) = e^{2\pi i\alpha}F(\tau), 0 \leqq \alpha < 1.\end{aligned}$$

In this paper we shall treat only the case $\alpha = 0$. In this case it can be shown that in order to obtain modular forms, we must assume that r is a nonnegative *even integer*, and that, in addition, $\varepsilon(a, b, c, d) = (-1)^{r/2}$ for all transformations of the modular group. (For a proof see [3], pp. 443–445.) Under these restrictions the conditions (1.03) for a modular form become

$$(1.04)\qquad \begin{aligned}&(a)\quad F(-1/\tau) = \tau^{-r}F(\tau),\\ &(b)\quad F(\tau+1) = F(\tau),\end{aligned}$$

and the Fourier series expansion (1.02) reduces to

$$F(\tau) = \sum_{m=-\mu}^{\infty} a_m \exp(2\pi i m\tau),$$

$$(1.05)\qquad \begin{aligned} a_m &= (-1)^{r/2}\sum_{\nu=1}^{\mu} a_{-\nu}\sum_{k=1}^{\infty} k^{-1}A_{k,\nu}(m)(\nu/m)^{(r+1)/2} I_{r+1}\left\{\frac{4\pi}{k}(m\nu)^{1/2}\right\}\\ &= (-1)^{r/2}\sum_{\nu=1}^{\mu} a_{-\nu}a_m(\nu), \quad\text{with}\end{aligned}$$

$$A_{k,\nu}(m) = \sum_{h(k)}{}' \exp\left[\frac{-2\pi i}{k}(\nu h' + mh)\right],\qquad \text{for } m \geqq 0.$$

In (1.05) we have written $\sum_{h(k)}'$ in place of $\sum_{\substack{0\leq h<k\\(h,k)=1}}'$. This practice will be followed throughout the paper. Also we will write $k \equiv a(b)$ in place of $k \equiv a \pmod b$.

The basic result of this paper is the following theorem.

THEOREM (1.06). *Let $F(\tau)$ be defined by the Fourier series (1.05), with r a nonnegative even integer. Then, in $\mathfrak{I}(\tau) > 0$, $F(\tau)$ is regular and satisfies the functional equation*

$$F(\tau) - \tau^r F(-1/\tau) = p(\tau), \tag{1.07}$$

where $p(\tau)$ is a polynomial in τ of degree at most r.

Remark. If $p(\tau) \equiv 0$, then by (1.04) and the fact that $F(\tau + 1) = F(\tau)$, $F(\tau)$ is a modular form of dimension r. We will later use Theorem (1.06) to construct modular forms. This theorem will be proved in Section III, and similar theorems for the groups $G(\sqrt{2})$, $G(\sqrt{3})$, $G(2)$ will be proved in Sections IV and V.

II. The Rademacher Lemma

In a lemma in [2], Rademacher rearranges the terms of a certain conditionally convergent double series. In this section we shall state and prove the several variations of Rademacher's Lemma that we require in our applications of the method of [2].

LEMMA (2.01). *Suppose $\tau = i\beta$ ($\beta > 0$), r is a nonnegative real number, and a, b, c, and ν are positive integers. Then*

$$\begin{aligned}\sum_{\substack{k=1\\k\equiv a(b)}}^{\infty} \lim_{N\to\infty} \sum_{\substack{|m|\leq N\\(m,k)=1}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} \\ = \lim_{K\to\infty} \sum_{\substack{k=1\\k\equiv a(b)}}^{cK} \sum_{\substack{|m|\leq K\\(m,k)=1}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)}.\end{aligned} \tag{2.02}$$

LEMMA (2.03). *Let τ, β, ν, and r be as in (2.01). Let a and b be positive integers such that $(a, b) = 1$. Then*

$$\begin{aligned}\sum_{\substack{k=1\\k\equiv a(b)}}^{\infty} \lim_{N\to\infty} \sum_{\substack{m\leq N\\(m,k)=1\\m\equiv 0(b)}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} \\ = \lim_{K\to\infty} \sum_{\substack{k=1\\k\equiv a(b)}}^{K} \sum_{\substack{|m|\leq K\\(m,k)=1\\m\equiv 0(b)}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)}.\end{aligned} \tag{2.04}$$

Remarks. In the proofs of these lemmas, the estimate [5]

$$A_{k,\nu}(m) = O(k^{2/3+\varepsilon}(k, \nu)^{1/3}), \tag{2.05}$$

where $A_{k,\nu}(m)$ is defined as in (1.05), is used in a very strong way when

$r = 0$. Actually by using (2.05), the results hold for any $r > -\frac{1}{3}$. If we use the improved estimate, $A_{k,\nu}(m) = O(k^{1/2+\varepsilon})$, due to Weil, we can prove the lemmas for any $r > -\frac{1}{2}$. For $r > 0$, the trivial estimate $A_{k,\nu}(m) = O(k)$, is enough.

The proof of (2.01) is exactly as given by Rademacher in [2], for the case $r = 0, a = b = c = \nu = 1$, and will not be included here. We should mention however, that Rademacher's proof holds for $\mathcal{I}(\tau) > 0$, not merely for purely imaginary τ. The proof of (2.03) requires some changes and is given below.

Proof of (2.03). We first show the convergence of the left-hand side of (2.04).

$$\sum_{\substack{m=-N \\ (m,k)=1 \\ m\equiv 0(b)}}^{N} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)}$$

$$= k^{-1-r} \sum_{h(k)}{}' \exp(-2\pi i h'\nu/k) \sum_{\substack{n \\ |uh+bkn|\leq N}} (k\tau - uh - bkn)^{-1},$$

where u is an integer uniquely determined mod bk satisfying $u \equiv 1(k)$, $u \equiv 0(b)$. Such a u can be found since $k \equiv a(b)$ and $(a, b) = 1$, and hence $(k, b) = 1$. Now $(u, k) = 1$, and this, together with $(h, k) = 1$ yields $(uh + bkn, k) = 1$. This justifies the replacement $m = uh + bkn$ above. Therefore,

$$\lim_{N\to\infty} \sum_{\substack{m=-N \\ (m,k)=1 \\ m\equiv 0(b)}}^{N} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)}$$

$$= b^{-1}k^{-r-2} \sum_{h(k)}{}' \exp(-2\pi i h'\nu/k) \cdot \lim_{N\to\infty} \sum_{\substack{n \\ |uk+bkn|\leq N}} \left(\frac{\tau}{b} - \frac{u}{b}\frac{h}{k} - n\right)^{-1}$$

$$= b^{-1}k^{-r-2} \sum_{h(k)}{}' \exp(-2\pi i h'\nu/k) \cdot 2\pi i \left[\frac{1}{2} - \left\{1 - \exp\left(2\pi i \frac{\tau}{b} - 2\pi i \frac{u}{b}\frac{h}{k}\right)\right\}^{-1}\right]$$

$$= \frac{\pi i}{bk^{r+2}} \sum_{\substack{d|\nu \\ d|k}} \mu\left(\frac{k}{d}\right) \cdot d - \frac{2\pi i}{bk^{r+2}} \sum_{h(k)}{}' \exp(-2\pi i h'\nu/k)$$
$$\cdot \sum_{p=0}^{\infty} \exp\left[2\pi i p \left(\frac{\tau}{b} - \frac{u}{b} \cdot \frac{h}{k}\right)\right]$$

$$= \frac{\pi i}{bk^{r+2}} \sum_{\substack{d|\nu \\ d|k}} \mu\left(\frac{k}{d}\right) \cdot d - \frac{2\pi i}{bk^{r+2}} \sum_{p=0}^{\infty} \exp(2\pi i p\tau/b)$$
$$\cdot \sum_{h(k)}{}' \exp\left[-2\pi i \left(h'\nu + \frac{u}{b} ph\right) \Big/ k\right].$$

Now, since u/b is an integer, it follows from (2.05) that

$$\sum_{h(k)}{}' \exp\left[-2\pi i \left(h'\nu + \frac{u}{b} ph\right) \Big/ k\right] = O(k^{2/3+\varepsilon}).$$

Consequently,

$$\sum_{p=0}^{\infty} \exp(2\pi i p\tau/b) \sum_{h(k)}' \exp\left[-2\pi i\left(h'\nu + \frac{u}{b}\,ph\right)\Big/ k\right]$$
$$= O\left[k^{2/3+\varepsilon}\left\{1 - \exp\left(\frac{-2\pi\beta}{b}\right)\right\}^{-1}\right].$$

Therefore,

$$\lim_{N\to\infty} \sum_{\substack{m=-N \\ (m,k)=1 \\ m\equiv 0\,(b)}}^{N} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} = O[k^{-4/3-r+\varepsilon}\{1 - \exp(-2\pi\beta/b)\}^{-1}],$$

so that the left-hand side of (2.04) converges.

