TWO-POINT BOUNDARY PROBLEMS INVOLVING A PARAMETER LINEARLY

BY

HYMAN J. ZIMMERBERG[1]

1. Introduction

The present paper is concerned with the extension of the concepts of adjointness, normality, symmetrizability, and definiteness of Bliss [1], [2] and Reid [4], [6] to linear differential systems, written in vector form

$$(1.1)\qquad \begin{aligned} y' - A(x)y &= \lambda B(x)y, \qquad a \leqq x \leqq b, \\ (M_0 + \lambda M_1)y(a) &+ (N_0 + \lambda N_1)y(b) = 0. \end{aligned}$$

In addition, the hypotheses imposed on the coefficients of the boundary conditions are analyzed, and necessary and sufficient conditions for testing these assumptions for individual problems (1.1) are developed. For real-valued coefficients, Bobonis [3] has extended the class of definite problems introduced by Bliss [1] and [2] to problems (1.1) in which corresponding assumptions on the boundary conditions are postulated. As for problems with boundary conditions not involving the parameter, the extension of the definite classes of Reid [4] to problems (1.1) also yields further results for the definite problems of Bobonis.

Section 2 introduces the basic assumptions made on the coefficients of (1.1), and a simple necessary and sufficient test for the conditions imposed on the boundary conditions both in this paper and by Bobonis [3] to hold is given. Adjoint boundary problems and their basic interrelations with the original problem are also discussed. In Section 3 the equivalence of two boundary problems of the form (1.1) under a nonsingular transformation is discussed; in particular, the equivalence of a problem with its adjoint. A problem (1.1) will be termed abnormal if there exist nontrivial vectors $y(x)$ satisfying

$$y' - A(x)y \equiv 0 \quad \text{and} \quad B(x)y \equiv 0 \qquad \text{on } ab,$$

$$M_0 y(a) + N_0 y(b) = 0 \quad \text{and} \quad M_1 y(a) + N_1 y(b) = 0,$$

and otherwise normal. For abnormal problems (1.1) equivalent to their adjoint under a nonsingular skew-hermitian transformation, Theorem 6.1 of Reid [6] is extended in Section 4 to establish the existence of an equivalent normal problem also equivalent to its adjoint under the same transformation

Received October 19, 1959.

[1] This paper contains, in part, results obtained in the author's doctoral dissertation written under the guidance of Professor W. T. Reid. That portion of the results was presented to the American Mathematical Society on September 15, 1945; abstract no. 94, Bull. Amer. Math. Soc., vol. 51 (1945), p. 369. Subsequent results were presented to the American Mathematical Society on June 18, 1960.

and satisfying the boundary assumptions of Section 2. The extension of the concept of symmetrizability to problems (1.1) is discussed in Section 5, and the results of Reid [6] are generalized to show that for an abnormal problem (1.1) equivalent to its adjoint under a nonsingular transformation, there exist an associated nonsingular skew-hermitian transformation and an equivalent normal problem symmetrizable under the associated transformation.

The extension of the classes of definite problems of Bobonis [3] and Reid [4] to problems (1.1) with complex-valued coefficients is developed in Section 6. In particular, for normal definite problems (1.1) we have the reality and the equality of index and multiplicity of proper values, existence theorems, and completeness and extremizing properties of the proper values and solutions. Moreover, for abnormal definite problems (1.1) there are shown to exist corresponding equivalent normal definite problems. Consequently, results for abnormal definite problems (1.1) follow from the application of the above results to the associated normal definite problem.

Matrix notation will be employed throughout this paper. The $n \times n$ identity matrix will be designated by E, while M^* shall denote the conjugate transpose of the matrix M. Vectors are treated as $n \times 1$ matrices, with (y, z) denoting the inner product z^*y of two n-dimensional vectors. In addition, let $\langle y, z\rangle$ denote $\int_a^b (y, z)\, dx$ for a pair of vectors $y(x)$, $z(x)$ for which (y, z) is integrable on ab.

2. Adjoint boundary problems

In the following it will be assumed that the elements of the $n \times n$ matrices $A(x)$ and $B(x)$ are complex-valued continuous functions of the real variable x on $a \leqq x \leqq b$, $B(x) \not\equiv 0$ on the interval, and the $n \times 2n$ matrix $\| M_0 + \lambda M_1 \quad N_0 + \lambda N_1 \|$ has rank n for every complex value of λ. The elements of the $n \times n$ coefficient matrices M_0, N_0, M_1, and N_1 may be complex-valued. The boundary problem under consideration is

$$\begin{aligned} L[y] &\equiv y' - A(x)y = \lambda B(x)y, \qquad a \leqq x \leqq b, \\ s[y; \lambda] &\equiv M(\lambda)y(a) + N(\lambda)y(b) = 0, \end{aligned} \tag{2.1}$$

with $M(\lambda) \equiv M_0 + \lambda M_1$, $N(\lambda) \equiv N_0 + \lambda N_1$.

If $n \times n$ matrices $P(\lambda)$, $Q(\lambda)$ are such that the $n \times 2n$ matrix $\| P^*(\bar\lambda) \quad Q^*(\bar\lambda) \|$, where $P^*(\bar\lambda) \equiv [P(\bar\lambda)]^*$, $Q^*(\bar\lambda) \equiv [Q(\bar\lambda)]^*$, is of rank n for all λ, and if, furthermore, they satisfy

$$M(\lambda)P(\lambda) - N(\lambda)Q(\lambda) \equiv 0 \qquad \text{for all } \lambda, \tag{2.2}$$

then the problem

$$\begin{aligned} L^\star[z] &\equiv z' + A^*(x)z = -\lambda B^*(x)z, \qquad a \leqq x \leqq b, \\ t[z; \lambda] &\equiv P^*(\bar\lambda)z(a) + Q^*(\bar\lambda)z(b) = 0 \end{aligned} \tag{2.3}$$

will be termed the boundary problem *adjoint* to (2.1).

Concerning the boundary conditions $s[y; \lambda] = 0$ *it will be assumed* throughout that there exist constant matrices M_2, N_2, P_2, and Q_2 such that for all values of λ the $2n \times 2n$ matrices

$$(2.4)\qquad \begin{Vmatrix} M(\lambda) & N(\lambda) \\ M_2 & N_2 \end{Vmatrix}, \qquad \begin{Vmatrix} -P_2 & -P(\lambda) \\ Q_2 & Q(\lambda) \end{Vmatrix}$$

are reciprocals. It is to be noted that this matrix hypothesis is also employed in [3]. The following result shows that $P(\lambda)$ and $Q(\lambda)$ must then necessarily be linear in λ.

THEOREM 2.1. *A necessary and sufficient condition that there exist matrices* $P(\lambda)$, $Q(\lambda)$ *and constant matrices* M_2, N_2, P_2, Q_2 *such that the matrices* (2.4) *are reciprocals is that the* $2n \times 2n$ *matrix*

$$(2.5)\qquad \begin{Vmatrix} M_0 & N_0 \\ M_1 & N_1 \end{Vmatrix}$$

have rank $n + \rho$, *where* ρ *is the rank of the* $n \times 2n$ *matrix* $\| M_1 \quad N_1 \|$. *Moreover, in this case* $P(\lambda)$ *and* $Q(\lambda)$ *must be linear in* λ.

