THE DUAL OF A SECONDARY COHOMOLOGY OPERATION

BY

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1. Introduction

In 1926, J. W. Alexander [3] showed that the homology groups of X determined the homology groups of $S^N - X$ whenever X was embedded in S^N. In [4], Peterson showed how the stable primary cohomology operations in X determine those of $S^N - X$. In this paper, we show how the stable secondary cohomology operations in X determine those of $S^N - X$.

Heuristically, if Φ is a secondary cohomology operation defined on the kernel of θ with values in the cokernel of θ' (i.e., corresponding to the relation $\theta'\theta = 0$), then the dual of Φ will be a secondary cohomology operation defined on the kernel of $\chi(\theta')$ with values in the cokernel of $\chi(\theta)$ (i.e., corresponding to the relation $\chi(\theta)\chi(\theta') = \chi(\theta'\theta) = 0$), where χ is the involution in the Steenrod algebra (see [4]). If Φ is nonzero in X, then the dual of Φ will be nonzero in $S^N - X$. As in [4], this will be used to prove "nonembedding theorems."

We will state and prove our theorem in the language of J. F. Adams ([1] or [2]). One of the key steps in the proof is the fact that the two formulas in [5] are "dual" to each other.

2. Secondary cohomology operations

In this section, we recall Adams's definition of stable secondary cohomology operations with coefficients Z_p, [2].

Let (d, z, m) be such that $d: C_1 \to C_0$, $z \in C_1$, $d(z) = 0$, where C_1 and C_0 are free graded modules over the Steenrod algebra A. Let c_0 and c_r, $r = 1, \cdots, R$, be bases for C_0 and C_1 respectively, and let $\deg c_0 = 0$, $\deg c_r = q(r)$, $\deg d = 0$, and $\deg z = n + 1$. Let

$$\varepsilon: C_0 \to H^*(X) \qquad (= \textstyle\sum_{q>0} H^q(X; Z_p))$$

be a map of degree m. Then Φ is a secondary cohomology operation associated with (d, z, m) if it satisfies the following four axioms:

1. $\Phi(\varepsilon)$ is defined if $\varepsilon d = 0$.
2. $\Phi(\varepsilon) \in H^{m+n}(X)/\sum_{r=1}^{R} \alpha_r H^{m+q(r)-1}(X)$, where $z = \sum_{r=1}^{R} \alpha_r c_r$.
3. If $f: X \to Y$, and $\varepsilon: C_0 \to H^*(Y)$, then
$$\Phi(f^*\varepsilon) = f^*\Phi(\varepsilon) \in H^{m+n}(X)/\textstyle\sum_{r=1}^{R} \alpha_r H^{m+q(r)-1}(X).$$
4. Let (X, Y) be a pair, and let $i: Y \to X$, and $j: X \to (X, Y)$ be the

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inclusion maps. If $\eta: C_0 \to H^*(X, Y)$ is such that $\Phi(j^*\eta)$ is defined, then

$$i^*\Phi(j^*\eta) = \{\zeta(z)\} \in H^{m+n}(Y)/i^* \textstyle\sum_{r=1}^{R} \alpha_r H^{m+q(r)-1}(X),$$

where $\zeta: C_1 \to H^*(Y)$ is such that $\delta\zeta = \eta d$.[1]

Let $\sigma: H^*(X) \to H^*(SX)$ be the suspension isomorphism. The formula $\sigma^{-1}\Phi(\sigma\varepsilon)$ defines a secondary cohomology operation associated with $(d, z, m - 1)$, called the suspension of Φ and denoted by $\tilde{\sigma}(\Phi)$.

$\{\Phi_m\}$ is a stable secondary cohomology operation associated with (d, z) if

(1) each Φ_m is a secondary cohomology operation associated with (d, z, m), and

(2) $\tilde{\sigma}(\Phi_m) = \Phi_{m-1}$ for all m.