We can now state the lemma as follows:

$$\lim_{k\to\infty} \sum_{\substack{k=1 \\ k\equiv a\,(b)}}^{K} \lim_{N\to\infty} \sum_{\substack{K<|m|\leq N \\ (m,k)=1 \\ m\equiv 0\,(b)}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} = 0. \tag{2.06}$$

Let

$$T_k(K) = \lim_{N\to\infty} T_k(K, N) = \lim_{N\to\infty} \sum_{\substack{K<|m|\leq N \\ (m,k)=1 \\ m\equiv 0\,(b)}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)}$$
$$= b^{-1} \lim_{N\to\infty} \sum_{\substack{K/b<|m|\leq N/b \\ (m,k)=1}} \frac{\exp[2\pi i(-bm)'\nu/k]}{k^{1+r}(k\tau/b - m)}.$$

Since $(b, k) = 1$, we can find an integer δ_k, uniquely determined mod k such that $b\cdot\delta_k \equiv 1(k)$. Of course $(\delta_k, k) = 1$. It follows that $(bm)' = \delta_k m'$, since $(bm)\delta_k m' \equiv mm' \equiv -1(k)$. Hence,

$$T_k(K, N) = b^{-1} \sum_{\substack{K/b<|m|\leq N/b \\ (m,k)=1}} \frac{\exp(-2\pi i\delta_k m'\nu/k)}{k^{1+r}(k\tau/b - m)}.$$

Define $g(m)$ by

$$g(m) = \exp(-2\pi i\delta_k m'\nu/k) \quad \text{if } (m, k) = 1,$$
$$= 0 \qquad\qquad \text{otherwise.}$$

We can write $g(m) = \sum_{j=1}^{k} B_{j,k} \exp(2\pi i jm/k)$ with

$$B_{j,k} = k^{-1} \sum_{\substack{m=1 \\ (m,k)=1}}^{k} \exp[-2\pi i(m'\delta_k\nu + jm)/k].$$

Since $(\delta_k, k) = 1$, we have from (2.05) that

$$B_{j,k} = O(k^{-1/3+\varepsilon}). \tag{2.07}$$

For the case $j = k$,

$$B_{j,k} = k^{-1} \sum_{m=1}^{k} \exp(-2\pi i m'\delta_k\nu/k) = k^{-1} \sum_{\substack{d|k \\ d|\nu\delta_k}} \mu\left(\frac{k}{d}\right)\cdot d = k^{-1} \sum_{\substack{d|k \\ d|\nu}} \mu\left(\frac{k}{d}\right)\cdot d.$$

Therefore, if we write $[K/b]$ for the greatest integer $\leqq K/b$,

$$T_k(K, N) = b^{-1} \sum_{K/b<|m|\leqq N/b} \sum_{j=1}^{k} B_{j,k} \frac{\exp(2\pi ijm/k)}{k^{1+r}(k\tau/b - m)},$$

and

$$\begin{aligned} T_k(K) &= \lim_{N\to\infty} T_k(K, N) = \frac{1}{bk^{1+r}} \sum_{j=1}^{k-1} B_{j,k} \sum_{m=[K/b]+1}^{\infty} \frac{\exp(2\pi ijm/k)}{(k\tau/b - m)} \\ &\quad + \frac{1}{bk^{1+r}} \sum_{j=1}^{k-1} B_{j,k} \sum_{m=[K/b]+1}^{\infty} \frac{\exp(-2\pi ijm/k)}{(k\tau/b + m)} \\ &\quad + \frac{1}{bk^{2+r}} \left\{ \sum_{\substack{d|k \\ d|\nu}} \mu\left(\frac{k}{d}\right) \cdot d \right\} \cdot \sum_{m=[K/b]+1}^{\infty} \left(\frac{1}{k\tau/b - m} + \frac{1}{k\tau/b + m} \right) \\ &= S_1 + S_2 + S_3. \end{aligned} \tag{2.08}$$

Now

$$\begin{aligned} S_3 &= b^{-1}k^{-r-2} \cdot \left\{ \sum_{\substack{d|k \\ d|\nu}} \mu\left(\frac{k}{d}\right) \cdot d \right\} \cdot \sum_{m=[K/b]+1} \left(\frac{1}{i\beta k/b - m} + \frac{1}{i\beta k/b + m} \right) \\ &= b^{-1}k^{-r-2} \left\{ \sum_{\substack{d|k \\ d|\nu}} \mu\left(\frac{k}{d}\right) \cdot d \right\} \cdot \sum_{m=[K/b]+1}^{\infty} \frac{2i\beta k/b}{-\beta^2k^2/b^2 - m^2}. \end{aligned}$$

Therefore, if $K > 3b$,

$$|S_3| < \frac{\nu^2}{b^2k^{1+r}} \sum_{m=[K/b]+1}^{\infty} \frac{2\beta}{m^2} < \frac{\nu^2}{b^2k^{1+r}} \int_{[K/b]}^{\infty} \frac{2\beta\, dx}{x^2} < \frac{3\nu^2\beta}{bk^{1+r}} K^{-1}.$$

Hence,

$$S_3 = O(1/k^{1+r}K). \tag{2.09}$$

In order to handle S_1, put

$$\begin{aligned} E_m &= \sum_{p=[K/b]+1}^{m} \exp(2\pi ijp/k) \\ &= \frac{\exp[2\pi ij(m + \frac{1}{2})/k] - \exp[2\pi ij([K/b] + \frac{1}{2})/k]}{\exp(\pi ij/k) - \exp(-\pi ij/k)}. \end{aligned}$$

Now

$$\begin{aligned} |E_m| &\leqq \frac{2}{|\exp(\pi ij/k) - \exp(-\pi ij/k)|} = \frac{1}{\sin(\pi j/k)} \\ &\leqq \frac{1}{\min\{2j/k, 2(k-j)/k\}} = \max\left\{\frac{k}{2j}, \frac{k}{2(k-j)}\right\} \leqq \frac{k}{2}\left(\frac{1}{j} + \frac{1}{k-j}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{m=[K/b]+1}^{\infty} \frac{\exp(2\pi ijm/k)}{(k\tau/b - m)} &= \sum_{m=[K/b]+1}^{\infty} \frac{(E_m - E_{m-1})}{ki\beta/b - m} \\ &= \sum_{m=[K/b]+1}^{\infty} E_m \cdot \left(\frac{1}{ki\beta/b - m} - \frac{1}{ki\beta/b - m - 1} \right) \end{aligned}$$

and

$$\left| \sum_{m=[K/b]+1}^{\infty} \frac{\exp(2\pi ijm/k)}{ki\beta/b - m} \right| \leqq \frac{k}{2}\left(\frac{1}{j} + \frac{1}{k-j}\right) \sum_{m=[K/b]+1}^{\infty} \frac{1}{m^2}$$

$$< \frac{k}{2}\left(\frac{1}{j} + \frac{1}{k-j}\right) [K/b]^{-1}.$$

It follows from this and (2.07) that

$$S_1 = O\left(k^{-1-r} \sum_{j=1}^{k-1} k^{-1/3+\varepsilon} \frac{k}{2}\left(\frac{1}{j} + \frac{1}{k-j}\right) K^{-1}\right),$$

and therefore

$$\text{(2.10)} \qquad S_1 = O\left(\frac{k^{-1/3+\varepsilon}}{k^r} K^{-1} \log k\right).$$

A similar estimate holds for S_2.

Using (2.08), (2.09), and (2.10) we get $T_k(K) = O(k^{-1/3-r+\varepsilon}K^{-1}\log k)$. Hence

$$\sum_{\substack{k=1 \\ k\equiv a(b)}}^{K} T_k(K) = O\left\{K^{-1} \sum_{\substack{k=1 \\ k\equiv a(b)}}^{K} (\log k)\cdot k^{-1/3-r+\varepsilon}\right\}$$

$$= O\{K^{-1}(1 + K^{2/3-r+\varepsilon}) \log K\} = O(K^{-\min(1/3+r+\varepsilon,1)} \cdot \log K).$$

Therefore (2.06) follows, and the lemma is proved.

III. The modular group

1. In this section we prove Theorem (1.06). We begin with

PROPOSITION (3.01). (a) *If $a_n(\nu)$ is defined as in* (1.05), *then as $n \to +\infty$*

$$a_n(\nu) \sim \nu^{r/2+1/4} \frac{\exp\{4\pi(n\nu)^{1/2}\}}{2^{1/2}n^{r/2+3/4}}.$$

(b) *If $|z| < 1$ and r is a nonnegative integer, then*

$$\sum_{m=1}^{\infty} z^m \sum_{p=0}^{\infty} \left\{\left(\frac{4\pi^2 m\nu}{k^2}\right)^p \Big/ p!\,(p+r+1)!\right\}$$

is absolutely convergent.

Proof. (a). Using (2.05), we have

$$\left| a_n(\nu) - \frac{2\pi\nu^{(r+1)/2}}{n^{(r+1)/2}} \cdot I_{r+1}\{4\pi(n\nu)^{1/2}\} \right|$$

$$= \left| \frac{2\pi\nu^{(r+1)/2}}{n^{(r+1)/2}} \sum_{k=2}^{\infty} A_{k,\nu}(n) \cdot I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\} \cdot k^{-1} \right.$$

$$\leqq C_1\left(\frac{\nu}{n}\right)^{(r+1)/2} \cdot \sum_{k=2}^{\infty} k^{-1/3+\varepsilon} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\}.$$

Using the fact that $I_{r+1}(t) \leqq t^r \sinh t$ for r a nonnegative even integer, and the fact that $\sinh t \leqq (t/B) \sinh B$, for $0 \leqq t \leqq B$, we have

$$\left| a_n(\nu) - 2\pi\left(\frac{\nu}{n}\right)^{(r+1)/2} \cdot I_{r+1}\{4\pi(n\nu)^{1/2}\}\right|$$
$$\leqq C_1(\nu/n)^{(r+1)/2} \sum_{k=2}^{\infty} k^{-1/3+\varepsilon}(4\pi(n\nu)^{1/2}/k)^{r+1} \frac{\sinh\,(4\pi(n\nu)^{1/2}/2)}{4\pi(n\nu)^{1/2}/2}$$
$$\leqq C_2 \frac{\nu^{r+1/2}}{n^{1/2}} \exp\,\{2\pi(n\nu)^{1/2}\}.$$

In [7, p. 203, formula (2)], it is shown that $I_{r+1}(t) \sim e^t/(2\pi t)^{1/2}$. Hence,

$$I_{r+1}\{4\pi(n\nu)^{1/2}\} \sim \exp\,\{4\pi(n\nu)^{1/2}\}/\{2\pi 2^{1/2}(n\nu)^{1/4}\},$$

and the result follows.

(b) $$\sum_{p=0}^{\infty} \{(4\pi^2 m\nu/k^2)^p/p!\,(p+r+1)!\} = \left\{\frac{k}{2\pi(m\nu)^{1/2}}\right\}^{r+1} \cdot I_{r+1}\left\{\frac{4\pi(m\nu)^{1/2}}{k}\right\}$$
$$\leqq \frac{2^r k}{2\pi(m\nu)^{1/2}} \sinh\left\{\frac{4\pi(m\nu)^{1/2}}{k}\right\} < \frac{2^r k}{2\pi(m\nu)^{1/2}} \exp\,\{4\pi(m\nu)^{1/2}/k\}.$$

The result follows.