If the matrices (2.4) are reciprocals, there exists a matrix V of rank ρ such that $M_1 = VM_2$, $N_1 = VN_2$. Then, if $(\breve{\xi}, \breve{\eta})$ is a $1 \times 2n$ vector orthogonal to each column of matrix (2.5),

$$0 = \breve{\xi}M_0 + \breve{\eta}M_1 = \breve{\xi}M_0 + \breve{\eta}VM_2, \qquad 0 = \breve{\xi}N_0 + \breve{\eta}N_1 = \breve{\xi}N_0 + \breve{\eta}VN_2,$$

and it follows that $\breve{\xi} = 0$, $\breve{\eta}V = 0$. Hence, $\breve{\eta}M_1 = \breve{\eta}N_1 = 0$, and, consequently, (2.5) has rank $n + \rho$.

Furthermore, as

$$(2.6)\qquad \begin{Vmatrix} M(\lambda) & N(\lambda) \\ M_2 & N_2 \end{Vmatrix} = \begin{Vmatrix} E & \lambda V \\ 0 & E \end{Vmatrix} \cdot \begin{Vmatrix} M_0 & N_0 \\ M_2 & N_2 \end{Vmatrix},$$

it follows, for the choices of P_0, Q_0, P_2, and Q_2 such that the matrices

$$(2.7)\qquad \begin{Vmatrix} M_0 & N_0 \\ M_2 & N_2 \end{Vmatrix}, \qquad \begin{Vmatrix} -P_2 & -P_0 \\ Q_2 & Q_0 \end{Vmatrix}$$

are reciprocals, that the reciprocal of (2.6) is

$$(2.8)\qquad \begin{Vmatrix} -P_2 & -P_0 \\ Q_2 & Q_0 \end{Vmatrix} \cdot \begin{Vmatrix} E & -\lambda V \\ 0 & E \end{Vmatrix} = \begin{Vmatrix} -P_2 & -(P_0 - \lambda P_2 V) \\ Q_2 & (Q_0 - \lambda Q_2 V) \end{Vmatrix};$$

and, thus, $P(\lambda)$ and $Q(\lambda)$ are necessarily linear in λ.

On the other hand, if the rank of (2.5) is $n + \rho$, let σ be a $\rho \times n$ matrix such that $\| \sigma M_1 \quad \sigma N_1 \|$ has rank ρ. Then there exist $(n - \rho) \times n$ matrices

μ, ν such that the $2n \times 2n$ matrix

$$\begin{Vmatrix} M_0 & N_0 \\ \sigma M_1 & \sigma N_1 \\ \mu & \nu \end{Vmatrix}$$

is nonsingular. On setting $M_2 = \begin{Vmatrix} \sigma M_1 \\ \mu \end{Vmatrix}$, $N_2 = \begin{Vmatrix} \sigma N_1 \\ \nu \end{Vmatrix}$, the first matrix of (2.7) is nonsingular, and there exists an $n \times n$ matrix V such that $M_1 = VM_2$, $N_1 = VN_2$. We then have the factorization (2.6), and the matrices (2.4) are reciprocals with the choice of the second matrix as (2.8), where the matrices (2.7) are reciprocals.

The condition that (2.5) have rank $n + \rho$ does not automatically hold whenever the $n \times 2n$ matrix $\| M_0 + \lambda M_1 \quad N_0 + \lambda N_1 \|$ has rank n for all λ. This may be seen, for example, for $n = 2$ from the choice of

$$M_0 = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix}, \quad N_0 = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \quad M_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \quad N_1 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$$

If we now set

$$s_i[y] \equiv M_i y(a) + N_i y(b), \qquad t_i[z] \equiv P_i^* z(a) + Q_i^* z(b) \qquad (i = 0, 1, 2)$$

it follows from the reciprocal character of (2.4) that

$$(2.9) \qquad \begin{aligned} (s_0[y], t_2[z]) + (s_2[y], t_0[z]) &= (y(b), z(b)) - (y(a), z(a)), \\ (s_1[y], t_2[z]) + (s_2[y], t_1[z]) &= 0 \end{aligned}$$

for arbitrary values $y(a)$, $y(b)$, $z(a)$, $z(b)$. Moreover,

$$(2.10) \qquad (L[y], z) + (y, L^\star[z]) \equiv (y, z)' \qquad \text{for } y, z \in C';$$

and, in particular, $\langle L[y], z\rangle + \langle y, L^\star[z]\rangle = 0$ for all vectors $y(x)$ of class C' satisfying $s[y; \lambda] = 0$ for a value λ and $z(x)$ of class C' if and only if $t[z; \bar{\lambda}] = 0$ for this value of λ. Consequently, with the further definitions of a *proper value* of (2.1) as a complex number λ for which there exist nonidentically vanishing solutions of (2.1), termed *proper solutions*, and the *index* of λ as the dimension $i(\lambda)$ of the linear space of all solutions of (2.1) for this value λ, we then have the following result from relations (2.9) and (2.10).

LEMMA 2.1. *A constant λ_0 is a proper value for* (2.1) *if and only if $\bar{\lambda}_0$ is a proper value for* (2.3) *of the same index.*

3. Equivalent boundary problems

Problem (2.1) will be said to be *equivalent* to a second boundary problem

$$(3.1) \qquad \begin{aligned} u' - A^0(x)u &= \lambda B^0(x)u, \qquad a \leqq x \leqq b, \\ M^0(\lambda)u(a) + N^0(\lambda)u(b) &= 0, \end{aligned}$$

with $M^0(\lambda)$, $N^0(\lambda)$ linear in λ, and coefficients satisfying the same conditions imposed on the corresponding coefficient matrices of (2.1), under a transformation $u(x) = H(x)y(x)$, $a \leqq x \leqq b$, provided $H(x)$ is a nonsingular matrix of class C' on ab such that, for an arbitrary value λ, the vector $y(x)$ satisfies the differential equations or the boundary conditions of (2.1) if and only if the corresponding $u(x)$ satisfies the respective differential equations or boundary conditions of (3.1).

THEOREM 3.1. *The boundary problem* (2.1) *is equivalent to* (3.1) *under* $H(x)$ *if and only if* $H(x)$ *is a nonsingular matrix of class* C' *satisfying*

$$H' - A^0H + HA \equiv 0, \qquad HB - B^0H \equiv 0, \qquad a \leqq x \leqq b,$$

$$M^0(\lambda)H(a)P(\lambda) - N^0(\lambda)H(b)Q(\lambda) \equiv 0 \qquad \textit{for all } \lambda,$$

where $P(\lambda)$, $Q(\lambda)$ *are* $n \times n$ *matrices with the* $n \times 2n$ *matrix* $\| P^*(\bar{\lambda}) \quad Q^*(\bar{\lambda}) \|$ *of rank* n *and* $M(\lambda)P(\lambda) - N(\lambda)Q(\lambda) \equiv 0$ *for all* λ.

As a special case of the above theorem we have the following result, the final conclusion of which also appears in Theorem 4.1 of Reid [6].