The following form of axiom 4 will be useful in what follows: Let $f: Y \to X$, and let $M = X \cup_f CY$ be the "shrunk mapping cylinder." Let $g: X \to M$ be the inclusion, and let $h: M \to SY$ be defined by collapsing X to a point. Let $\eta: C_0 \to H^*(M)$ and $\zeta: C_1 \to H^*(Y)$ be such that $h^*\sigma\zeta = \eta d$. Then

$$f^*\Phi(g^*\eta) = \{\zeta(z)\} \in H^{m+n}(Y)/f^* \textstyle\sum_{r=1}^{R} \alpha_r H^{m+q(r)-1}(X).$$

This is obviously equivalent to axiom 4 with M taking the place of the pair (X, Y).

COROLLARY 2.1. *Let $\{\Phi_m\}$ be a stable secondary cohomology operation associated with (d, z). Let $\eta: C_0 \to H^*(X)$ and $\zeta: C_1 \to H^*(M)$ be such that $g^*\zeta = \eta d$. Then*

$$h^*\sigma\Phi_m(f^*\eta) = \{\zeta(z)\} \in H^{m+n+1}(M)/h^*\sigma \textstyle\sum_{r=1}^{R} \alpha_r H^{m+q(r)-1}(Y).$$

Proof. Apply the construction above to $h: M \to SY$. That is, let

$$\bar{M} = SY \cup_h CM.$$

$\bar{M}$ is of the same homotopy type as SX (see [6]). Applying the alternate form of axiom 4 to Φ_{m+1} and this $\bar{M}$ gives the corollary.

Remark. Corollary 2.1 is another formulation of Theorem 6.3 in [5], as axiom 4 is another formulation of Theorem 6.1 in [5].

3. Duality

In this section we recall some properties of Spanier-Whitehead duality [7].

Let $X \subset S^N$ be embedded as a subpolyhedron. Let $\mathfrak{D}X$ be an N-dual to X. Define

$$\nu: H^q(X) \to H^{N-q-1}(\mathfrak{D}X)^*$$

to be the composition of the Alexander and Pontrjagin duality isomorphisms.

[1] We eliminate the signs that occur in Adams's axioms by using as the coboundary operator in dimension i, $(-1)^i$ times the usual coboundary.

That is,

$$H^q(X) = H^q(X; Z_p) \to H_{N-q-1}(\mathfrak{D}X; Z_p) \to H^{N-q-1}(\mathfrak{D}X; Z_p)^*,$$

where V^* is the dual vector space of V. If $f: Y \to X$, and $\mathfrak{D}f: \mathfrak{D}X \to \mathfrak{D}Y$ (defined under suitable conditions), then

$$\begin{array}{ccc} H^q(X) & \xrightarrow{f} & H^q(Y) \\ \downarrow \nu & & \downarrow \nu \\ H^{N-q-1}(\mathfrak{D}X)^* & \xrightarrow{(\mathfrak{D}f)^*} & H^{N-q-1}(\mathfrak{D}Y)^* \end{array}$$

is a commutative diagram. (For convenience, we will denote by f the induced homomorphism $H^q(X) \to H^q(Y)$, and by f^* the dual homomorphism $H^q(Y)^* \to H^q(X)^*$.)

Let $\chi: A \to A$ be defined as in [4]. χ has the following properties:

(1) $\sum_{i=0}^{n} \chi(\mathrm{Sq}^{n-i})\mathrm{Sq}^i = 1$, the unit element of A, and

(2)
$$\begin{array}{ccc} H^q(X) & \xrightarrow{a} & H^{q+i}(X) \\ \downarrow \nu & & \downarrow \nu \\ H^{N-q-1}(\mathfrak{D}X)^* & \xrightarrow{\chi(a)^*} & H^{N-q-i-1}(\mathfrak{D}X)^* \end{array}$$

is a commutative diagram for $a \in A$.

We shall also need the following standard properties of Alexander and Pontrjagin dualities.