2. Let r be a nonnegative even integer. Let F_ν be defined as follows:

$$F_\nu(\tau) = \exp\,(-2\pi i\nu\tau) + (-1)^{r/2} \sum_{n=1}^{\infty} a_n(\nu) \exp\,(2\pi i n\tau), \tag{3.02}$$

where $a_n(\nu)$ is defined by (1.05).

Remark. It is clear that with $F(\tau)$ defined by (1.05), $F(\tau) = \sum_{\nu=1}^{\mu} a_{-\nu} F_\nu(\tau) +$ constant. Thus we need only prove Theorem (1.06) for $F_\nu(\tau)$. From Proposition (3.01a) we have immediately that $F_\nu(\tau)$ is regular for $\mathcal{I}(\tau) > 0$. We must derive (1.07) for $F_\nu(\tau)$. In order to apply Lemma (2.01) we first restrict τ to be purely imaginary and derive (1.07) for such τ only. However, the result then follows directly for $\mathcal{I}(\tau) > 0$ by analytic continuation, since we have already seen that $F_\nu(\tau)$ is regular in $\mathcal{I}(\tau) > 0$.

Putting $x = \exp\,(2\pi i\tau)$, we have from (3.02)

$$F_\nu(\tau) = x^{-\nu} + \sum_{n=1}^{\infty} x^n \frac{(-1)^{r/2} 2\pi}{n^{(r+1)/2}} \sum_{k=1}^{\infty} k^{-1} A_{k,\nu}(n) \nu^{(r+1)/2} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\}$$
$$= x^{-\nu} + (-1)^{r/2} 2\pi\nu^{(r+1)/2} \sum_{k=1}^{\infty} k^{-1} \sum_{h(k)}{}' \exp\,(-2\pi i\nu h'/k)$$
$$\cdot \sum_{n=1}^{\infty} \{x \exp\,(-2\pi i h/k)\}^n \frac{1}{n^{(r+1)/2}} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\},$$

where the interchange in the order of summation is easily justified by examining the proof of Proposition (3.01a). Now

$$F_\nu(\tau) = x^{-\nu} + 2\pi\nu^{(r+1)/2} \sum_{k=1}^{\infty} (-1)^{r/2} k^{-1} \sum_{h(k)}{}' \exp\,(-2\pi i\nu h'/k) \cdot \phi_k\{x \cdot \exp\,(-2\pi i h/k)\}, \tag{3.03}$$

where

$$\begin{aligned}\phi_k(z) &= \sum_{n=1}^{\infty} \frac{1}{n^{(r+1)/2}} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\} \cdot z^n \\ &= \frac{\nu^{(r+1)/2}(2\pi)^{r+1}}{k^{r+1}} \sum_{n=1}^{\infty} z^n \sum_{p=0}^{\infty} \left\{\left(\frac{4\pi^2 n\nu}{k^2}\right)^p \Big/ p!\,(p+r+1)!\right\} \\ &= \frac{(2\pi)^{r+1}\nu^{(r+1)/2}}{k^{r+1}} \sum_{p=0}^{\infty} \left\{\left(\frac{2\pi\nu^{1/2}}{k}\right)^{2p} \Big/ p!\,(p+r+1)!\right\} \sum_{n=1}^{\infty} n^p z^n.\end{aligned}$$

The interchange is justified by Proposition (3.01b). Now by Lipschitz' formula (see [1]),

$$\begin{aligned}&\textstyle\sum_{n=1}^{\infty} n^p \exp(-2\pi t n) \\ (3.04) \qquad &= (p!/(2\pi)^{p+1}) \cdot \textstyle\sum_{l=-\infty}^{\infty} (t+li)^{-p-1} \qquad \text{for } p > 0, \\ &= -\tfrac{1}{2} + (1/2\pi) \lim_{N\to\infty} \textstyle\sum_{l=-N}^{N} (t+li)^{-1} \qquad \text{for } p = 0,\end{aligned}$$

for $\mathcal{R}(t) > 0$. Using this with $z = e^{-2\pi t}$, we obtain

$$\begin{aligned}\phi_k(z) &= \left(\frac{2\pi}{k}\right)^{r+1} \nu^{(r+1)/2} \left\{-\frac{1}{2(r+1)!} + \sum_{p=0}^{\infty} \frac{(2\pi\nu^{1/2}k^{-1})^{2p}}{p!\,(p+r+1)!}\right. \\ &\qquad \left. \cdot \frac{p!}{(2\pi)^{p+1}} \cdot \lim_{N\to\infty} \sum_{l=-N}^{N} (t+li)^{-p-1}\right\} \\ &= \frac{-(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} + \lim_{N\to\infty} \frac{(2\pi)^r \nu^{(r+1)/2}}{k^{r+1}} \sum_{l=-N}^{N} (t+li)^{-1} \\ (3.05) \qquad &\qquad \cdot \sum_{p=0}^{\infty} \left(\frac{2\pi\nu}{k^2(t+li)}\right)^p \Big/ (p+r+1)! \\ &= \frac{-(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} + \frac{k^{r+1}}{2\pi\nu^{(r+1)/2}} \lim_{N\to\infty} \sum_{l=-N}^{N} (t+li)^r \\ &\qquad \cdot \left\{\exp\left(\frac{2\pi\nu}{k^2(t+li)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi\nu}{k^2(t+li)}\right)^p\right\}.\end{aligned}$$

In (3.03) we need $z = x\exp(-2\pi ih/k) = \exp 2\pi i(\tau - h/k)$. But in order to apply (3.04) we have put $z = \exp(-2\pi t)$. Comparing these two equations we find $t = -i\tau + ih/k$. Using this in (3.05), we obtain

$$\begin{aligned}\phi_k\{x\exp(-2\pi ih/k)\} &= \frac{-(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} \\ &\qquad + \frac{k^{r+1}}{2\pi\nu^{(r+1)/2}} \lim_{N\to\infty} \sum_{l=-N}^{N} \left(-i\tau + i\frac{h}{k} + li\right)^r \\ &\cdot \left\{\exp\left(\frac{2\pi\nu}{k^2(-i\tau + ih/k + li)}\right) - \sum_{p=0}^{r} \left(\frac{2\pi\nu}{k^2(-i\tau + ih/k + li)}\right)^p \Big/ p!\right\}\end{aligned}$$

$$= \frac{-(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} + \lim_{N\to\infty} \frac{(-1)^{r/2}k}{2\pi\nu^{(r+1)/2}} \cdot \sum_{\substack{m\equiv h(k)\\ |m-h|\leq kN}} (k\tau - m)^r$$

$$\cdot \left\{\exp \frac{2\pi i\nu}{k(k\tau-m)} - \sum_{p=0}^{r} \left(\frac{2\pi i\nu}{k(k\tau-m)}\right) \Big/ p!\right\}$$

$$= \frac{-(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} + \lim_{M\to\infty} \frac{(-1)^{r/2}k}{2\pi\nu^{(r+1)/2}} \sum_{\substack{m\equiv h(k)\\ |m|\leq M}} (k\tau - m)^r$$

$$\cdot \left\{\exp\left(\frac{2\pi i\nu}{k(k\tau-m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau-m)}\right)^p\right\}.$$

Therefore,

$$(-1)^{r/2} \sum_{h(k)}{}' \exp(-2\pi i\nu h'/k)\, \phi_k\{x \exp(-2\pi i h/k)\}$$

$$= \frac{-(-1)^{r/2}(2\pi)^{r+1}\nu^{(r+1)/2}}{2(r+1)!\,k^{r+1}} \sum_{\substack{d|\nu\\ d|k}} \mu\left(\frac{k}{d}\right) \cdot d$$

$$+ \lim_{M\to\infty} \frac{k}{2\pi\nu^{(r+1)/2}} \sum_{\substack{|m|\leq M\\ (m,k)=1}} \exp(-2\pi i m'\nu/k)\cdot(k\tau-m)^r$$

$$\cdot \left\{\exp\left(\frac{2\pi i\nu}{k(k\tau-m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau-m)}\right)^p\right\}.$$

Going back to (3.03) we obtain

$$F_\nu(\tau) = \exp(-2\pi i\nu\tau) + c_\nu \;\; + \sum_{k=1}^{\infty} \lim_{M\to\infty} \sum_{\substack{|m|\leq M\\ (m,k)=1}} \exp(-2\pi i m'\nu/k)(k\tau-m)^r$$

$$\cdot \left\{\exp\left(\frac{2\pi i\nu}{k(k\tau-m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau-m)}\right)^p\right\},$$

where

$$c_\nu = \frac{-(-1)^{r/2}(2\pi)^{r+2}\nu^{(r+1)}}{2(r+1)!} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{r+2}} \left\{\sum_{\substack{d|\nu\\ d|k}} \mu\left(\frac{k}{d}\right)\cdot d\right\}.$$

If we expand the expression in braces in a power series in terms of $2\pi i\nu/k(k\tau - m)$, we obtain

$$\sum_{k=1}^{\infty} \lim_{M\to\infty} \sum_{\substack{|m|\leq M\\ (m,k)=1}} \exp(-2\pi i m'\nu/k) \sum_{p=r+1}^{\infty} \frac{(2\pi i\nu)^p}{p!\,k^p(k\tau-m)^{p-r}}$$

$$(3.06) \qquad = \sum_{k=1}^{\infty} \lim_{M\to\infty} \sum_{\substack{|m|\leq M\\ (m,k)=1}} \exp(-2\pi i m'\nu/k) \frac{(2\pi i\nu)^{r+1}}{(r+1)!\,k^{1+r}(k\tau-m)^{p-r}}$$

$$+ \sum_{k=1}^{\infty} \lim_{M\to\infty} \sum_{\substack{|m|\leq M\\ (m,k)=1}} \exp(-2\pi i m'\nu/k) \sum_{p=r+2}^{\infty} \frac{(2\pi i\nu)^p}{p!\,k^p(k-m)^{p-r}}.$$

Lemma (2.01), with $a = b = c = 1$, yields

$$\sum_{k=1}^{\infty} \lim_{M\to\infty} \sum_{\substack{|m|\leq M \\ (m,k)=1}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} = \lim_{K\to\infty} \left\{ \sum_{k=1}^{K} \sum_{\substack{|m|\leq K \\ (m,k)=1}} \frac{\exp(-2\pi i m'\nu/k)}{k^{1+r}(k\tau - m)} \right\}.$$