THEOREM 3.2. *A necessary and sufficient condition that the system* (2.1) *be equivalent to its adjoint* (2.3) *under* $T(x)$ *is that*

$$T' + A^*T + TA \equiv 0, \qquad TB + B^*T \equiv 0, \qquad a \leqq x \leqq b, \tag{3.2}$$

$$M(\lambda)T^{-1}(a)M^*(\bar{\lambda}) \equiv N(\lambda)T^{-1}(b)N^*(\bar{\lambda}) \qquad \textit{for all } \lambda. \tag{3.3}$$

Moreover, the general solution of the matrix differential equation of (3.2) *is* $T(x) = Y^{*-1}(x)CY^{-1}(x)$, *where* $Y(x)$ *is a nonsingular matrix solution of* $L[Y] \equiv Y' - AY = 0$ *and* C *is an arbitrary* $n \times n$ *constant matrix.*

From the relation

$$P^*(\bar{\lambda})T(a)P(\lambda) \equiv Q^*(\bar{\lambda})T(b)Q(\lambda) \qquad \text{for all } \lambda, \tag{3.3'}$$

equivalent to (3.3), it follows that a system equivalent to its adjoint under $T(x)$ is also equivalent to its adjoint under $T^*(x)$ and under $T_1(x) = c_1T(x) + c_2T^*(x)$, provided $T_1(x)$ is nonsingular for some x_0 on ab. In addition, the Corollary to Theorem 4.2 of Reid [6] with $A_1^0(x) \equiv A_1(x) \equiv E$ is also valid for systems (2.1), (3.1) and their adjoints.

COROLLARY. *If the boundary conditions* $s[y;\lambda] = 0$ *are equivalent to their adjoint conditions* $t[z;\lambda] = 0$ *under* $z = T(x)y$, *then the condition that the matrix* (2.5) *have rank* $n + \rho$, *where* ρ *is the rank of* $\| M_1 \quad N_1 \|$, *is equivalent to the condition that the matrix*

$$W \equiv M_0T^{*-1}(a)M_1^* - N_0T^{*-1}(b)N_1^* \tag{3.4}$$

have rank ρ.

From relation (3.3) for $\lambda = 0$ it follows that $W\eta = 0$ for a vector η if and only if there exists a vector ξ such that $M_1^*\eta = M_0^*\xi$, $N_1^*\eta = N_0^*\xi$. In this case the rank of (2.5) is equal to n plus the rank of W, and, hence, (2.5) has rank $n + \rho$ if and only if W has rank ρ.

4. Normal and abnormal boundary problems

Let $\Lambda_{0,0}$ denote the linear space of vector functions $y(x)$ for which $L[y] \equiv 0$ and $By \equiv 0$ on ab, $s_0[y] = 0$ and $s_1[y] = 0$; correspondingly, let $\Lambda^{\star}_{0,0}$ denote the totality of vectors $z(x)$ satisfying $L^{\star}[z] \equiv 0$ and $B^*z \equiv 0$ on ab, $t_0[z] = 0$ and $t_1[z] = 0$. A boundary problem (2.1) will be termed *normal* if $\Lambda_{0,0}$ is zero-dimensional, and *abnormal* with order of abnormality r if $\dim \Lambda_{0,0} = r > 0$. A nontrivial element of $\Lambda_{0,0}$ will be designated an *abnormal solution* of (2.1). As all values of λ are proper values of (2.1) in case $\dim \Lambda_{0,0} = r > 0$, let $i_n(\lambda) = i(\lambda) - r$ denote the *normal index* of λ as a proper value of (2.1) in case $i(\lambda) > r$, and let a *normal proper solution* $y(x)$ be a proper solution of (2.1) for which not both $By \equiv 0$ on ab and $s_1[y] = 0$ hold.

In addition, let Λ_0 denote the linear space of vectors $y(x)$ satisfying $L[y] \equiv 0$ and $By \equiv 0$ on ab; and, similarly, $\Lambda^{\star}_0$ will denote the linear space of vectors $z(x)$ for which $L^{\star}[z] \equiv 0$ and $B^*z \equiv 0$ on ab. Then, from (2.2) and the reciprocal character of the matrices of (2.4), it follows that a pair of end values $y(a)$, $y(b)$ satisfies $s_0[y] = 0$ and $s_1[y] = 0$ if and only if there is a constant vector ξ such that $y(a) = P_0 K\xi$, $y(b) = -Q_0 K\xi$, where the $n \times n$ constant matrix K is of rank $n - \rho$ and satisfies $VK = 0$ with the constant $n \times n$ matrix V of rank ρ for which $M_1 = VM_2$, $N_1 = VN_2$. Thus, if $\rho = n$ for a problem (2.1) the problem is normal, while the case $\rho = 0$ is the class of problems studied in Reid [6]. Consequently, if $\dim \Lambda_0 = p > 0$ and η denotes an $n \times p$ matrix whose column vectors form a basis for Λ_0, then $\dim \Lambda_{0,0} = r \geqq 0$ is equivalent to the condition that the $2n \times (n + p)$ matrix

$$\left\| \begin{matrix} P_0 K & \eta(a) \\ -Q_0 K & \eta(b) \end{matrix} \right\|$$

is of rank $n - \rho + p - r$.

Finally, if Λ_1 denotes the linear space of vectors $y(x)$ for which there exists a vector $g(x)$ with continuous components such that $L[y] \equiv Bg$ on ab, it follows from (2.10) that

$$(4.1) \qquad z^*(a)y(a) - z^*(b)y(b) = 0 \qquad \text{for } y(x) \in \Lambda_1, \quad z(x) \in \Lambda^{\star}_0.$$

Now, if (2.1) is equivalent to its adjoint (2.3) under $T(x)$, then $y(x)$ belongs to Λ_0 or $\Lambda_{0,0}$ if and only if $z(x) = T(x)y(x)$ belongs to $\Lambda^{\star}_0$ or $\Lambda^{\star}_{0,0}$, respectively. In particular, if under such equivalence $\dim \Lambda_{0,0} = \dim \Lambda^{\star}_{0,0} = r > 0$ and $\eta(x)$ is an $n \times r$ matrix whose column vectors form a basis for $\Lambda_{0,0}$, then the columns of $\zeta(x) = T(x)\eta(x)$ form a basis for $\Lambda^{\star}_{0,0}$. Moreover, as $M_0 P_1 - N_0 Q_1 = -M_1 P_0 + N_1 Q_0 = V(-M_2 P_0 + N_2 Q_0) = V$ from (2.2), it then follows from $t_0[\zeta] = t_1[\zeta] = 0$ and rank $V = \text{rank} \| M_1 \;\; N_1 \| = \rho$ that there exists an $n \times r$ constant matrix σ of rank r such that

$$\zeta^*(a) = \sigma^* J M_0, \qquad \zeta^*(b) = -\sigma^* J N_0,$$

where J is an $n \times n$ constant matrix of rank $n - \rho$ satisfying $JV = 0$, and σ^*J has rank $r \leqq n - \rho$. Hence, for τ an $n \times (n - r)$ constant matrix of rank $n - r$ such that $\sigma^*J\tau = 0$, the boundary conditions $s[y; \lambda] = 0$ are equivalent to