LEMMA 3.1. *Let* $X \subset S^N \subset S^{N'}$,

$$\sigma^{N'-N}: H^q(\mathfrak{D}_N X) \to H^{q+N'-N}(S^{N'-N}\mathfrak{D}_N X) = H^{q+N'-N}(\mathfrak{D}_{N'} X),$$

where $\mathfrak{D}_N X$ *is an* N*-dual of* X. *If* $u \in H^m(X)$, *then*

$$\nu_N(u) = (\sigma^{N'-N})^* \nu_{N''}(u) \in H^{N-m-1}(\mathfrak{D}_N X)^*.$$

LEMMA 3.2. *Let* $X \subset S^N \subset S^{N'}$. *If*

$$v \in H^m(\mathfrak{D}_N \mathfrak{D}_N X)^* = H^m(X)^* = H^m(\mathfrak{D}_{N'} \mathfrak{D}_{N'} X)^*,$$

then

$$\overset{-1}{\nu_N}(v) = (\sigma^{N'-N})^{-1} \overset{-1}{\nu_{N''}}(v) \in H^{N-m-1}(\mathfrak{D}_N X),$$

where $\sigma^{N'-N}: H^{N-m-1}(\mathfrak{D}_N X) \to H^{N'-m-1}(\mathfrak{D}_{N'} X)$.

4. The main theorem

We now define a dual $\chi(\Phi)$ for stable secondary cohomology operations analogous to $\chi(a)$.

Let $\{\Phi_m\}$ be a stable secondary cohomology operation associated with (d, z). Let $\tilde{C}_1$ be the free module over A on generators $\tilde{c}_r$, $r = 1, \cdots, R$, $\tilde{C}_0$ the free module over A on the generator $\tilde{c}_0$. Define $\tilde{d}:\tilde{C}_1 \to \tilde{C}_0$ by

$$\tilde{d}(\tilde{c}_r) = \chi(\alpha_r)\tilde{c}_0,$$

where we recall that $z = \sum_{r=1}^{R} \alpha_r c_r$. Let

$$\tilde{z} = \sum_{r=1}^{R} \chi(a_r)\tilde{c}_r \in \tilde{C}_1.$$

Then

$$\tilde{d}(\tilde{z}) = \sum_{r=1}^{R} \chi(a_r)\chi(\alpha_r)\tilde{c}_0 = \Big(\sum_{r=1}^{R} \chi(\alpha_r a_r)\Big)\tilde{c}_0 = \chi\Big(\sum_{r=1}^{R} \alpha_r a_r\Big)\tilde{c}_0 = 0.$$

Let $X \subset S^N$. Let $\varepsilon:\tilde{C}_0 \to H^*(X)$ be such that $\varepsilon\tilde{d} = 0$; that is, let $\varepsilon(\tilde{c}_0) = u$ be such that $\chi(\alpha_r)(u) = 0$ for all r. Let $v \in H^{m+n}(X)^* = H^{m+n}(\mathfrak{D}\mathfrak{D}X)^*$ be such that $\chi(a_r)^*(v) = 0 \in H^{m+n-q(r)}(X)^*$ for all r.

THEOREM 4.1. *The formula*

$$\langle \chi(\Phi)_m(u), v\rangle = \langle \Phi_{N-m-n-1}(\nu^{-1}(v)), \nu(u)\rangle$$

defines a stable secondary cohomology operation associated with $(\tilde{d}, \tilde{z})$.