Also, the triple sum in (3.06) is absolutely convergent and thus permits any rearrangement of its terms. Therefore, (3.06) can be rewritten

$$\lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|\leq K \\ (m,k)=1}} \exp(-2\pi i m'\nu/k) \sum_{p=r+1}^{\infty} \frac{(2\pi i\nu)^p}{p!\,k^p(k\tau - m)^{p-r}}$$
$$= \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|\leq K \\ (m,k)=1}} \exp(-2\pi i m'\nu/k)\cdot(k\tau - m)^r \cdot \left\{ \exp\frac{2\pi i\nu}{k(k\tau - m)} - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau - m)}\right)^p \right\}.$$

Hence we can write

$$(3.07)\quad \begin{aligned} F_\nu(\tau) &= \exp(-2\pi i\nu\tau) + c_\nu + \tau^r \left\{ \exp\left(\frac{2\pi i\nu}{\tau}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{\tau}\right)^p \right\} \\ &+ \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau - m)^r \exp(-2\pi i m'\nu/k) \\ &\cdot \left\{ \exp\frac{2\pi i\nu}{k(k\tau - m)} - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau - m)}\right)^p \right\}. \end{aligned}$$

Here we have separated out the term for $m = 0$, $k = 1$. Let now

$$(3.08)\quad \begin{aligned} S_K(\tau) &= \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau - m)^r \exp(-2\pi i m'\nu/k) \exp\left(\frac{2\pi i\nu}{k(k\tau - m)}\right) \\ &= \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau - m)^r \exp\left\{ 2\pi i\nu \frac{(mm'+1)/k - m'\tau}{k\tau - m} \right\}. \end{aligned}$$

Let $-k' = (mm' + 1)/k$. Then $kk' + mm' + 1 = 0$, so we also have $kk' \equiv -1(m)$. Then (3.08) becomes

$$S_K(\tau) = \sum_{k=1}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} (k\tau - m)^r \exp\left\{ 2\pi i\nu \frac{-k' - m'\tau}{k\tau - m} \right\} + \sum_{k=1}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} (k\tau + m)^r \exp\left\{ 2\pi i\nu \frac{-k' + m'\tau}{k\tau + m} \right\}.$$

Hence

$$(3.09)\quad \begin{aligned} \tau^r S_K\left(-\frac{1}{\tau}\right) &= \sum\sum (k + m\tau)^r \exp\left\{ 2\pi i\nu \frac{-k'\tau + m'}{-k - m\tau} \right\} \\ &\quad + \sum\sum (-k + m\tau)^r \exp\left\{ 2\pi i\nu \frac{-k'\tau - m'}{-k + m\tau} \right\} \\ &= \sum\sum (m\tau - k)^r \exp\left\{ 2\pi i\nu \frac{-m' - k'\tau}{m\tau - k} \right\} \\ &\quad + \sum\sum (m\tau + k)^r \exp\left\{ 2\pi i\nu \frac{-m' + k'\tau}{m\tau + k} \right\} = S_K(\tau), \end{aligned}$$

if we interchange the roles of m and k. Going back to (3.07) we see that

$$\tau^r F_\nu\left(-\frac{1}{\tau}\right) = \tau^r \exp\left(\frac{2\pi i\nu}{\tau}\right) + c_\nu \tau^r + \exp(-2\pi i\nu\tau) - \sum_{p=0}^{r} \frac{1}{p!}(-2\pi i\nu\tau)^p$$
$$+ \lim_{K\to\infty}\left\{\tau^r S_K\left(\frac{-1}{\tau}\right) - \sum_{k=1}^{K}\sum_{\substack{|m|=1\\(m,k)=1}}^{K}(m\tau+k)^r \exp(-2\pi i m'\nu/k) \cdot \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(m\tau+k)}\right)^p\right\}.$$

Comparing this with (3.07) and using (3.09) we have

$$(3.10)\quad \begin{aligned} F_\nu(\tau) - \tau^r F_\nu\left(-\frac{1}{\tau}\right) &= c_\nu(1-\tau^r) + \sum_{p=0}^{r}\frac{1}{p!}\left\{(-2\pi i\nu\tau)^p - \tau^r\left(\frac{2\pi i\nu}{\tau}\right)^p\right\} \\ &+ \lim_{K\to\infty}\sum_{k=1}^{K}\sum_{\substack{|m|=1\\(m,k)=1}}^{K}\exp(-2\,\pi i m'\nu/k)\left\{(m\tau+k)^r\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(m\tau+k)}\right)^p\right. \\ &\left. - (k\tau - m)^r \cdot \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau-m)}\right)^p\right\}. \end{aligned}$$

Now the right-hand side of (3.10) is a polynomial in τ of degree at most r. Therefore, by the remark following (3.02), the proof of Theorem (1.06) is complete.

3. If $r = 0$, the right-hand side of (3.10) is identically zero. Thus if we put $F(\tau) = a_0 + \sum_{\nu=1}^{\mu} a_{-\nu} F_\nu(\tau)$, with $a_0, a_{-1}, \cdots, a_{-\mu}$ any constants, $F(\tau)$ is a modular function, that is, a modular form of dimension zero. We can in fact state the following stronger result.

THEOREM (3.11). *A function $F(\tau)$ analytic in the upper half plane is a modular function if and only if there exist constants $a_0, a_{-1}, \cdots, a_{-\mu}$ such that $F(\tau) = a_0 + \sum_{\nu=1}^{\mu} a_{-\nu} F_\nu(\tau)$, where $F_\nu(\tau)$ is defined by* (3.02) *with $r = 0$.*

The "only if" part of the theorem follows from the fact that a modular function which is bounded at ∞ must be constant.

From now on, r is a fixed *positive* even integer. Denote the polynomial occurring in (3.10) by $p_\nu(\tau)$. Let us now form

$$(3.12)\qquad F(\tau) = a_0 + a_{-1}F_1(\tau) + \cdots + a_{-\mu}F_\mu(\tau).$$

It follows from (3.10) that

$$(3.13)\quad F(\tau) - \tau^r F(-1/\tau) = a_0(1-\tau^r) + \textstyle\sum_{\nu=1}^{\mu} a_{-\nu}\, p_\nu(\tau) \equiv p(\tau),$$

where $p(\tau)$ is a polynomial in τ of degree at most r. Replacing τ by $-1/\tau$ in (3.13) we see that

$$(3.14)\qquad \tau^r p(-1/\tau) = -p(\tau).$$

If we can choose $a_0, \cdots, a_{-\mu}$ in such a way that $p(\tau_k) = 0$ for $r/2 + 1$ distinct points τ_k, then by (3.14), $p(\tau) \equiv 0$. We must assume, of course, that the set of τ_k's and the set of $-1/\tau_k$'s have no point in common.

We therefore write the system of linear equations

$$(3.15)\quad p(\tau_k) = a_0(1 - \tau_k^r) + \sum_{\nu=1}^{\mu} a_{-\nu}\, p_\nu(\tau_k) = 0 \quad (k = 1, \cdots, r/2 + 1),$$

where τ_k are chosen as described above. This has a nontrivial solution as long as $\mu \geqq r/2 + 1$. In fact (3.15) has $\mu - r/2$ linearly independent solutions. Hence we have the following result.

THEOREM (3.16). *Let μ be an integer such that $\mu \geqq r/2 + 1$. If we define $F(\tau)$ as in (3.12) with $(a_0, a_{-1}, \cdots, a_{-\mu})$ chosen to satisfy (3.15), then $F(\tau)$ is a modular form of dimension r, with principal part*

$$a_{-\mu} \exp(-2\pi i \mu\tau) + \cdots + a_{-1} \exp(-2\pi i\tau).$$

Professor Bateman has made several interesting observations which are restated in the following remarks.

The general linear combination (3.12) has $\mu + 1$ parameters. The vector space of modular forms of even integral dimension r and principal part at ∞ of order no greater than μ has dimension (see [3], §8)

$$\mu + 1 - \left[\frac{r + 10}{12}\right] \qquad \text{if} \quad r \equiv 0, 2, 4, 6, 8 \pmod{12},$$

$$\mu - \left[\frac{r + 10}{12}\right] \qquad \text{if} \quad r \equiv 10 \pmod{12}.$$

Thus (3.12) gives a modular form under

$$\left[\frac{r + 10}{12}\right] \qquad \text{if} \quad r \equiv 0, 2, 4, 6, 8 \pmod{12},$$

$$\left[\frac{r + 10}{12}\right] + 1 \qquad \text{if} \quad r \equiv 10 \pmod{12},$$

linear relations on $a_0, a_{-1}, \cdots, a_{-\mu}$. Theorem (3.16) gives us modular forms if we impose at least $r/2 + 1$ linear relations on $a_0, a_{-1}, \cdots, a_{-\mu}$, which is of course more than is actually needed. In particular, if $r = 2, 4, 6, 8, 12$, only one linear relation is needed. In fact the single relation $a_0 + \sum_{\nu=1}^{\mu} a_{-\nu}\, p_\nu(0) = 0$ should suffice in these cases. However this seems hard to prove by our approach.

The following partially explains the discrepancy between the actual number of linear relations needed and the number we have to impose.

Remark. For special values of r the polynomial $p(\tau)$ has certain fixed roots which are *independent* of the choice of $a_0, a_{-1}, \cdots, a_{-\mu}$. That is, no matter how these constants are chosen, $p(\tau)$ will have as roots the solutions of

$$(\tau^2 - \tau + 1)(\tau^2 + \tau + 1)(\tau^2 + 1) = 0 \quad \text{if} \quad r \equiv 0 \pmod{12},$$

$$\tau^2 + 1 = 0 \quad \text{if} \quad r \equiv 4, 8 \pmod{12},$$

$$(\tau^2 - \tau + 1)(\tau^2 + \tau + 1) = 0 \quad \text{if} \quad r \equiv 6 \pmod{12}.$$

Therefore, in these cases, correspondingly fewer linear relations will suffice in Theorem (3.16).