$$\tau^*s[y; \lambda] = 0,$$

$$\sigma^*Js[y; \lambda] \equiv \zeta^*(a)y(a) - \zeta^*(b)y(b) = 0.$$

In addition, as $\sigma^*JM_1 = \sigma^*JVM_2 = 0$, $\sigma^*JN_1 = \sigma^*JVN_2 = 0$ it follows from the nonsingularity of $\| \tau \quad J^*\sigma \|$ that the $(n - r) \times 2n$ matrix $\| \tau^*M_1 \quad \tau^*N_1 \|$ has rank ρ and its rows are linearly independent of the rows of $\| \tau^*M_0 \quad \tau^*N_0 \|$. Consequently, if θ and ϕ are $n \times r$ constant matrices such that the $r \times r$ matrix $\theta^*\eta(a) + \phi^*\eta(b)$ is nonsingular, then the boundary problem

$$L[y] = \lambda By, \qquad \tau^*s[y; \lambda] = 0, \qquad \theta^*y(a) + \phi^*y(b) = 0 \tag{4.2}$$

is a normal problem whose boundary conditions satisfy the matrix hypotheses of Section 2. Furthermore, as relation (4.1) implies that $\sigma^*Js[y; \lambda] = 0$ for any proper solution $y(x)$ of $L[y] = \lambda By$, problem (4.2) is equivalent to (2.1) in the sense that, if $y(x)$ is a proper solution of (4.2) for a value λ, then $y(x)$ is a normal solution of (2.1) for this value λ, while if $y(x)$ is a solution of (2.1) for a value λ, then

$$y(x) + \eta(x)\gamma, \qquad \gamma = -[\theta^*\eta(a) + \phi^*\eta(b)]^{-1}\cdot[\theta^*y(a) + \phi^*y(b)],$$

is a solution of (4.2) for the same value λ. Finally, λ is a proper value of (4.2) of index k if and only if λ is a proper value of (2.1) with normal index $i_n(\lambda) = k$.

THEOREM 4.1. *If* (2.1) *is an abnormal problem equivalent to its adjoint* (2.3) *under a nonsingular skew-hermitian transformation* $T(x)$ *and the matrices* (2.4) *are reciprocals, then there exists an equivalent normal problem,* (4.4) *below, that is also equivalent to its adjoint under the same* $T(x)$ *and for which the matrices corresponding to* (2.4) *are reciprocals.*

For a problem (2.1) equivalent to its adjoint under $T(x)$ one may choose

$$P(\lambda) = T^{*-1}(a)M^*(\bar{\lambda}), \qquad Q(\lambda) = T^{*-1}(b)N^*(\bar{\lambda}) \tag{4.3}$$

in view of (3.3). Then, for $\eta(x)$, J, σ, and τ as above and

$$R \equiv -\tfrac{1}{2}[M_2 T^{-1}(a)M_2^* - N_2 T^{-1}(b)N_2^*],$$

$\sigma^*J(s_2[\eta] - Rs_0[\eta]) = -\sigma^*J(\sigma^*J)^*$ is nonsingular, and, hence, the boundary problem

$$\begin{aligned} L[y] &= \lambda By, \\ \tau^*s[y; \lambda] &= 0, \\ \sigma^*J(s_2[y] - Rs_0[y]) &= 0 \end{aligned} \tag{4.4}$$

is a normal problem equivalent to (2.1). Furthermore, if $T(x)$ is skew-hermitian on ab, then R is also skew-hermitian, while the matrix W given by (3.4) is hermitian in view of (3.3). Consequently, as $W = V$ and $0 = VR = WR = RW$ from the choice in (4.3), we have, by direct computation, that the boundary conditions of (4.4) satisfy with $T(x)$ a relation corresponding to (3.3), and, hence, problem (4.4) is also equivalent to its adjoint under the same $T(x)$ as the original problem. Moreover, for problem (4.4) it also follows from the choice (4.3) and $RW = 0$ that the matrix W_1 corresponding to the matrix W of problem (2.1) is of the form

$$W_1 = \begin{Vmatrix} \tau^* W \tau & 0 \\ 0 & 0 \end{Vmatrix}$$

and has rank equal to rank $W = \rho$ as $\sigma^* J W = \sigma^* J V = 0$ and the matrix $\| \tau \quad J^* \sigma \|$ is nonsingular.

5. Symmetrizable boundary problems

A problem (2.1) will be termed *symmetrizable* under $T(x)$ if it satisfies the matrix assumption of Section 2 that the matrices (2.4) are reciprocals, is equivalent to its adjoint (2.3) under $T(x)$, $S(x) \equiv T^*(x)B(x)$ is hermitian on ab, and the $2n \times 2n$ constant matrix

$$(5.1) \qquad \mathfrak{G} = \begin{Vmatrix} T^*(a)P_2 M_1 & T^*(a)P_2 N_1 \\ T^*(b)Q_2 M_1 & T^*(b)Q_2 N_1 \end{Vmatrix}$$

belonging to the bilinear form $\mathcal{G}[u; v] \equiv (s_1[v], t_2[Tu])$ is hermitian.

Now, for a boundary problem (2.1) equivalent to its adjoint (2.3) under $T(x)$, it follows from (3.3) that the most general form of $P(\lambda)$ and $Q(\lambda)$ is

$$(5.2) \quad P(\lambda) = T^{*^{-1}}(a)M^*(\bar{\lambda})C(\lambda), \qquad Q(\lambda) = T^{*^{-1}}(b)N^*(\bar{\lambda})C(\lambda),$$

where $C(\lambda)$ is nonsingular for all λ. From the reciprocal character of the matrices in (2.4) we then have that

$$(5.3) \qquad C^{-1}(\lambda) \equiv -M_2 T^{*^{-1}}(a)M^*(\bar{\lambda}) + N_2 T^{*^{-1}}(b)N^*(\bar{\lambda});$$

that is, $C^{-1}(\lambda)$ is linear in λ. Writing $C^{-1}(\lambda) = D_0 + \lambda D_1$, we have

$$(5.4) \qquad D_1 = -M_2 T^{*^{-1}}(a)M_1^* + N_2 T^{*^{-1}}(b)N_1^*.$$

LEMMA 5.1. *Suppose that the boundary conditions* $s[y; \lambda] = 0$ *are equivalent to their adjoint conditions* $t[z; \lambda] = 0$ *under* $z = T(x)y$ *and that the rank of* (2.5) *is* $n + \rho$, *where* ρ *is the rank of* $\| M_1 \quad N_1 \|$. *Then the matrix* (5.1) *is hermitian if and only if the matrix* $C(\lambda)$ *defined by* (5.2) *is independent of* λ *and the matrix* W *given by* (3.4) *is hermitian. Moreover, under these conditions* $P(\lambda)$ *and* $Q(\lambda)$ *can be chosen as in* (4.3).