Proof. $v \in \bigcap_{r=1}^{R} \operatorname{Ker}\big(H^{m+n}(X)^* \xrightarrow{\chi(a_r)^*} H^{m+n-q(r)}(X)^*\big)$, which has as dual

$$H^{m+n}(X)\Big/\sum_{r=1}^{R} \chi(a_r)H^{m+n-q(r)}(X).$$

$$\nu^{-1}(v) \in \bigcap_{r=1}^{R} \operatorname{Ker}\big(H^{N-m-n-1}(\mathfrak{D}X) \xrightarrow{a_r} H^{N-m-n+q(r)}(\mathfrak{D}X)\big).$$

Hence $\Phi_{N-m-n-1}(\nu^{-1}(v))$ is defined and is a member of

$$H^{N-m-1}(\mathfrak{D}X)\Big/\sum_{r=1}^{R} \alpha_r H^{N-m-n-2+q(r)}(\mathfrak{D}X),$$

which has as dual

$$\bigcap_{r=1}^{R} \operatorname{Ker}\big(H^{N-m-1}(\mathfrak{D}X)^* \xrightarrow{\alpha_r^*} H^{N-m-n-2+q(r)}(\mathfrak{D}X)^*\big).$$

$\nu(u)$ belongs to this latter group; hence our formula makes sense, and $\chi(\Phi)_m(u)$ is defined.

Axioms 1 and 2 follow immediately from the construction. The proof of axiom 3 will imply that $\chi(\Phi)_m(u)$ is independent of how X is embedded in S^N.

Let $f:X \to Y$, $X \subset S^N$, $Y \subset S^{N'}$. Assume that N'' is large enough so that $S^{N''} = S^{N''-N}(S^N) = S^{N''-N'}(S^{N'})$ contains X, Y, and the mapping cylinder of f. First note that

$$\begin{aligned}\langle \chi(\Phi)_m(u), v\rangle &= \langle \Phi_{N-m-n-1}(\nu_N^{-1}(v)), \nu_N(u)\rangle \\ &= \langle \Phi_{N-m-n-1}((\sigma^{N''-N})^{-1}\nu_{N''}^{-1}(v)), (\sigma^{N''-N})^*\nu_{N''}(u)\rangle\end{aligned}$$

by Lemmas 3.1 and 3.2

$$= \langle (\sigma^{N''-N})\Phi_{N-m-n-1}((\sigma^{N''-N})^{-1}\nu_{N''}^{-1}(v)),\ \nu_{N''}(u)\rangle$$
$$= \langle \Phi_{N''-m-n-1}(\nu_{N''}^{-1}(v)),\ \nu_{N''}(u)\rangle.$$

Hence $\chi(\Phi)_m$ is independent of N. Thus we may assume that the mapping cylinder of f is embedded in $S^{N''}$, or that f is an inclusion.

Let $u \in H^m(Y)$ be such that $\chi(\alpha_r)(u) = 0$ for all r. Then

$$\begin{aligned}
\langle \chi(\Phi)_m(fu), v\rangle &= \langle \Phi_{N''-m-n-1}(\nu_{N''}^{-1}(v)),\ \nu_{N''}(fu)\rangle \\
&= \langle \Phi_{N''-m-n-1}(\nu_{N''}^{-1}(v)),\ (\mathfrak{D}f)^*\nu_{N''}(u)\rangle \\
&= \langle (\mathfrak{D}f)\Phi_{N''-m-n-1}(\nu_{N''}^{-1}(v)),\ \nu_{N''}(u)\rangle \\
&= \langle \Phi_{N''-m-n-1}(\mathfrak{D}f(\nu_{N''}^{-1}(v))),\ \nu_{N''}(u)\rangle \\
&= \langle \Phi_{N''-m-n-1}(\nu_{N''}^{-1}f^*(v)),\ \nu_{N''}(u)\rangle \\
&= \langle \chi(\Phi)_m(u), f^*(v)\rangle \\
&= \langle f\chi(\Phi)_m(u), v\rangle,
\end{aligned}$$

and axiom 3 is verified.