We now use Theorem (1.06) to construct modular forms of *negative* even integral dimensions. First, we need the following result.

LEMMA (3.17). *If $\tau \to (a\tau + b)/(c\tau + d)$ is any transformation of the modular group, and F is any complex function with sufficiently many derivatives, then*

$$\frac{d^{r+1}}{d\tau^{r+1}}\left\{(c\tau + d)^r \cdot F\left(\frac{a\tau + b}{c\tau + d}\right)\right\} = (c\tau + d)^{-r-2} \cdot F^{(r+1)}\left(\frac{a\tau + b}{c\tau + d}\right),$$

for any integer $r \geqq 0$.

The proof is simple and proceeds by induction on r.

Applying this Lemma for $a = 0$, $b = -1$, $c = 1$, $d = 0$ and using Theorem (1.06), we obtain

THEOREM (3.18). *Let $F(\tau)$ be defined by the Fourier series* (1.06), *with r a nonnegative even integer. Then*

$$F^{(r+1)}(\tau) - \tau^{-r-2} F^{(r+1)}(-1/\tau) = 0,$$

i.e., $F^{(r+1)}(\tau)$ is a modular form of dimension $-r - 2$.

IV. The groups $G(\surd 2)$ and $G(\surd 3)$

In this section we construct functions which behave under substitutions of the groups $G(\surd 2)$ and $G(\surd 3)$ in the same way as the functions $F_\nu(\tau)$, defined by (3.02), behave under substitutions of the modular group. These functions were given by Raleigh [4] for the case $r = 0$. Here we let r be any nonnegative even integer. Let $\varepsilon_0 = \pm 1$.

1. *The group $G(\surd 2)$.* Let $F_{\nu,2}$ be defined as follows

$$\begin{aligned} F_{\nu,2}(\tau) &= \exp(-2\pi i\nu\tau/2^{1/2}) + \sum_{n=1}^{\infty} a_n(\nu) \exp(2\pi i n\tau/2^{1/2}), \\ a_n(\nu) &= (-1)^{r/2} 2\pi \sum_{k=1}^{\infty} \frac{1}{2k} A_{2k,\nu}(n)(\nu/n)^{(r+1)/2} I_{r+1}\left\{\frac{2\pi(n\nu)^{1/2}}{k}\right\} \\ &\quad + \varepsilon_0(-1)^{r/2} \frac{2\pi}{2^{1/2}} \sum_{k=1}^{\infty} \frac{1}{2k-1} A_{2k-1,\nu}\{(1-k)n\} \\ &\quad \cdot (\nu/n)^{(r+1)/2} I_{r+1}\left\{\frac{2\pi(2n\nu)^{1/2}}{2k-1}\right\} = a_{n,1}(\nu) + a_{n,2}(\nu). \end{aligned} \tag{4.01}$$

Remark. A result analogous to Proposition (3.01a), modified to handle the present situation, shows that $F_{\nu,2}(\tau)$ is regular for $\mathfrak{I}(\tau) > 0$. As before, we derive the transformation properties of $F_{\nu,2}(\tau)$ for τ on the positive imaginary axis, and then the result follows for $\mathfrak{I}(\tau) > 0$ by analytic continuation. The same remark can be made in connection with $F_{\nu,3}(\tau)$ to be defined in (4.13).

Put

$$x = \exp(2\pi i\tau/2^{1/2}), \qquad f_1(x) = \textstyle\sum_{n=1}^{\infty} a_{n,1}(\nu) x^n.$$

As before,

$$
\begin{aligned}
f_1(x) &= (-1)^{r/2}\cdot 2\pi\nu^{(r+1)/2}\cdot \sum_{\substack{k=1\\ k\equiv 0(2)}}^{\infty} \frac{1}{k}\sum_{h(k)}{}' \exp(-2\pi i\nu h'/k) \\
&\qquad \cdot \sum_{n=1}^{\infty} \{x \exp(-2\pi i h/k)^n\} \frac{1}{n^{(r+1)/2}} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\}.
\end{aligned} \tag{4.02}
$$

If we keep in mind that here $x \exp(-2\pi i h/k) = \exp\{2\pi i(\tau/2^{1/2} - h/k)\}$, the argument of the previous section with the use of Lemma (2.01) for $a = 0$, $b = 2$, $c = 1$, yields

$$
\begin{aligned}
f_1(x) &= c_{\nu,1} + \lim_{K\to\infty} \sum_{\substack{k=1\\ k\equiv 0(2)}}^{K} \sum_{\substack{|m|=1\\ (m,k)=1}}^{K} (k\tau/2^{1/2} - m)^r \exp(-2\pi i m'\nu/k) \\
&\qquad \cdot \left\{\exp\left(\frac{2\pi i\nu}{k(k\tau/2^{1/2} - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/2^{1/2} - m)}\right)^p\right\}.
\end{aligned} \tag{4.03}
$$

Since $k \equiv 0(2)$, $(m, k) = 1$, there are no terms with $m = 0$.

Let $f_2(x) = x^{-\nu} + \sum_{n=1}^{\infty} a_{n,2}(\nu)x^n$. Then

$$
\begin{aligned}
f_2(x) &= x^{-\nu} + \frac{\varepsilon_0}{2^{1/2}}(-1)^{r/2}\cdot 2\pi\nu^{(r+1)/2} \sum_{\substack{k=1\\ k\equiv 1(2)}}^{\infty} \frac{1}{k}\sum_{h(k)}{}' \exp(-2\pi i h'\nu/k) \\
&\qquad \cdot \sum_{n=1}^{\infty}\left\{x \exp\left(-2\pi i\,\frac{h(1-k)}{2k}\right)\right\}^n \cdot \frac{1}{n^{(r+1)/2}} I_{r+1}\left\{\frac{2\pi(2n\nu)^{1/2}}{k}\right\}.
\end{aligned} \tag{4.04}
$$

Here,

$$
x \exp\left\{-2\pi i\,\frac{h(1-k)}{2k}\right\} = \exp\{2\pi i(\tau/2^{1/2} - h(1-k)/2k)\}.
$$

Therefore applying the previous argument and using Lemma (2.03) with $a = 1$, $b = 2$, we obtain

$$
\begin{aligned}
f_2(x) &= \exp(-2\pi i\nu\tau/2^{1/2}) + c_{\nu,2} \\
&\quad + \varepsilon_0\tau^r\left\{\exp\left(\frac{2\pi i\nu}{2^{1/2}\tau}\right) - \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i\nu}{2^{1/2}\tau}\right)^p\right\} \\
&\quad + \varepsilon_0 \lim_{K\to\infty} 2^{-r/2} \sum_{\substack{k=1\\ k\equiv 1(2)}}^{K} \sum_{\substack{|m|=1\\ m\equiv 0(2)\\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)(2^{1/2}k\tau - m)^r \\
&\qquad \cdot \left\{\exp\left(\frac{2\pi i\nu}{k(2^{1/2}k\tau - m)}\right) - \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i\nu}{k(2^{1/2}k\tau - m)}\right)^p\right\}.
\end{aligned} \tag{4.05}
$$

Here we have separated out the term for $m = 0$, $k = 1$.

By (4.03) and (4.05), it follows that

$$
\begin{aligned}
F_{\nu,2}(\tau) &= f_1(x) + f_2(x) = \exp(-2\pi i\nu\tau/2^{1/2}) + c_{\nu,1} + c_{\nu,2} \\
&\quad + \varepsilon_0\,\tau^r\left\{\exp(2\pi i\nu/2^{1/2}\tau) - \sum_{p=0}^{r}\frac{1}{p!}(2\pi i\nu/2^{1/2}\tau)^p\right\} \\
&\quad + \lim_{K\to\infty}\sum_{\substack{k=1\\k\equiv0(2)}}^{K}\ \sum_{\substack{|m|=1\\(m,k)=1}}^{K}(k\tau/2^{1/2}-m)^r\exp(-2\pi i m'\nu/k) \\
&\qquad\cdot\left\{\exp\left(\frac{2\pi i\nu}{k(k\tau/2^{1/2}-m)}\right) - \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/2^{1/2}-m)}\right)^p\right\} \\
&\quad + \varepsilon_0\lim_{K\to\infty}2^{-r/2}\sum_{\substack{k=1\\k\equiv1(2)}}^{K}\ \sum_{\substack{|m|=1\\m\equiv0(2)\\(m,k)=1}}^{K}(2^{1/2}k\tau-m)^r\exp(-2\pi i m'\nu/k) \\
&\qquad\cdot\left\{\exp\left(\frac{2\pi i\nu}{k(2^{1/2}k\tau-m)}\right) - \sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i\nu}{k(2^{1/2}k\tau-m)}\right)^p\right\}.
\end{aligned} \tag{4.06}
$$

Now let

$$
\begin{aligned}
S_{K,1}(\tau) &= \sum_{\substack{k=1\\k\equiv0(2)}}^{K}\ \sum_{\substack{|m|=1\\(m,k)=1}}^{K}(k\tau/2^{1/2}-m)^r\exp\left\{\frac{-2\pi i\nu}{k}\left(m'-\frac{1}{k\tau/2^{1/2}-m}\right)\right\} \\
&= \sum\sum(k\tau/2^{1/2}-m)^r\exp\left\{2\pi i\nu\left(\frac{(mm'+1)/k - m'\tau/2^{1/2}}{k\tau/2^{1/2}-m}\right)\right\}.
\end{aligned} \tag{4.07}
$$

Let $-k' = (mm'+1)/k$. Then $kk' + mm' + 1 = 0$, and $kk' \equiv -1(m)$. Then (4.07) becomes

$$
\begin{aligned}
S_{K,1}(\tau) &= \sum_{\substack{k=1\\k\equiv0(2)}}^{K}\ \sum_{\substack{m=1\\(m,k)=1}}^{K}(k\tau/2^{1/2}-m)^r\exp\left\{2\pi i\nu\,\frac{-k'-m'\tau/2^{1/2}}{k\tau/2^{1/2}-m}\right\} \\
&\quad + \sum_{\substack{k=1\\k\equiv0(2)}}^{K}\ \sum_{\substack{m=1\\(m,k)=1}}^{K}(k\tau/2^{1/2}+m)^r\exp\left\{2\pi i\nu\,\frac{-k'+m'\tau/2^{1/2}}{k\tau/2^{1/2}+m}\right\}.
\end{aligned} \tag{4.08}
$$