Under the one-to-one transformation between values $u(a)$, $u(b)$ and constant vectors ξ, η

$$(5.5)\quad T(a)u(a) = M_0^*\xi + M_2^*\eta, \qquad -T(b)u(b) = N_0^*\xi + N_2^*\eta,$$

we have from the reciprocal character of the matrices (2.4) that $t_2^*[Tu] = -\xi^*$ and $s_1[u] = W^*\xi - D_1^*\eta$, where D_1 is given by (5.4). Now, if ξ_α, η_α $(\alpha = 1, 2)$ are arbitrary constant vectors and $u_\alpha(a)$, $u_\alpha(b)$ $(\alpha = 1, 2)$ corresponding sets of values related by (5.5), then

$$\overline{\mathcal{G}[u_2\,;\,u_1]} - \mathcal{G}[u_1\,;\,u_2] = \xi_1^*(W^* - W)\xi_2 + \eta_1^* D_1\,\xi_2 - \xi_1^* D_1^*\eta_2\,.$$

Thus, $\mathcal{G}[u; v]$ is hermitian if and only if $W = W^*$ and $D_1 = 0$. Now, as the matrices (2.4) are reciprocals from Theorem 2.1, the corresponding matrices obtained on replacing $P(\lambda)$, $Q(\lambda)$, M_2, and N_2 by $P(\lambda)D_0$, $Q(\lambda)D_0$, $D_0^{-1}M_2$, and $D_0^{-1}N_2$, respectively, are also reciprocals. Hence, without loss of generality, we may choose $P(\lambda)$ and $Q(\lambda)$ of the form (4.3).

COROLLARY. *Under the conditions of Lemma* 5.1 *the hermitian form* $\mathcal{G}[u; v]$ *has the representation*

$$(5.6)\qquad \mathcal{G}[u; v] = -(Wt_2[Tv], t_2[T\bar{u}]).$$

From $D_1 = 0$ it follows that there exists an $n \times n$ constant matrix F such that $T^{*^{-1}}(a)M_1^* = P_2F$, $T^{*^{-1}}(b)N_1^* = Q_2F$, and from $-M_0P_2 + N_0Q_2 = E$ we have that $F = -W$. Consequently, $s_1[v] = -Wt_2[Tv]$ as W is hermitian.

THEOREM 5.1. *If the boundary conditions* $s[y; \lambda] \doteq 0$ *satisfy* (3.3) *with a nonsingular* $T(x)$, *then necessary and sufficient conditions that there exist matrices* $P(\lambda) = P_0 + \lambda P_1$, $Q(\lambda) = Q_0 + \lambda Q_1$ *and constant matrices* M_2, N_2, P_2, Q_2 *such that the matrices* (2.4) *are reciprocals and the* $2n \times 2n$ *matrix* (5.1) *is hermitian are that the matrix* W *given in* (3.4) *be hermitian and of rank* ρ, *the rank of* $\| M_1 \quad N_1 \|$.

The necessity follows at once from Theorem 2.1, Lemma 5.1, and the Corollary to Lemma 3.1. To establish the sufficiency, let σ be a $r \times n$ matrix such that σW is of rank ρ. As $\sigma W^* = \sigma W$, it follows that $\| \sigma M_1 \quad \sigma N_1 \|$ has rank ρ, and if τ is a $(n - \rho) \times n$ matrix, of rank $n - \rho$, such that $\tau W = 0$, there exist $(n - \rho) \times n$ matrices μ, ν such that $\mu T^{*^{-1}}(a)M_1^* - \nu T^{*^{-1}}(b)N_1^* = 0$ while the $(2n - \rho) \times 2n$ matrix

$$(5.7)\qquad \left\| \begin{matrix} \tau M_0 & \tau N_0 \\ \sigma M_1 & \sigma N_1 \\ \mu & \nu \end{matrix} \right\|$$

is of rank $2n - \rho$. The rows of $\| \sigma M_0 \quad \sigma N_0 \|$ are linearly independent of the rows of (5.7), for else a nonnull linear combination of its rows, $\| \check{\mathbf{x}} \quad \check{\mathbf{y}} \|$, would be dependent on the rows of (5.7) and would, therefore, satisfy $\check{\mathbf{x}}T^{*^{-1}}(a)M_1^* - \check{\mathbf{y}}T^{*^{-1}}(b)N_1^* = 0$, implying that the rows of σW are linearly dependent. Moreover, if we set $M_2^1 = \left\| \begin{matrix} \sigma M_1 \\ \mu \end{matrix} \right\|$, $N_2^1 = \left\| \begin{matrix} \sigma N_1 \\ \nu \end{matrix} \right\|$, then $M_2^1 T^{*^{-1}}(a) M_1^* - N_2^1 T^{*^{-1}}(b) N_1^* = 0$, and $\left\| \begin{matrix} M_0 & N_0 \\ M_2^1 & N_2^1 \end{matrix} \right\|$ is nonsingular as $\left\| \begin{matrix} \tau \\ \sigma \end{matrix} \right\|$ is nonsingular. Hence, there exists a matrix V^1 such that $M_1 = V^1 M_2^1$, $N_1 = V^1 N_2^1$, and $H \equiv -M_2^1 T^{*^{-1}}(a) M_0^* + N_2^1 T^{*^{-1}}(b) N_0^*$ is nonsingular. Consequently, if we define $M_2 = H^{-1} M_2^1$, $N_2 = H^{-1} N_2^1$, the first matrix of (2.7) is nonsingular, and there exists a matrix $V = V^1 H$ such that $M_1 = V M_2$, $N_1 = V N_2$, while $-M_2 T^{*^{-1}}(a) M^*(\bar{\lambda}) + N_2 T^{*^{-1}}(b) N^*(\bar{\lambda}) \equiv E$ for all λ. Now, let P_2 and Q_2 be determined by the relations

$$-M_0 P_2 + N_0 Q_2 = E, \qquad -M_2 P_2 + N_2 Q_2 = 0.$$

As $-M_1 P_2 + N_1 Q_2 = V(-M_2 P_2 + N_2 Q_2) = 0$, it then follows that for the further choices $P(\lambda) = T^{*^{-1}}(a) M^*(\bar{\lambda})$, $Q(\lambda) = T^{*^{-1}}(b) N^*(\bar{\lambda})$, the matrices (2.4) are reciprocals. The final desired conclusion on the hermitian character of (5.1) is then assured by Lemma 5.1.

An immediate consequence of the above result and relation (3.3) is that if the problem (2.1) is symmetrizable under $T(x)$, then (2.1) is also symmetrizable under $T^*(x)$. Moreover, under the assumption that the matrices (2.4) are reciprocals, the Corollary to Theorem 3.2 and Theorems 2.1 and 5.1 imply that (2.1) is symmetrizable under a skew-hermitian transformation $T(x)$ whenever (2.1) is equivalent to its adjoint (2.3) under such a $T(x)$, as the matrix W is then hermitian.

Theorem 5.3 of Reid [6] can now be extended.