For axiom 4, let

$$f:Y \to X, \qquad g:X \to M = X \cup_f CY, \qquad h:M \to SY, \qquad Sf:SY \to SX,$$

with M, SX, and SY embedded in S^N. Let $u \in H^m(M)$ be such that $\chi(\Phi)_m(gu)$ is defined. Let $w_r \in H^{m+n-q(r)+1}(SY)$ be such that $h(w_r) = \chi(\alpha_r)(u)$. We are to show that

$$f\chi(\Phi)_m(gu) = \left\{\textstyle\sum_{r=1}^R \chi(\alpha_r)w_r\right\} \in H^{m+n}(Y)/f\textstyle\sum_{r=1}^R \chi(\alpha_r)H^{m+n-q(r)}(X).$$

Let

$$v \in \bigcap_{r=1}^{R} \operatorname{Ker}\left(H^{m+n}(Y)^* \xrightarrow{\chi(a_r)^*f^*} H^{m+n-q(r)}(X)^*\right).$$

Also, let $x_r \in H^{N-m-n+q(r)}(\mathfrak{D}M)$ be such that

$$(\mathfrak{D}h)(x_r) = a_r(\nu^{-1}(v)) = \nu^{-1}(\chi(a_r)^*)(v),$$

or

$$h^*\nu(x_r) = \nu(\mathfrak{D}h)x_r = \chi(a_r)^*(v).$$

Then

$$\begin{aligned}
\langle f\chi(\Phi)_m(gu), v\rangle &= \langle \chi(\Phi)_m(gu), f^*(v)\rangle \\
&= \langle \Phi_{N-m-n-1}(\nu^{-1}(f^*(v))),\ \nu(gu)\rangle \\
&= \langle \Phi_{N-m-n-1}((\mathfrak{D}f)(\nu^{-1}(v))),\ (\mathfrak{D}g)^*\nu(u)\rangle \\
&= \langle (\mathfrak{D}g)\Phi_{N-m-n-1}((\mathfrak{D}f)(\nu^{-1}(v))),\ \nu(u)\rangle \\
&= \langle \textstyle\sum_{r=1}^R \alpha_r\, x_r,\ \nu(u)\rangle, && \text{by axiom 4 for } \Phi \\
&= \langle \nu(\textstyle\sum_{r=1}^R \alpha_r\, x_r), u\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \textstyle\sum_{r=1}^{R} \chi(\alpha_r)^* \nu(x_r),\, u\rangle = \sum_{r=1}^{R} \langle \chi(\alpha_r)^* \nu(x_r),\, u\rangle \\
&= \textstyle\sum_{r=1}^{R} \langle \nu(x_r),\, \chi(\alpha_r)(u)\rangle = \sum_{r=1}^{R} \langle \nu(x_r),\, hw_r\rangle \\
&= \textstyle\sum_{r=1}^{R} \langle hw_r,\, \nu(x_r)\rangle = \sum_{r=1}^{R} \langle w_r,\, h^*\nu(x_r)\rangle \\
&= \textstyle\sum_{r=1}^{R} \langle w_r,\, \nu(\mathfrak{D}h)x_r\rangle = \sum_{r=1}^{R} \langle w_r,\, \chi(a_r)^*(v)\rangle \\
&= \textstyle\sum_{r=1}^{R} \langle \chi(a_r)w_r,\, v\rangle = \langle \sum_{r=1}^{R} \chi(a_r)w_r,\, v\rangle,
\end{aligned}
$$

and axiom 4 is proved.

We must also show that $\tilde{\sigma}\chi(\Phi)_m = \chi(\Phi)_{m-1}$.

$$
\begin{aligned}
\langle \tilde{\sigma}\chi(\Phi)_m(u),\, v\rangle &= \langle \sigma^{-1}\chi(\Phi)_m(\sigma u),\, v\rangle \\
&= \langle \chi(\Phi)_m(\sigma u),\, (\sigma^{-1})^*(v)\rangle \\
&= \langle \Phi_{N-m-n-1}(\nu^{-1}(\sigma^{-1})^*(v)),\, \nu\sigma(u)\rangle \\
&= \langle \Phi_{N-m-n-1}(\sigma^{-1}\nu^{-1}(v)),\, \sigma^*\nu(u)\rangle \\
&= \langle \sigma\Phi_{N-m-n-1}(\sigma^{-1}\nu^{-1}(v)),\, \nu(u)\rangle \\
&= \langle \Phi_{N-m-n}(\nu^{-1}(v)),\, \nu(u)\rangle \\
&= \langle \chi(\Phi)_{m-1}(u),\, v\rangle.
\end{aligned}
$$

To finish the proof of Theorem 4.1, we remark that it is sufficient to prove the Adams axioms for X a finite complex, and this we have done.