Let

$$
\begin{aligned}
S_{K,2}(\tau) &= \varepsilon_0 2^{-r/2}\sum_{\substack{k=1\\k\equiv1(2)}}^{K}\ \sum_{\substack{|m|=1\\m\equiv0(2)\\(m,k)=1}}^{K}(2^{1/2}k\tau-m)^r\exp\left\{2\pi i\nu\,\frac{-k'-2^{1/2}m'\tau}{2^{1/2}k\tau-m}\right\} \\
&= \varepsilon_0\sum_{\substack{k=1\\k\equiv1(2)}}^{K}\ \sum_{\substack{m=1\\m\equiv0(2)\\(m,k)=1}}^{K}(k\tau-m/2^{1/2})^r\exp\left\{2\pi i\nu\,\frac{-k'-2^{1/2}m'\tau}{2^{1/2}k\tau-m}\right\} \\
&\quad + \varepsilon_0\sum\sum(k\tau+m/2^{1/2})^r\exp\left\{2\pi i\nu\,\frac{-k'+2^{1/2}m'\tau}{2^{1/2}k\tau+m}\right\}.
\end{aligned} \tag{4.09}
$$

Now

$$\varepsilon_0 \tau^r S_{K,1}(-1/\tau) = \varepsilon_0 \sum_{\substack{k=1 \\ k\equiv 0(2)}}^{K} \sum_{\substack{m=1 \\ (m,k)=1}}^{K} (m\tau + k/2^{1/2})^r \exp\left\{2\pi i\nu \frac{-k'\tau + m'/2^{1/2}}{-k/2^{1/2} - m\tau}\right\}$$

$$+ \varepsilon_0 \sum\sum (m\tau - k/2^{1/2})^r \exp\left\{2\pi i\nu \frac{-k' - m'/2^{1/2}}{-k/2^{1/2} + m}\right\}$$

$$= \varepsilon_0 \sum\sum (m\tau - k/2^{1/2})^r \exp\left\{2\pi i\nu \frac{-m' - 2^{1/2}k'\tau}{2^{1/2}m\tau - k}\right\}$$

$$+ \varepsilon_0 \sum\sum (m\tau + k/2^{1/2})^r \exp\left\{2\pi i\nu \frac{-m' + 2^{1/2}k'\tau}{2^{1/2}m\tau + k}\right\}.$$

Therefore, if we interchange the roles of m and k and notice that $k \equiv 0(2)$, $(m, k) = 1$ implies $m \equiv 1(2)$, we have

$$\varepsilon_0 \tau^r S_{K,1}(-1/\tau) = S_{K,2}(\tau). \tag{4.10}$$

It follows directly that

$$\varepsilon_0 \tau^r S_{K,2}(-1/\tau) = S_{K,1}(\tau). \tag{4.11}$$

Going back to (4.06) we see that

$$\varepsilon_0 \tau^r F_{\nu,2}(-1/\tau) = \varepsilon_0 \tau^r \exp(2\pi i\nu/2^{1/2}\tau) + \varepsilon_0(c_{\nu,1} + c_{\nu,2})\tau^r$$

$$+ \exp(-2\pi i\nu\tau/2^{1/2}) - \sum_{p=0}^{r} \frac{1}{p!}(-2\pi i\nu\tau/2^{1/2})^p$$

$$+ \lim_{K\to\infty}\left\{\varepsilon_0 \tau^r S_{K,1}(-1/\tau) + \varepsilon_0 \tau^r S_{K,2}(-1/\tau)\right.$$

$$- \varepsilon_0 \sum_{\substack{k=1 \\ k\equiv 0(2)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)(m\tau + k/2^{1/2})^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(k/2^{1/2} + m\tau)}\right)^p$$

$$\left. - \sum_{\substack{k=1 \\ k\equiv 1(2)}}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(2) \\ (m,k)=1}}^{K} (m\tau/2^{1/2} + k)^r \exp(-2\pi i m'\nu/k) \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(2^{1/2}k + m\tau)}\right)^p\right\}.$$

Comparing this with (4.06) and using (4.10) and (4.11) we have

$$\begin{aligned} &F_{\nu,2}(\tau) - \varepsilon_0 \tau^r F_{\nu,2}(-1/\tau) \\ &= (c_{\nu,1} + c_{\nu,2})(1 - \varepsilon_0 \tau^r) + \sum_{p=0}^{r} \frac{1}{p!}\left\{\left(\frac{-2\pi i\nu\tau}{2^{1/2}}\right)^p - \varepsilon_0 \tau^r\left(\frac{2\pi i\nu}{2^{1/2}\tau}\right)^p\right\} \\ &+ \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 0(2)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k) \\ &\cdot \left\{\varepsilon_0(m\tau + k/2^{1/2})^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(k/2^{1/2} + m\tau)}\right)^p\right. \end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&\qquad\qquad - (k\tau/2^{1/2} - m)^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/2^{1/2} - m)}\right)^p\Bigg\} \\
&+ \lim_{K\to\infty} \sum_{\substack{k=1\\k\equiv 1(2)}}^{K} \sum_{\substack{|m|=1\\m\equiv 0(2)\\(m,k)=1}}^{K} \exp(-2\pi i m'\nu/k) \\
&\cdot \Bigg\{ (m\tau/2^{1/2} + k)^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(2^{1/2}k + m\tau)}\right)^p \\
&\qquad\qquad - \varepsilon_0 (k\tau - m/2^{1/2})^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(2^{1/2}k\tau - m)}\right)^p\Bigg\}.
\end{aligned}
\tag{4.12}
$$

2. *The group* $G(\sqrt{3})$. Let $F_{\nu,3}$ be defined as follows

$$
F_{\nu,3}(\tau) = \exp(-2\pi i\nu\tau/3^{1/2}) + \sum_{n=1}^{\infty} a_n(\nu)\exp(2\pi i n\tau/3^{1/2}),
$$

$$
\begin{aligned}
a_n(\nu) &= (-1)^{r/2}\, 2\pi \sum_{k=1}^{\infty} \frac{1}{3k} A_{3k,\nu}(n) \left(\frac{\nu}{n}\right)^{(r+1)/2} \cdot I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{3k}\right\} \\
&+ \frac{\varepsilon_0(-1)^{r/2} 2\pi}{3^{1/2}} \sum_{k=1}^{\infty} \frac{1}{3k-1} A_{3k-1,\nu}(kn) \left(\frac{\nu}{n}\right)^{(r+1)/2} I_{r+1}\left\{\frac{4\pi(3n\nu)^{1/2}}{3(3k-1)}\right\} \\
&+ \frac{\varepsilon_0(-1)^{r/2} 2\pi}{3^{1/2}} \sum_{k=1}^{\infty} \frac{1}{3k-2} A_{3k-2,\nu}((1-k)n) \left(\frac{\nu}{n}\right)^{(r+1)/2} \\
&\qquad\qquad \cdot I_{r+1}\left\{\frac{4\pi(3n\nu)^{1/2}}{3(3k-2)}\right\} \\
&= a_{n,1}(\nu) + a_{n,2}(\nu) + a_{n,3}(\nu).
\end{aligned}
\tag{4.13}
$$

Put $x = \exp(2\pi i\tau/3^{1/2})$, $f_1(x) = \sum_{n=1}^{\infty} a_{n,1}(\nu)x^n$. Applying the same argument used above, and using Lemma (2.01) with $a = 0$, $b = 3$, $c = 1$, we obtain

$$
\begin{aligned}
f_1(x) &= c_{\nu,1} + \lim_{K\to\infty} \sum_{\substack{k=1\\k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1\\(m,k)=1}}^{K} (k\tau/3^{1/2} - m)^r \exp(-2\pi i m'\nu/k) \\
&\cdot \left\{\exp\left(\frac{2\pi i\nu}{k(k\tau/3^{1/2} - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/3^{1/2} - m)}\right)^p\right\}.
\end{aligned}
\tag{4.14}
$$

Let $f_2(x) = \sum_{n=1}^{\infty} a_{n,2}(\nu)x^n$. Using once again the same argument, and applying Lemma (2.03) with $a = 2$, $b = 3$, we obtain

$$
\begin{aligned}
f_2(x) &= c_{\nu,2} + \varepsilon_0 \lim_{K\to\infty} \sum_{\substack{k=1\\k\equiv 2(3)}}^{K} \sum_{\substack{|m|=1\\m\equiv 0(3)\\(m,k)=1}}^{K} (k\tau - m/3^{1/2})^r \exp(-2\pi i m'\nu/k) \\
&\cdot \left\{\exp\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right)^p\right\}.
\end{aligned}
\tag{4.15}
$$

Finally, we let $f_3(x) = x^{-\nu} + \sum_{n=1}^{\infty} a_{n,3}(\nu)x^n$. Once more we apply the above argument. We use Lemma (2.03) with $a = 1$, $b = 3$ and obtain

$$\begin{aligned} f_3(x) = \exp(-2\pi i\nu\tau/3^{1/2}) + c_{\nu,3} + \varepsilon_0 \tau^r \\ \cdot \left\{ \exp\left(\frac{2\pi i\nu}{3^{1/2}\tau}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{3^{1/2}\tau}\right)^p \right\} \\ + \varepsilon_0 \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 1(3)}}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} (k\tau - m/3^{1/2})^r \exp(-2\pi i m'\nu/k) \\ \cdot \left\{ \exp\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right)^p \right\}. \end{aligned} \tag{4.16}$$

Here we have separated out the term for $m = 0$, $k = 1$.