THEOREM 5.2. *For a problem* (2.1) *equivalent to its adjoint* (2.3) *under* $T(x)$ *and satisfying the condition that the associated matrices* (2.4) *are reciprocals, there exist constants* c_1, c_2 *such that* (2.1) *is symmetrizable under* $T_1(x) \equiv c_1 T(x) + c_2 T^*(x)$ *and* $T_1(x)$ *is a nonsingular skew-hermitian transformation on ab. Moreover, if* (2.1) *is symmetrizable under* $T(x)$, *then for each such pair* c_1, c_2 *there is an associated nonzero real constant* k_1 *such that the matrix* $S_1(x) \equiv T_1^*(x) B(x)$ *and the form* $\mathcal{G}_1[u; v]$, *corresponding to* $\mathcal{G}[u; v]$, *satisfy* $S_1(x) \equiv k_1 S(x)$ *on ab and* $\mathcal{G}_1[u; v] = k_1 \mathcal{G}[u; v]$ *for arbitrary vectors* $u(a)$, $u(b)$, $v(a)$, $v(b)$.

In view of the remarks immediately prior to the theorem above, Theorem 2.1, and the remarks following Theorem 3.2, the first result follows as in the proof of the corresponding result of Theorem 5.3 of Reid [6]. Then, if (2.1) is symmetrizable under $T(x)$, it also follows, as in the proof of Theorem 5.3 of [6], on setting $A_1 = E$, that for any pair of constants c_1, c_2 such that (2.1) is symmetrizable under $T_1(x) = c_1 T(x) + c_2 T^*(x)$ with $T_1(x)$ nonsingular and skew-hermitian on ab, that $S_1(x) \equiv T_1^*(x) B(x) \equiv k_1 S(x)$ on ab for $k_1 = c_1 - c_2$ a nonzero real constant. Now, as in the proof of Theorem 5.1 above, $P_1(\lambda) = T_1^{*^{-1}}(a) M^*(\bar{\lambda})$, $Q_1(\lambda) = T_1^{*^{-1}}(b) N^*(\bar{\lambda})$ may be chosen as the matrices corresponding to $P(\lambda) = T^{*^{-1}}(a) M^*(\bar{\lambda})$, $Q(\lambda) = T^{*^{-1}}(b) N^*(\bar{\lambda})$

and, hence, in view of relation (3.3) for both T and T_1, it follows that there exists an $n \times n$ matrix $C(\lambda)$, nonsingular for all λ, such that

$$P(\lambda) \equiv P_1(\lambda)C(\lambda), \qquad Q(\lambda) \equiv Q_1(\lambda)C(\lambda).$$

Thus,

$$\begin{aligned} M^*(\bar{\lambda})C(\lambda) &= T_1^*(a)T^{*^{-1}}(a)M^*(\bar{\lambda}) \\ &= T(a)[\bar{c}_1 T^{-1}(a) + \bar{c}_2 T^{*^{-1}}(a)]M^*(\bar{\lambda}), \\ N^*(\bar{\lambda})C(\lambda) &= T_1^*(b)T^{*^{-1}}(b)N^*(\bar{\lambda}) \\ &= T(b)[\bar{c}_1 T^{-1}(b) + \bar{c}_2 T^{*^{-1}}(b)]N^*(\bar{\lambda}). \end{aligned} \tag{5.8}$$

On multiplying the first equation of (5.8) on the left by $-WP_2^*$, and the second on the left by WQ_2^*, and adding, it then follows from (3.3), the reciprocal property of the matrices (2.4), the hermitian character of W, and the relations $M_1 = -WP_2^*T(a)$, $N_1 = -WQ_2^*T(b)$, established in the proof of the Corollary to Lemma 5.1, that

$$WC(\lambda) = (\bar{c}_1 - \bar{c}_2)W = k_1 W.$$

Moreover,

$$\begin{aligned} W = M_0 P_1 - N_0 Q_1 &= M_0 P(\lambda) - N_0 Q(\lambda) \\ &= [M_0 P_1(\lambda) - N_0 Q_1(\lambda)]C(\lambda) = W_1 C(\lambda), \end{aligned}$$

where W_1 designates for the transformation $T_1(x)$ the matrix corresponding to W. Consequently,

$$W = k_1 WC^{-1}(\lambda) = k_1 W_1. \tag{5.9}$$

Finally, if P_2^1, Q_2^1, $t_2^1[u]$, and $\mathcal{G}_1[u; v]$ denote the matrices and forms for the transformation $T_1(x)$ corresponding to P_2, Q_2, $t_2[u]$, and $\mathcal{G}[u; v]$, respectively, then

$$-M_1 = W_1 P_2^{1*} T_1(a) = WP_2^*T(a), \qquad -N_1 = W_1 Q_2^{1*} T_1(b) = WQ_2^*T(b)$$

and, hence, $W_1 t_2^1[T_1 u] = Wt_2[Tu]$ for arbitrary vectors $u(a)$, $u(b)$. It now follows from the representation (5.6) and relation (5.9) that

$$\begin{aligned} \mathcal{G}_1[u; v] &= -(W_1 t_2^1[T_1 v], t_2^1[T_1 u]) = -(Wt_2[Tv], t_2^1[T_1 u]) \\ &= -k_1(t_2[Tv], W_1 t_2^1[T_1 u]) = -k_1(t_2[Tv], Wt_2[Tu]) = k_1 \mathcal{G}[u; v] \end{aligned}$$

for arbitrary end values $u(a)$, $u(b)$, $v(a)$, $v(b)$.

It is to be noted that the reality and nonvanishing of $k_1 = c_1 - c_2$ allows the representation $c_1 = \alpha + i\gamma$, $c_2 = \beta + i\gamma$, α and β real and distinct, while the skew-hermitian character of $T_1(x)$ implies that $(\alpha + \beta)T(x)$ is skew-hermitian on ab. In particular, if $\alpha + \beta = c_1 + \bar{c}_2 \neq 0$ for a suitable pair c_1, c_2 above, then $T(x)$ is also skew-hermitian on ab, and

$$T_1(x) \equiv (c_1 - c_2)T(x) \equiv k_1 T(x)$$

on ab.

Combining the results of Theorems 4.1 and 5.2 with the argument below yields the following extension of Theorem 6.2 of Reid [6].

THEOREM 5.3. *If* (2.1) *is an abnormal problem equivalent to its adjoint under* $T(x)$, *and if the associated matrices* (2.4) *are reciprocals, then for an associated transformation* $T_1(x)$ *of Theorem* 5.2 *there is an equivalent normal problem that is symmetrizable under* $T_1(x)$; *moreover, if the original problem* (2.1) *is symmetrizable under* $T(x)$, *then for each such* $T_1(x)$ *there is a nonzero real constant* k_1 *such that the corresponding matrix* $S_1(x)$ *and the corresponding form* $\mathcal{G}_1[u; v]$ *satisfy* $S_1(x) \equiv k_1 S(x)$ *on ab and* $\mathcal{G}_1[u; v] = k_1 \mathcal{G}[u; v]$ *for arbitrary vectors* $u(a)$, $u(b)$, $v(a)$ *and* $v(b)$.