5. Applications

For a finite complex K, let $d(K)$ denote the least integer N such that K can be embedded in S^N up to homotopy type. In this section we will compute $d(K)$ in two particular examples. We do this not for the intrinsic interest of these particular results, but as examples of the general technique developed by Theorem 4.1. It seems likely that, with more information about secondary cohomology operations, similar results can be obtained for more interesting spaces.

THEOREM 5.1. *Let $X_n = S^n \cup e^{n+3}$ $(n \geqq 2)$, where e^{n+3} is attached by the nontrivial element in $\pi_{n+2}(S^n)$. Then $d(X_n) \geqq n + 6$.*

Proof. Let $\Phi_{1,1}$ be the secondary cohomology operation described in [2]. $\Phi_{1,1}$ is associated with the relation

$$\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^2\,\mathrm{Sq}^2 = 0.$$

(That is, $a_1 = \mathrm{Sq}^1$, $a_2 = \mathrm{Sq}^2$, $\alpha_1 = \mathrm{Sq}^3$, $\alpha_2 = \mathrm{Sq}^2$.) Let $\Psi = \chi(\Phi_{1,1})$. Then Ψ is associated with the relation

$$\mathrm{Sq}^1(\mathrm{Sq}^2\,\mathrm{Sq}^1) + \mathrm{Sq}^2\,\mathrm{Sq}^2 = 0.$$

Ψ is defined on $\mathrm{Ker}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 \cap \mathrm{Ker}\,\mathrm{Sq}^2 \subset H^m(X)$, and has values in

$$H^{m+3}(X)/\mathrm{Sq}^1 H^{m+2}(X) + \mathrm{Sq}^2 H^{m+1}(X).$$

Thus in X_n, we have

$$\Psi: H^n(X_n\,; Z_2) \to H^{n+3}(X_n\,; Z_2).$$

We assert that here Ψ is an isomorphism. To see this, let $Y = S^{n+1} \cup e^{n+3}$, where e^{n+3} is attached by the nontrivial element of $\pi_{n+2}(S^{n+1})$. Define a map $f: Y \to X_n$ by mapping S^{n+1} onto S^n by a generator of $\pi_{n+1}(S^n)$ and extending over e^{n+3} as a homeomorphism on the interior. Then

$$f^*: H^{n+3}(X_n\,; Z_2) \to H^{n+3}(Y; Z_2)$$

is an isomorphism. Let $h \in H^n(X_n\,; Z_2)$ be the generator. By Theorem 6.1 of [5], we see that

$$\begin{aligned} f^*\Psi(h) &= \mathrm{Sq}^2\,\mathrm{Sq}^2_f(h) + \mathrm{Sq}^1(\mathrm{Sq}^2\,\mathrm{Sq}^1)_f(h) \\ &= \mathrm{Sq}^2\,\mathrm{Sq}^2_f(h) = \text{the generator of } H^{n+3}(Y; Z_2). \end{aligned}$$

Hence $\Psi(h)$ generates $H^{n+3}(X_n\,; Z_2)$.