By (4.14), (4.15), and (4.16) it follows that

$$\begin{aligned} F_{\nu,3}(\tau) = \exp(-2\pi i\nu\tau/3^{1/2}) + c_{\nu,1} + c_{\nu,2} + c_{\nu,3} \\ + \varepsilon_0 \tau^r \left\{ \exp\left(\frac{2\pi i\nu}{3^{1/2}\tau}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{3^{1/2}\tau}\right)^p \right\} \\ + \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau/3^{1/2} - m)^r \exp(-2\pi i m'\nu/k) \\ \cdot \left\{ \exp\left(\frac{2\pi i\nu}{k(k\tau/3^{1/2} - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/3^{1/2} - m)}\right)^p \right\} \\ + \varepsilon_0 \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} (k\tau - m/3^{1/2})^r \exp(-2\pi i m'\nu/k) \\ \cdot \left\{ \exp\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right)^p \right\}. \end{aligned} \tag{4.17}$$

In this last infinite sum we have combined those k such that $k \equiv 1(3)$ and $k \equiv 2(3)$, and made use of the fact that $m \equiv 0(3)$, $(m, k) = 1$ implies $k \equiv 1(3)$ or $k \equiv 2(3)$.

Now let

$$\begin{aligned} S_{K,1}(\tau) &= \sum_{\substack{k=1 \\ k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau/3^{1/2} - m)^r \exp\left\{ \frac{-2\pi i\nu}{k}\left(m' - \frac{1}{k\tau/3^{1/2} - m}\right) \right\} \\ &= \sum_{\substack{k=1 \\ k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} (k\tau/3^{1/2} - m)^r \exp\left\{ 2\pi i\nu \frac{-k' - m'\tau/3^{1/2}}{k\tau/3^{1/2} - m} \right\}, \end{aligned}$$

where, as before, $-k' = (mm' + 1)/k$. Let

$$S_{K,2}(\tau) = \varepsilon_0 \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} (k\tau - m/3^{1/2})^r \exp\left\{ -\frac{2\pi i\nu}{k}\left(m' - \frac{1}{3^{1/2}k\tau - m}\right) \right\}$$

$$= \varepsilon_0 \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} (k\tau - m/3^{1/2})^r \exp\left\{2\pi i\nu \frac{-k' - 3^{1/2}m'\tau}{3^{1/2}k\tau - m}\right\}.$$

If we interchange the roles of m and k, it is easy to see that

$$\varepsilon_0\, \tau^r S_{K,1}(-1/\tau) = S_{K,2}(\tau), \tag{4.18}$$

$$\varepsilon_0\, \tau^r S_{K,2}(-1/\tau) = S_{K,1}(\tau). \tag{4.19}$$

Going back to (4.17), we see that

$$\varepsilon_0 \tau^r F_{\nu,3}(-1/\tau) = \varepsilon_0 \tau^r \exp(2\pi i\nu/3^{1/2}\tau) + \varepsilon_0(c_{\nu,1} + c_{\nu,2} + c_{\nu,3})\tau^r$$
$$+ \exp(-2\pi i\nu\tau/3^{1/2}) - \sum_{p=0}^{r} \frac{1}{p!}(-2\pi i\nu\tau/3^{1/2})^p$$
$$+ \lim_{K\to\infty}\left\{\varepsilon_0 \tau^r S_{K,1}(-1/\tau) - \varepsilon_0 \sum_{\substack{k=1 \\ k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)(k/3^{1/2} + m\tau)^r \cdot \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(k/3^{1/2} + m\tau)}\right)^p\right\}$$
$$+ \lim_{K\to\infty}\left\{\varepsilon_0 \tau^r S_{K,2}(-1/\tau) - \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)(k + m\tau/3^{1/2})^r \cdot \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(3^{1/2}k + m\tau)}\right)^p\right\}.$$

Comparing this with (4.17) and using (4.18) and (4.19) we have

$$F_{\nu,3}(\tau) - \varepsilon_0 \tau^r F_{\nu,3}(-1/\tau) = (c_{\nu,1} + c_{\nu,2} + c_{\nu,3})(1 - \varepsilon_0 \tau^r)$$
$$+ \sum_{p=0}^{r} \frac{1}{p!}\{(-2\pi i\nu\tau/3^{1/2})^p - \varepsilon_0 \tau^r (2\pi i\nu/3^{1/2}\tau)^p\}$$
$$+ \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 0(3)}}^{K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)\left\{\varepsilon_0(k/3^{1/2} + m\tau)^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(k/3^{1/2} + m\tau)}\right)^p - (k\tau/3^{1/2} - m)^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/3^{1/2} - m)}\right)^p\right\}$$
$$+ \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{\substack{|m|=1 \\ m\equiv 0(3) \\ (m,k)=1}}^{K} \exp(-2\pi i m'\nu/k)\left\{(k + m\tau/3^{1/2})^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{-2\pi i\nu\tau}{k(3^{1/2}k + m\tau)}\right)^p - \varepsilon_0(k\tau - m/3^{1/2})^r \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(3^{1/2}k\tau - m)}\right)^p\right\}. \tag{4.20}$$

3. We summarize the results of this section with the following theorem.

THEOREM (4.21). *Let* $F_l(\tau) = a_0 + \sum_{\nu=1}^{\mu} a_{-\nu} F_{\nu,l}(\tau)$, *where* $F_{\nu,l}(\tau)$ $(l = 2, 3)$ *is defined by* (4.01) *or* (4.13). *Then in* $\mathfrak{I}(\tau) > 0$, $F_l(\tau)$ *is regular and satisfies*

$$(4.22) \qquad F_l(\tau) - \varepsilon_0\, \tau^r F_l(-1/\tau) = p_l(\tau),$$

where $p_l(\tau)$ *is a polynomial of degree at most* r.

This follows by noting that the right-hand sides of (4.12) and (4.20) are polynomials of degree at most r. If $r = 0$ and $\varepsilon_0 = +1$, the right-hand sides of (4.12) and (4.20) are zero, and (4.22) becomes $F_l(-1/\tau) = F_l(\tau)$. More particularly, if $\mu = 1$, $a_{-1} = 1$, $a_0 = 0$, $r = 0$, $\varepsilon_0 = +1$, (4.01) and (4.13) reduce to the expansions for $j(2^{1/2}; \tau)$ and $j(3^{1/2}; \tau)$ respectively as given by Raleigh in [4], and (4.22) becomes $j(l^{1/2}; -1/\tau) = j(l^{1/2}; \tau)$, $(l = 2, 3)$. Thus we have given another proof of the validity of Raleigh's formulas.

It is now clear that results analogous to Theorems (3.16) and (3.18) are true for the groups $G(\sqrt{2})$ and $G(\sqrt{3})$. In particular the result analogous to Theorem (3.16) enables us to construct automorphic forms of even integral dimension r for the groups $G(\sqrt{l})$, $(l = 2, 3)$. That is, we can construct functions $F_l(\tau)$ such that

$$(4.23) \qquad F_l(\tau + l^{1/2}) = F_l(\tau), \qquad \tau^r F_l(-1/\tau) = F_l(\tau).$$

V. The group $G(2)$

In this section we construct functions which behave under substitutions of $G(2)$ in much the same way that the functions $F_\nu(\tau)$, defined by (3.02), behave under modular substitutions. These functions were given by Simons [6] for the case $r = 0$.

Let λ_ν be defined as follows, with r again a nonnegative even integer.

$$(5.01) \qquad \begin{aligned} \lambda_\nu(\tau) &= \sum_{n=1}^{\infty} a_n(\nu) \exp(\pi i n \tau), \\ a_n(\nu) &= (-1)^{r/2} \frac{\pi}{8} \sum_{\substack{k=1 \\ k \equiv 2(4)}}^{\infty} k^{-1} A_{k,\nu}(n) (\nu/n)^{(r+1)/2} I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\}. \end{aligned}$$

The method of proof of Proposition (3.01a) yields the following result.

PROPOSITION (5.02). *If* $a_n(\nu)$ *is defined as in* (5.01), *then as* $n \to +\infty$

$$a_n(\nu) \sim \nu^{r/2+1/4} \frac{\exp(\pi i(\nu + n)) \exp(2\pi (n\nu)^{1/2})}{32 n^{r/2+3/4}}.$$

As in the previous sections, Proposition (5.02) shows that $\lambda_\nu(\tau)$ is regular in $\mathcal{I}(\tau) > 0$. Again we derive the transformation properties of $\lambda_\nu(\tau)$ for τ purely imaginary and extend the result to $\mathcal{I}(\tau) > 0$ by analytic continuation.

Put

$$x = \exp(\pi i \tau), \qquad f(x) = \textstyle\sum_{n=1}^{\infty} a_n(\nu) x^n.$$

Then

$$f(x) = (-1)^{r/2} \frac{\pi}{8} \sum_{\substack{k=1 \\ k \equiv 2(4)}}^{\infty} \sum_{h(k)}{}' \exp(-2\pi i \nu h'/k) \cdot \sum_{n=1}^{\infty} \{x \exp(-2\pi i h/k)\}^n \left(\frac{\nu}{n}\right)^{(r+1)/2} \cdot I_{r+1}\left\{\frac{4\pi(n\nu)^{1/2}}{k}\right\}.$$

We can apply the previous argument, this time using Lemma (2.01) with $a = 2$, $b = 4$, $c = 2$, to obtain

$$\lambda_\nu(\tau) = c_\nu + \frac{1}{16} \lim_{K\to\infty} \sum_{\substack{k=1\\ k\equiv 2(4)}}^{2K} \sum_{\substack{|m|=1\\ (m,k)=1}}^{K} (k\tau/2 - m)^r \exp(-2im'\nu/k) \cdot \left\{ \exp\left(\frac{2\pi i\nu}{k(k\tau/2 - m)}\right) - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i\nu}{k(k\tau/2 - m)}\right)^p \right\}. \tag{5.03}$$