To establish the final conclusion we shall show that the matrix (5.1) remains invariant when we pass from an abnormal problem (2.1) to its equivalent normal problem (4.4), with each problem equivalent to its adjoint under a skew-hermitian, nonsingular transformation $T(x)$. Let the superscript 1 following a matrix associated with problem (2.1) denote the corresponding matrix for the problem (4.4). Then, from Lemma 5.1 and the comments prior to Theorem 5.2 we may choose $P^1(\lambda) = T^{*-1}(a)M^{1*}(\bar{\lambda})$, $Q^1(\lambda) = T^{*-1}(b)N^{1*}(\bar{\lambda})$. Moreover, with R, J, σ, and τ as in the proof of Theorem 4.1, $P(\lambda)$ and $Q(\lambda)$ as in (4.3), and the choices

$$\left\| \begin{matrix} P_2^1 \\ Q_2^1 \end{matrix} \right\| = \left\| \begin{matrix} P_2 & P_0 \\ Q_2 & Q_0 \end{matrix} \right\| \cdot \left\| \begin{matrix} E \\ R + J^*\sigma\sigma^*J \end{matrix} \right\| \cdot \left\| \begin{matrix} \tau^* \\ \varepsilon\sigma^*J \end{matrix} \right\|^{-1},$$

$$\| M_2^1 \quad N_2^1 \| = \left\| \begin{matrix} \tau^*\tau & 0 \\ 0 & \varepsilon \end{matrix} \right\|^{-1} \cdot \left\| \begin{matrix} -\tau^*R & \tau^* \\ -\sigma^*J - \varepsilon^{-1}\sigma^*JR & \varepsilon^{-1}\sigma^*J \end{matrix} \right\| \cdot \left\| \begin{matrix} M_0 & N_0 \\ M_2 & N_2 \end{matrix} \right\|,$$

where ε is the $r \times r$ nonsingular matrix $\sigma^*JJ^*\sigma$, it follows by direct calculation, in view of the relations $V = W = W^*$, $0 = JV = WJ^*$, $R = -R^*$, $0 = VR = RV$, $\sigma^*J\tau = 0$, $0 = JM_1 = JN_1$, and $0 = RM_1 = RN_1$, that for the problem (4.4) the matrices corresponding to (2.4) are reciprocals. Furthermore, as

$$\| M_1^1 \quad N_1^1 \| = \| \tau \quad 0 \|^* \| M_1 \quad N_1 \| = \| \tau \quad J^*\sigma\varepsilon^* \|^* \| M_1 \quad N_1 \|$$

we then have that

$$\left\| \begin{matrix} P_2^1 \\ Q_2^1 \end{matrix} \right\| \cdot \| M_1^1 \quad N_1^1 \| = \left\| \begin{matrix} P_2 \\ Q_2 \end{matrix} \right\| \cdot \| M_1 \quad N_1 \|,$$

and, consequently, $\mathfrak{G}^1 = \mathfrak{G}$.

6. Definite boundary problems

For a boundary problem (2.1) let Λ denote the linear class of vectors $y(x)$ satisfying $L[y] \equiv Bg$ on ab and $s_0[y] + s_1[g] = 0$ with a continuous vector $g(x)$.

LEMMA 6.1. *For a problem* (2.1) *symmetrizable under* $T(x)$, *the bilinear functional*

$$\mathcal{J}[u; v] \equiv -(s_0[v], t_2[Tu]) + \langle L[v], Tu \rangle$$

is hermitian on Λ *in the sense that* $\mathcal{J}[u; v] = \overline{\mathcal{J}[v; u]}$ *for arbitrary vectors* u *and* v *of* Λ; *in particular,* $\mathcal{J}[u] \equiv \mathcal{J}[u; u]$ *is real-valued on the space* Λ.

For suppose that $u(x)$ and $v(x)$ belong to Λ with $g(x)$ and $h(x)$, respectively. Then, with the choice (4.3), $w \equiv T(x)u$ belongs to the corresponding space $\Lambda^\star$ for the adjoint problem (2.3) with the vector $T(x)g$; i.e., $L^\star[w] = -B^*Tg = -Sg$, $t_0[w] + t_1[Tg] = 0$. From relation (2.10) and the hermitian character of S it now follows that

$$\begin{aligned}\langle L[v], Tu\rangle - \overline{\langle L[u], Tv\rangle} &= \langle L[v], w\rangle - \langle v, Sg\rangle \\ &= \langle L[v], w\rangle + \langle v, L^\star[w]\rangle \\ &= (v(b), w(b)) - (v(a), w(a)).\end{aligned}$$

Moreover, from relations (2.9) and the hermitian character of (5.1) we have that

$$\begin{aligned}&-(s_0[v], t_2[Tu]) + \overline{(s_0[u], t_2[Tv])} \\ &\quad = (s_2[v], t_0[w]) - \overline{(s_1[g], t_2[Tv])} + (v(a), w(a)) - (v(b), w(b)) \\ &\quad = -(s_2[v], t_1[Tg]) - (s_1[v], t_2[Tg]) + (v(a), w(a)) - (v(b), w(b)) \\ &\quad = (v(a), w(a)) - (v(b), w(b)),\end{aligned}$$

and, thus, $\mathcal{J}[u; v] = \overline{\mathcal{J}[v; u]}$.

Now, with

$$\mathcal{K}[y] \equiv \mathcal{G}[y; y] + \langle Sy, y\rangle,$$

it follows that for a problem (2.1) symmetrizable under $T(x)$ the functional

$$\mathcal{I}[y; c_1, c_2; T] \equiv c_1\,\mathcal{J}[y] + c_2\,\mathcal{K}[y] \tag{6.1}$$

is real-valued for vectors $y \in \Lambda$ and arbitrary real constants c_1, c_2. The boundary problem (2.1) will be termed definite $[c_1, c_2; T]$ whenever (2.1) is symmetrizable under $T(x)$ and there exist real constants c_1, c_2 such that (6.1) is positive for arbitrary vectors $y(x) \in \Lambda$ unless $B(x)y(x) \equiv 0$ on ab and $s_1[y] = 0$.

For symmetrizable problems (2.1) with real coefficients, the condition of definiteness considered by Bobonis [3] is the positive semidefiniteness of $\mathcal{K}[y]$ for arbitrary continuous vectors. From Lemma 5.3 of [3] such problems are clearly definite $[c_1, c_2; T]$ with $c_1 = 0$, $c_2 = 1$. On the other hand, for a problem (2.1) definite $[c_1, c_2; T]$ with $c_1 \neq 0$, the associated problem obtained on replacing λ by $\lambda - c_2/c_1$ is definite either $[1, 0; T]$ or $[1, 0; -T]$ according as $c_1 > 0$ or $c_1 < 0$. A problem (2.1) that is normal and definite $[0, 1; T]$ may be treated by methods corresponding to those of Bobinis [3], while a problem that is normal and definite $[1, 0; T]$ may be handled by an extension of the methods employed by Reid [4] for the class of problems in which λ does not appear in the boundary conditions.

For a normal and definite $[c_1, c_2; T]$ problem (2.1), it follows, from the relation $\mathcal{J}[y] = \lambda\mathcal{K}[y]$ for a proper solution $y(x)$ corresponding to a proper value λ, that $\mathcal{K}[y] \neq 0$ for all proper solutions. By methods analogous to those employed in the proof of Theorem 4.2 of [3] it then follows that for such a problem (2.1) all proper values are real and at most denumerably infinite in number as they are the zeros of an entire function

$$\Delta(\lambda) \equiv \det [M(\lambda)Y(a;\lambda) + N(\lambda)Y(b;\lambda)],$$

where $Y(x;\lambda)$ denotes a fundamental matrix solution of $L[y] = \lambda B(x)y$ with elements entire functions of λ for fixed x on ab. Furthermore, the index of each proper value is equal to its multiplicity as a zero of $\Delta(\lambda)$, as may be established by a method analogous to that used in the proof of Theorem 5.2 of [3].