We now assume that X_n is embedded up to homotopy type in S^{n+5} and consider $D = \mathfrak{D}_{n+5}(X_n)$. By the Alexander duality theorem

$$H^1(D; Z_2) \approx Z_2 \approx H^4(D; Z_2),$$

and by Theorem 4.1, $\chi(\Psi) = \Phi_{1,1}$ must give an isomorphism between these groups. We shall now show that $\Phi_{1,1}$ is 0 on this one-dimensional class. To obtain $\Phi_{1,1}$, we begin with a space $K(Z_2\,, 1)$ and kill the class

$$\mathrm{Sq}^1 \in H^2(Z_2\,, 1; Z_2).$$

The resulting space is $K(Z_4\,, 1)$. Since Sq^2 is already zero in $H^3(Z_4\,, 1; Z_2)$, $\Phi_{1,1}$ corresponds to an element of $H^4(Z_4\,, 1; Z_2) \approx Z_2$. This group is generated by $\mathrm{Sq}^2\,\delta_2^*\,\iota$, where ι generates $H^1(Z_4\,, 1; Z_2)$ and δ_2^* is the secondary Bockstein operator. Hence, if $h \in H^1(D; Z_2)$, then $\Phi_{1,1}(h) = \alpha\mathrm{Sq}^2\,\delta_2^*\,h$, where $\alpha = 0$ or 1. However, in D, $\delta_2^*\,h = 0$ because $H^2(D; Z_2) = 0$. Thus X_n cannot be embedded in S^{n+5}.

COROLLARY 5.2. $d(X_n) = n + 6$.

Proof. In order to prove that X_n can be embedded in S^{n+6}, it is enough to show that X_2 can be embedded in S^8 as $SX_n = X_{n+1}$. Let M be the mapping cylinder of a nontrivial map $S^4 \to S^2$. $X_2 = M \cup e^5$, where $e^5 \cap M = S^4$, the boundary of e^5 and the upper cap in M. M is a subspace of the join $S^4 * S^2 = S^7$, where S^4 is a subequator of S^7, and we embed S^7 as an equator in S^8. Clearly, we may embed e^5 in S^8 so that $e^5 \cap S^7 = S^4$. This embeds X_2 in S^8.

THEOREM 5.3. *Let $K_n = S^n \cup e^{n+7}$ $(n \geqq 4)$, where e^{n+7} is attached by the composition $S^{n+6} \to S^{n+3} \to S^n$ of the suspensions of the Hopf class $\nu \in \pi_7(S^4)$. Then $d(K_n) \geqq n + 12$.*

Proof. The proof of this theorem is, in part, very similar to that of Theorem 5.1 and hence will be only sketched. Let Φ be the secondary cohomology operation associated with the relation

$$\mathrm{Sq}^4\,\mathrm{Sq}^4 + \mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^7\,\mathrm{Sq}^1 = 0.$$

Then $\Psi = \chi(\Phi)$ is associated with the relation

$$(\mathrm{Sq}^4 + \mathrm{Sq}^3\,\mathrm{Sq}^1)(\mathrm{Sq}^4 + \mathrm{Sq}^3\,\mathrm{Sq}^1) + \mathrm{Sq}^2(\mathrm{Sq}^4\,\mathrm{Sq}^2) + \mathrm{Sq}^1(\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) = 0.$$

By considering a space analogous to Y in the proof of Theorem 5.1 and using Theorem 6.1 of [5], we see that Ψ is nonzero in K_n. Thus, if $K_n \subset S^{n+11}$, Φ is nonzero on the three-dimensional cohomology class in $\mathfrak{D}_{n+11}(K_n)$.

To show that this is impossible, we consider the space W constructed from $K(Z_2, 3)$ by killing Sq^1, Sq^2, and Sq^4. By explicitly computing $H^{10}(W; Z_2)$, we see that Φ on a three-dimensional class is decomposable into cup products and Steenrod squares of cohomology classes of dimensions between 4 and 9. Hence Φ is 0 in $\mathfrak{D}_{n+11}(K_n)$.

Corollary 5.4. $d(K_n) = n + 12$.

Proof. The construction in Corollary 5.2 can be carried over to a similar construction for K_n.

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