We have made use of the fact that $k \equiv 2(4)$ and $(m, k) = 1$ together imply that $m = 0$ does not appear in the sum. Let

$$\begin{aligned} S_K(\tau) &= \sum_{\substack{k=1\\ k\equiv 2(4)}}^{2K} \sum_{\substack{|m|=1\\ (m,k)=1}}^{K} (k\tau/2 - m)^r \exp\left\{\frac{-2\pi i\nu}{k}\left(m' - \frac{1}{k(k\tau/2 - m)}\right)\right\} \\ &= \sum_{\substack{k=1\\ k\equiv 2(4)}}^{2K} \sum_{\substack{|m|=1\\ (m,k)=1}}^{K} (k\tau/2 - m)^r \exp\left\{2\pi i\nu \frac{-k' - m'\tau/2}{k\tau/2 - m}\right\} \\ &= \sum_{\substack{k=1\\ k\equiv 2(4)}}^{2K} \sum_{\substack{m=1\\ (m,k)=1}}^{K} (k\tau/2 - m)^r \exp\left\{2\pi i\nu \frac{-k' - m'\tau/2}{k\tau/2 - m}\right\} \\ &\quad + \sum\sum (k\tau/2 + m)^r \exp\left\{2\pi i\nu \frac{-k' + m'\tau/2}{k\tau/2 + m}\right\}. \end{aligned} \tag{5.04}$$

Here again we have put $-k' = (mm' + 1)/k$. From (5.04) it follows that

$$\begin{aligned} \tau^r S_K(-1/\tau) &= \sum\sum (m\tau + k/2)^r \exp\left\{2\pi i\nu \frac{-k'\tau + m'/2}{-k/2 - m\tau}\right\} \\ &\quad + \sum\sum (m\tau - k/2)^r \exp\left\{2\pi i\nu \frac{-k'\tau - m'/2}{-k/2 + m\tau}\right\} \\ &= \sum\sum (m\tau - k/2)^r \exp\left\{2\pi i\nu \frac{-m'/2 - k'\tau}{m\tau - k/2}\right\} \\ &\quad + \sum\sum (m\tau + k/2)^r \exp\left\{2\pi i\nu \frac{-m'/2 + k'\tau}{m\tau + k/2}\right\}. \end{aligned} \tag{5.05}$$

Now put $l = k/2$ and $n = 2m$. We wish to find integers l' and n' such that $ll' + nn' + 1 = 0$. Since $k \equiv 2(4)$, $k/2$ is odd and m' is odd. Thus $(k/2 + m')/2$ is an integer, and we put $l' = 2k' - m$ and $n' = (k/2 + m')/2$. It is easy to see that, with this choice, $nn' + ll' + 1 = mm' + kk' + 1 = 0$. Now (5.05) becomes

$$\begin{aligned} \tau^r S_K(-1/\tau) &= \sum_{\substack{2l=1\\ 2l\equiv 2(4)}}^{2K} \sum_{\substack{n/2=1\\ (n/2,2l)=1}}^{K} (n\tau/2 - l)^r \cdot \exp\left\{2\pi i\nu \frac{-n' + l/2 - (l' + n/2)\tau/2}{n\tau/2 - l}\right\} \\ &\quad + \sum_{\substack{2l=1\\ 2l\equiv 2(4)}}^{2K} \sum_{\substack{n/2=1\\ (n/2,2l)=1}}^{K} (n\tau/2 + l)^r \exp\left\{2\pi i\nu \frac{-n' + l/2 + (l' + n/2)\tau/2}{n\tau/2 + l}\right\}. \end{aligned} \tag{5.06}$$

In (5.04) we have $k \equiv 2(4)$, $(m, k) = 1$. Hence m is odd. Therefore $n = 2m$ satisfies $n \equiv 2(4)$, and (5.06) becomes

$$\tau^r S_K(-1/\tau) = \sum_{l=1}^{K} \sum_{\substack{n=1 \\ n\equiv 2(4) \\ (n,l)=1}}^{2K} (n\tau/2 - l)^r \exp\left\{2\pi i\nu \left(\frac{-n' + l'\tau/2}{n\tau/2 - l} - \frac{1}{2}\right)\right\}$$

$$+ \sum_{l=1}^{K} \sum_{\substack{n=1 \\ n\equiv 2(4) \\ (n,l)=1}}^{2K} (n\tau/2 + l)^r \exp\left\{2\pi i\nu \left(\frac{-n' + l'\tau/2}{n\tau/2 + l} + \frac{1}{2}\right)\right\}.$$

Comparing this with (5.04), we see that

$$\tau^r S_K(-1/\tau) = (-1)^\nu S_K(\tau). \tag{5.07}$$

Going back to (5.03) we obtain

$$(-1)^\nu \tau^r \lambda_\nu(-1/\tau) = (-1)^\nu \cdot c_\nu \tau^r$$

$$+ \frac{1}{16} \lim_{K\to\infty} \left\{(-1)^\nu \tau^r S_K(-1/\tau) - (-1)^\nu \sum_{\substack{k=1 \\ k\equiv 2(4)}}^{2K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i\nu m'/k)\right.$$

$$\left. \cdot (m\tau + k/2)^r \cdot \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{-2\pi i\nu\tau}{k(k/2 + m\tau)}\right)^p\right\}.$$

Comparing this with (5.03) and using (5.07) we have

$$\lambda_\nu(\tau) - (-1)^\nu \tau^r \lambda_\nu(-1/\tau) = c_\nu(1 - (-1)^\nu \tau^r)$$

$$+ \frac{1}{16} \lim_{K\to\infty} \sum_{\substack{k=1 \\ k\equiv 2(4)}}^{2K} \sum_{\substack{|m|=1 \\ (m,k)=1}}^{K} \exp(-2\pi i\nu m'/k)$$

$$\cdot \left\{(-1)^\nu (m\tau + k/2)^r \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{-2\pi i\nu\tau}{k(k/2 + m\tau)}\right)^p \right. \tag{5.08}$$

$$\left. - (k\tau/2 - m)^r \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i\nu}{k(k\tau/2 - m)}\right)^p\right\}.$$

The right-hand side of (5.08) is a polynomial in τ of degree at most r. Let $r = 0$. Then (5.08) becomes $\lambda_\nu(\tau) - (-1)^\nu \lambda_\nu(-1/\tau) =$ constant, and in particular if ν is even, the right-hand side of (5.08) is zero, and we have $\lambda_\nu(-1/\tau) = \lambda_\nu(\tau)$. Thus for even ν, $\lambda_\nu(\tau)$ is an invariant for the group generated by $\tau' = \tau + 2$, $\tau' = -1/\tau$. This group contains $G(2)$. If ν is odd, we have $\lambda_\nu(\tau) + \lambda_\nu(-1/\tau) =$ constant. Since $\tau/(2\tau + 1) = -1/(-2 - 1/\tau)$, $\lambda_\nu(\tau/(2\tau + 1)) = \lambda_\nu(\tau)$, so that in any case $\lambda_\nu(\tau)$ is an invariant for $G(2)$. In the case $r = 0$ and $\nu = 1$, (5.01) reduces to the Fourier expansion of $\lambda(\tau)$ as given by Simons in [6]. Thus we have shown directly from the Fourier series expansion that $\lambda(\tau)$ is invariant with respect to $G(2)$.

If r is a *positive* even integer, denote the right-hand side of (5.08) by $\rho_\nu(\tau)$. We can now state the following theorem.

THEOREM (5.09). *If* $\lambda_\nu(\tau)$ *is defined as in* (5.01), *then in* $\mathcal{I}(\tau) > 0$, $\lambda_\nu(\tau)$ *is regular and satisfies*

$$\lambda_\nu(\tau) - (-1)^\nu \tau^r \lambda_\nu(-1/\tau) = \rho_\nu(\tau), \tag{5.10}$$

where $\rho_\nu(\tau)$ *is a polynomial of degree at most* r.

It is an immediate consequence of (5.10) that

$$\lambda_\nu(\tau) - (2\tau + 1)^r \lambda_\nu(\tau/(2\tau + 1)) = \rho_\nu(\tau) + (-1)^\nu \tau^r \rho_\nu((-2\tau - 1)/\tau),$$

which is again a polynomial of degree at most r. Thus we have the following result.

COROLLARY (5.11). *Let*

$$\Lambda(\tau) = a_0 + \textstyle\sum_{\nu=1}^{\mu} a_{-\nu} \lambda_\nu(\tau), \tag{5.12}$$

with $a_0, a_{-1}, \cdots, a_{-\mu}$ *constants and* $\lambda_\nu(\tau)$ *as in* (5.01). *Then in* $\mathcal{I}(\tau) > 0$, $\Lambda(\tau)$ *is regular and satisfies*

$$\Lambda(\tau) - (2\tau + 1)^r \Lambda(\tau/(2\tau + 1)) = \rho(\tau), \tag{5.13}$$

where $\rho(\tau)$ *is a polynomial of degree at most* r.

We now apply Corollary (5.11) to construct automorphic forms $\Lambda(\tau)$ of dimension r for $G(2)$. That is, we choose μ sufficiently large and $a_0, a_{-1}, \cdots, a_{-\mu}$ in such a fashion that $\rho(\tau) \equiv 0$, exactly as we did in Section III, where we constructed forms for the full modular group. Then,

$$\Lambda(\tau + 2) = \Lambda(\tau), \qquad (2\tau + 1)^r \Lambda(\tau/(2\tau + 1)) = \Lambda(\tau), \tag{5.14}$$

and since $\tau' = \tau + 2$, $\tau' = \tau/(2\tau + 1)$ generate $G(2)$, $\Lambda(\tau)$ is an automorphic form of dimension r for $G(2)$. Now, in contrast to the functions $F(\tau)$ and $F_l(\tau)$, previously constructed, $\Lambda(\tau)$ has no principal part at ∞, and it is not immediately clear that $\Lambda(\tau)$ is not identically zero. However, the following result implies that $\Lambda(\tau)$ cannot reduce to a constant, so that the forms we have constructed are nontrivial.

PROPOSITION (5.15). *Let* b_n *be the* n^{th} *Fourier coefficient of the function* $\Lambda(\tau)$ *defined by* (5.12). *Then, as* $n \to +\infty$

$$b_n \sim a_{-\mu} \cdot \mu^{r/2+1/4} \frac{\exp(\pi i(\mu + n)) \exp(2\pi(n\mu)^{1/2})}{32 n^{r/2+3/4}}.$$

Proof. It is immediate from (5.12) that $b_n = \sum_{\nu=1}^{\mu} a_{-\nu}\, a_n(\nu)$, for $n > 0$, where $a_n(\nu)$ is defined by (5.01). The result is now a trivial consequence of Proposition (5.02).

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