Now, if $\lambda = \lambda_0$ is not a proper value for a problem (2.1), it follows, by methods entirely analogous to those of Bliss [1, Section 5] for real-valued coefficients, that

$$G(x, t; \lambda_0) \equiv \tfrac{1}{2}Y(x;\lambda_0)\left[\frac{|x-t|}{x-t}E + D^{-1}(\lambda_0)\Omega(\lambda_0)\right]Y^{-1}(t;\lambda_0),$$

$$a \leqq x, t \leqq b, x \neq t,$$

with

$$D(\lambda_0) \equiv M(\lambda_0)Y(a;\lambda_0) + N(\lambda_0)Y(b;\lambda_0),$$

$$\Omega(\lambda_0) \equiv M(\lambda_0)Y(a;\lambda_0) - N(\lambda_0)Y(b;\lambda_0),$$

is the unique Green's matrix for the incompatible homogeneous system

$$L[y] - \lambda_0 B(x)y = 0, \qquad s[y;\lambda_0] = 0.$$

Furthermore, by an argument similar to the one employed by Bobonis [3, Section 6] we have, for arbitrary vectors $g(x)$ with components continuous on ab and arbitrary constant vectors h, that the nonhomogeneous problem

$$L[y] - \lambda_0 B(x)y = g(x), \qquad s[y;\lambda_0] = h$$

has a unique solution given by

$$y(x) = -[G(x, a;\lambda_0)P_2 + G(x, b;\lambda_0)Q_2]h + \int_a^b G(x, t;\lambda_0)g(t)\,dt,$$

where

$$G(a, a;\lambda_0) \equiv \lim_{x\to a^+} G(x, a;\lambda_0),$$

$$G(b, b;\lambda_0) \equiv \lim_{x\to b^-} G(x, b;\lambda_0).$$

Consequently, if λ_0 is not a proper value for a problem (2.1), it follows, on rewriting (2.1) in the form

$$L[y] - \lambda_0 B(x)y = (\lambda - \lambda_0)B(x)y, \qquad s[y;\lambda_0] = -(\lambda - \lambda_0)s_1[y],$$

that problem (2.1) is equivalent to the integral system

$$(6.2)\quad y(x) = (\lambda - \lambda_0)\{[G(x, a; \lambda_0) P_2 + G(x, b; \lambda_0) Q_2] s_1[y] + \int_a^b G(x, t; \lambda_0) B(t) y(t) dt\}.$$

The integral equation (6.2) is of the form of problems considered in [7, Section 8], wherein $H(x, t) \equiv G(x, t; \lambda_0)T^{*-1}(t)$, $S(x) \equiv T^*(x)B(x)$ and $\mathcal{G} \equiv \mathfrak{G}$. Moreover, such integral equations are equivalent to a system of $3n$ integral equations of Fredholm type, as indicated in [7, p. 387]. If (2.1) is normal and definite [1, 0; T], then λ_0 may be chosen as 0, while for a normal and definite [0, 1; T] problem (2.1), there exists a real constant λ_0 not a proper value of (2.1). For each of these normal and definite problems the results of Reid [5] on symmetrizable completely continuous linear transformations in Hilbert space provide, for the integral system equivalent to (6.2), results on the existence and extremizing properties of proper values, integral expansions of Hilbert type, and convergence results of associated Fourier series.

For a problem (2.1) that is definite [c_1, c_2 ; T] and abnormal, let

$$L[y] = \lambda By, \qquad s^1[y; \lambda] \equiv s_0^1[y] + \lambda s_1^1[y] = 0$$

be an equivalent normal problem that is symmetrizable under an associated nonsingular skew-hermitian transformation $T_1(x)$, as guaranteed by Theorem 5.3. If Λ^1 denotes the class of vectors $y(x)$ for which there exists a corresponding vector $g(x)$ with continuous components on ab such that $L[y] \equiv Bg$ on ab and $s_0^1[y] + s_1^1[g] = 0$, it follows from relation (4.1) and the discussion preceding Theorem 4.1 that $\Lambda^1 \subset \Lambda$, Λ denoting the corresponding class for the original problem (2.1). Moreover, if $\mathcal{J}_1$ and $\mathcal{G}_1$ denote the functionals for the problem $L[y] = \lambda By$, $s^1[y; \lambda] = 0$ corresponding to $\mathcal{J}$ and $\mathcal{G}$, respectively, for problem (2.1) it then follows from Theorem 5.3 that for an element $y \in \Lambda^1$ we have $\mathcal{J}[y; c_1, c_2; T] = \mathcal{J}_1[y; c_1/k_1, c_2/k_1; T_1]$, where k_1 is the nonzero real constant such that $T_1^*B \equiv k_1 T^*B$ on ab and $\mathcal{G}_1[y; y] = k_1 \mathcal{G}[y; y]$. Furthermore, if $y \in \Lambda$, then there is an abnormal solution y_0 of (2.1) such that $y^1 = y + y_0$ is an element of Λ^1 and

$$\mathcal{J}_1[y^1; c_1/k_1, c_2/k_1; T_1] = \mathcal{J}[y^1; c_1, c_2; T] = \mathcal{J}[y; c_1, c_2; T].$$

Consequently, the normal problem $L[y] = \lambda By$, $s^1[y; \lambda] = 0$ is definite [c_1/k_1, c_2/k_1 ; T_1], and results on the existence and extremizing properties of normal proper values, integral expansions of Hilbert type, and convergence in mean of associated generalized Fourier series in terms of normal proper solutions for the abnormal definite problem (2.1) follow from the application of the above results to the associated normal definite problem.

Bibliography

1. G. A. Bliss, *A boundary value problem for a system of ordinary linear differential equations of the first order*, Trans. Amer. Math. Soc., vol. 28 (1926), pp. 561–584.

2. ———, *Definitely self-adjoint boundary value problems*, Trans. Amer. Math. Soc., vol. 44 (1938), pp. 413–428.
3. A. Bobonis, *Differential systems with boundary conditions involving the characteristic parameter*, Contributions to the Calculus of Variations 1938–1941, University of Chicago Press, 1942, pp. (99)–(138).
4. W. T. Reid, *A new class of self-adjoint boundary value problems*, Trans. Amer. Math. Soc., vol. 52 (1942), pp. 381–425.
5. ———, *Symmetrizable completely continuous linear transformations in Hilbert space*, Duke Math. J., vol. 18 (1951), pp. 41–56.
6. ———, *A class of two-point boundary problems*, Illinois J. Math., vol. 2 (1958), pp. 434–453.
7. H. J. Zimmerberg, *Definite integral systems*, Duke Math. J., vol. 15 (1948), pp. 371–388.

Rutgers, The State University
New Brunswick, New Jersey